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THE SPACES $\mathcal{L}^{(p,\lambda)}$, $N^{(p,\lambda)}$ AND INTERPOLATION (*)

by GUIDO STAMPACCHIA

Recently properties of certain spaces of functions $\mathcal{L}^{(p,\lambda)}$ (see definition 1.1) have been studied by various authors (F. John and L. Nirenberg [6], Campanato [1], Meyers [7], Stampacchia [11], Peetre [9]) (see § 1). It has been shown by Campanato [1] and Meyers [7] independently that the classes of Hölder continuous functions are contained in this family of spaces; F. John and L. Nirenberg have given a characterization of the functions belonging to $\mathcal{L}^{(p,0)} \equiv \mathcal{E}_0$.

Making use of these results the author has given in [11] some theorems of interpolation for this family of spaces which permits us to collegate the spaces L^p with the spaces of Hölder continuous functions. Successively more general theorems have been proved by Campanato and Murthy [4], Peetre [9], [9′] and Grisvard [5], [5′].

In the paper [11] the author established, among others, a theorem of interpolation for linear operations whose image spaces vary from L^p to $\mathcal{E}_0$ (see theorem 3.1 of [11]). However, the proof of this theorem was not complete, as was indicated to the author by Campanato. Here our main object is to give a complete proof of this theorem (see theorem 4.1, here).

We make use of a lemma due to F. John and L. Nirenberg [6] in connection with the spaces $N^{(p,\lambda)}$ defined in § 2.

In section 3 we introduce subclasses of the spaces $\mathcal{L}^{(p,\lambda)}$ and we use them in order to improve some inclusion properties of the Morrey's spaces, proved by Campanato in [3] (see Theorem 3.2.).

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In section 4 we give applications of the interpolation theorems of [11] and of section 2.

In section 5 we indicate some simple results on functions which are Hölder continuous of strong type connecting them with the classical theorem of Rademacher on the differentiability in the strong sense of Lipschitz functions.

In a paper in collaboration with Campanato we are going to apply the interpolation theorem of section 2 together with some recent results of Campanato [2] to the theory of partial differential equations of elliptic type.

The author wishes to thank Nirenberg, Campanato and Murthy for the discussions he had with them in connection with the results considered here.

§ 1. The $\mathcal{L}^{(p,\lambda)}$-spaces.

We shall always consider, for the sake of simplicity, real valued (unless otherwise explicitely stated) integrable functions on a fixed bounded cube Q_0 in E^n.

A generic subcube of Q_0 having its sides parallel to those of Q_0 will be denoted by Q and its measure by $|Q|$. The mean value on a subcube Q of Q_0 of a function u will be denoted by u_Q:

$$u_Q = \frac{1}{|Q|}\int_Q u(x)\, dx.$$

DEFINITION 1.1. A function u is said to belong to $\mathcal{L}^{(p,\lambda)}(Q_0) \equiv \mathcal{L}^{(p,\lambda)}$, where $p \geq 1$, $-\infty < \lambda < +\infty$, if there exists a constant $K(u) \equiv K$ such that

$$\int_Q |u - u_Q|^p\, dx \leq K^p\, |Q|^{1-\frac{\lambda}{n}} \tag{1.1}$$

for every $Q \subset Q_0$.

A semi norm in $\mathcal{L}^{(p,\lambda)}$ is given by

$$[u]_{\mathcal{L}^{(p,\lambda)}} = \sup_{Q \subset Q_0}\left\{|Q|^{\frac{\lambda}{n}-1}\int_Q |u - u_Q|^p dx\right\}^{1/p}. \tag{1.2}$$

and a norm is obtained by setting

$$\| u \|_{\mathcal{L}^{(p,\lambda)}} = \| u \|_{L^p} + [u]_{\mathcal{L}^{(p,\lambda)}} \tag{1.3}$$

which renders $\mathcal{L}^{(p,\lambda)}$ with the structure of a Banach space.

We recall some results.

LEMMA 1.1 [11]. *The Banach space $\mathcal{L}^{(p,n)}$ is isomorphic to $L^p(Q_0)$ and* (1.3) *is equivalent to the norm $\| u \|_{L^p}$.*

THEOREM 1.1 (Campanato [1] and Meyers [7]). *If $\lambda < 0$ then $\mathcal{L}^{(p,\lambda)}$ is isomorphic to $C_{0,-\frac{\lambda}{p}}(Q_0)$ and the norm* (1.3) *is equivalent to the norm*

$$\| u \|_{C_{0,-\frac{\lambda}{p}}} = \max_{Q_0} | u(x) | + [u]_{C_{0,-\frac{\lambda}{p}}}$$

where, for $0 < \alpha < 1$,

$$[u]_{C_{0,\alpha}} = \sup_{x', x'' \in Q_0} \frac{| u(x') - u(x'') |}{| x' - x'' |^{\alpha}} .$$

DEFINITION 1.2. A function u is said to belong to $\mathcal{E}_0$ if there exist positive constants H and β such that

$$\text{meas}\, \{x \in Q \, ; \, | u(x) - u_Q | > \eta\} \leq H e^{-\beta\eta} \, | Q |$$

for every $Q \subset Q_0$.

THEOREM 1.2 (F. John and L. Nirenberg [6]). *A function u belongs to $\mathcal{E}_0$ if and only if u belongs to $\mathcal{L}^{(p,0)}$ for some $p \geq 1$.*

LEMMA 1.2. *If $q \geq p$ and $\mu/q \leq \lambda/p$ then*

$$\mathcal{L}^{(q,\mu)} \subset \mathcal{L}^{(p,\lambda)}.$$

DEFINITION 1.3. A linear operation T on functions f defined over Q_0 is said to be of strong type $\mathcal{L}[p,(q,\mu)]$ if there exists a constant K, independent of f, such that

$$[Tf]_{\mathcal{L}^{(q,\mu)}} \leq K \, \| f \|_{L^p} ; \tag{1.4}$$

the smallest of the constants K in (1.4) is called the strong $\mathcal{L}[p,(q,\mu)]$ norm of T.

We now introduce the following expression:

$$\Phi_\mu(u, \sigma) = \sup_{Q \subset Q_0} [| Q |^{\mu/n-1} \operatorname{meas} \{x \in Q ; | u(x) - u_Q | > \sigma\}].$$

DEFINITION 1.3′. A linear operation T on functions defined over Q_0 is said to be of weak type $\mathcal{L}[p, (q, \mu)]$ if there exists a constant K, independent of f, such that

$$\Phi_\mu(Tf, \sigma) \leq \left(\frac{K \| f \|_{L^p}}{\sigma}\right)^q; \tag{1.5}$$

the smallest of the constants K in (1.5) is called the weak $\mathcal{L}[p, (q, \mu)]$ norm of T.

THEOREM 1.3 [11]. *Let* $[p_i, q_i, \mu_i]$ *be real numbers satisfying the conditions*

$$p_i \geq 1, p_i \leq q_i \ (i = 1, 2) ; \ p_1 \neq p_2 \ \text{and} \ q_1 \neq q_2 .$$

For $0 < t < 1$ *let* $[p(t), q(t), \mu(t)] = [p, q, \mu]$ *be defined by the relations*

$$\begin{cases} \dfrac{1}{p} = \dfrac{(1-t)}{p_1} + \dfrac{t}{p_2}, \quad \dfrac{1}{q} = \dfrac{(1-t)}{q_1} + \dfrac{t}{q_2}, \\ \dfrac{\mu}{q} = (1-t)\dfrac{\mu_1}{q_1} + t\,\dfrac{\mu_2}{q_2}. \end{cases} \tag{1.6}$$

If T *is a linear operation which is simultaneously of weak types* $\mathcal{L}[p_i, (q_i, \mu_i)]$ *with respective norms* $K_i \ (i = 1, 2)$ *then* T *is of strong type* $\mathcal{L}[p, (q, \mu)]$ *for* $0 < t < 1$ *and*

$$[Tf]_{\mathcal{L}^{(q,\mu)}} \leq \mathcal{K}\, K_1^{(1-t)} K_2^t \| f \|_{L^p(Q_0)}$$

where $\mathcal{K}$ *is a constant, independent of* f, *but depending on* t, p_i, q_i, μ_i *and it is bounded for* t *away from* 0 *and* 1.

THEOREM 1.4 (Campanato and Murthy [4]). *Let* $[p_i, q_i, \mu_i]$ *be real numbers such that* $p_i, q_i \geq 1 \ (i = 1, 2)$. *If* T *is a linear operation (in general on complex valued functions on* Q_0*) which is simultaneously of strong types* $\mathcal{L}[p_i, (q_i, \mu_i)]$ *with respective norms* $K_i \ (i = 1, 2)$ *then* T *is of strong type* $\mathcal{L}[p, (q, \mu)]$ *where* p, q, μ *are defined for* $0 \leq t \leq 1$ *by* (1.6) *and further the* *following estimate holds :*

$$[u]_{\mathcal{L}^{(q,\mu)}} \leq K_1^{1-t} K_2^t \| u \|_{L^p} .$$

§ 2. **The $N^{(p,\lambda)}$ spaces.**

We fix a bounded cube in E^n as in § 1. We shall denote by $\bar{S}$ the family of systems S of a finite number of subcubes Q_i no two of which have an interior point in common and having their sides parallel to those of Q_0 $(\bigcup_i Q_i \subset Q_0)$.

For any (real or complex valued) function $u \in L^1(Q_0)$ and for any $1 < p < +\infty$ we consider the expressions of the form

$$\sum_i \left| \int_{Q_i} | u - u_{Q_i} | \, dx \right|^p | Q_i |^{(1-p-\lambda)}$$

where Q_i runs through a system $S \in \bar{S}$.

For $1 < p < +\infty$ and $-\infty < \lambda < +\infty$ set

$$[u]_{N^{(p,\lambda)}} = \sup_{\{Q_i\} \equiv S \in \bar{S}} \left\{ \sum_i \left| \int_{Q_i} | u - u_{Q_i} | \, dx \right|^p | Q_i |^{(1-p-\lambda)} \right\}^{1/p}$$

and the following

Definition 2.1. A function u is said to belong to $N^{(p,\lambda)}$ $(1 < p < +\infty, -\infty < \lambda < +\infty)$ if $[u]_{N^{(p,\lambda)}} < +\infty$. We observe that $[u]_{N^{(p,\lambda)}}$ defines a semi norm in $N^{(p,\lambda)}$ and we obtain a Banach space by taking

$$\| u \|_{N^{(p,\lambda)}} = \| u \|_{L^1} + [u]_{N^{(p,\lambda)}}$$

as the norm in $N^{(p,\lambda)}$.

Remark. It can be verified that

$$[u]_{N^{(p,\lambda)}} = \sup_{\{Q_i\} \equiv \Sigma} \left\{ \sum_i \left| \int_{Q_i} | u - u_{Q_i} | \, dx \right|^p | Q_i |^{(1-p-\lambda)} \right\}^{1/p} .$$

where Σ is any decomposition of the cube Q_0 into an infinite number of subcubes Q_i no two of which have a common interior point and having their sides parallel to those of Q_0.

LEMMA 2.1. *If* $q \geq p$ *and* $\mu/q \geq \lambda/p$ *then* $N^{(q,\mu)} \subset N^{(p,\lambda)}$.
For the proof it is enough to apply Holder's inequality.

We now set

$$F[u, v, S] = \sum_i \int_{Q_i} (u - u_{Q_i})\, v\, dx \cdot |Q_i|^{-\lambda/p}$$

for every $v \in L^\infty(Q)$ and for every $S \in \bar{S}$.

LEMMA 2.2. *For every* $S \in \bar{S}$ *we have*

$$|F[u, v, S]| \leq [u]_{N^{(p,\lambda)}} \sup_{\{Q_i\} \equiv S \in \bar{S}} \left\{ \sum_i |v_i|^{p'} |Q_i| \right\}^{1/p'} \tag{2.1}$$

where $v_i = \sup_{Q_i} |v(x)|$.

LEMMA 2.3. *For any* $u \in N^{(p,\lambda)}$ *we have*

$$[u]_{N(p,\lambda)} = \sup_{S \in \bar{S}} \sup_{v \in L^\infty} \{|F[u, v, S]|, \sum_i |v_i|^{p'} |Q_i| = 1\}.$$

PROOF. If $u \in N^{(p,\lambda)}$, $v \in L^\infty(Q)$ and $S \in \bar{S}$ then

$$|F[u, v, S]| \leq \sum_i \int_{Q_i} |u - u_{Q_i}|\, dx\, |Q_i|^{\frac{1-\lambda}{p} - 1} v_i\, |Q_i|^{1/p'} \leq$$

$$\leq \left\{ \sum_i \left| \int_{Q_i} |u - u_{Q_i}|\, dx \right|^p |Q_i|^{1-\lambda-p} \right\}^{1/p} \left\{ \sum_i |v_i|^{p'} |Q_i| \right\}^{1/p'}.$$

We observe that it is possible to choose a v in each Q_i in such a manner that the first inequality above becomes actually an equality; for this purpose it is enough to take

$$v(x) = c_i [\overline{\text{sign}\ (u - u_{Q_i})}] \text{ in } Q_i$$

where c_i are arbitrary constants.

It is possible to choose the constants c_i in such a manner that in the second inequality also we have an equality.

From these observations the lemma 2.3 follows.

A function is said to be simple if it assumes only a finite number of values. We shall denote by $\mathcal{F}$ the set of simple functions on Q_0.

We observe that for simple functions equality holds in (2.1). Therefore the lemma 2.3 holds good also when the supremun is taken over simple functions $v \in \mathcal{F}$.

LEMMA 2.4. *If* $u \in L^1(Q_0)$, *one has*

$$\lim_{p \to +\infty} [u]_{N^{(p,0)}} = [u]_{\mathcal{L}^{(1,0)}} .$$

We have the following more general

LEMMA 2.5. *If* $u \in L^1(Q_0)$, *one has*

$$\lim_{p \to +\infty} [u]_{N\left(p, \frac{-p\lambda(p)}{n}\right)} = [u]_{\mathcal{L}^{(1,\lambda)}}$$

where

$$\lambda(p) \geq \lambda \quad \textit{and} \quad \lambda(p) \to \lambda \quad \textit{for} \quad p \to +\infty .$$

PROOF. Let $M = [u]_{\mathcal{L}^{(1,\lambda)}}$. If $M = 0$ then it follows that $[u]_{N\left(p, \frac{-p\lambda(p)}{n}\right)} = 0$ and hence we can without loss of generality assume that $M > 0$. Let M' be a number such that $0 < M' < M$. Then there exists at least one subcube Q of Q_0 for which

$$\int_Q | u - u_Q | \, dx \, | Q |^{\frac{\lambda}{n} - 1} \geq M'$$

and hence

$$\Phi(p) = [u]_{N\left(p, \frac{-p\lambda(p)}{n}\right)} \geq M' \, | Q |^{\frac{1}{p} + \frac{1}{n}(\lambda(p) - \lambda)}$$

consequently we see that

$$\min \lim_{p \to +\infty} \Phi(p) \geq M.$$

Hence when $M = +\infty$ the lemma is proved. If, instead, $M < +\infty$ then

$$\Phi(p) \leq M \left\{ \sum_i | Q_i |^{1 + \frac{p}{n}(\lambda(p) - \lambda)} \right\}^{1/p}$$

$$\leq M \, | Q_0 |^{1/p} \, | Q_0 |^{\frac{1}{n}(\lambda(p) - \lambda)}$$

and therefore

$$\max_{p\to+\infty} \lim \Phi(p) \leq M$$

proving the lemma.

REMARK. Lemma 2.5 enables us to put

$$[u]_{N^{(\infty,0)}} = [u]_{\mathcal{L}^{(1,0)}} \tag{2.2}$$

and as a consequence the lemmas 2.2 and 2.3 can be extended also to the case where in $p = +\infty$, $\lambda = 0$.

DEFINITION 2.2. A function u is said to belong to the space L^p-weak ($p \geq 1$) if there exists a constant K such that for any $\sigma > 0$

$$\text{meas } \{x \in Q_0 ; |u(x)| > \sigma\} \leq \left(\frac{K}{\sigma}\right)^p . \tag{2.3}$$

We shall denote this in notation by writing $u \in M^p$. The smallest constant K in (2.3) is called the norm of u in M^p and we denote it by $\|u\|_{M^p}$.

We have the following fundamental theorem due to F. John and L. Nirenberg [6].

THEOREM 2.1. *If* $u \in N^{(p,0)}$ *(real) with* $p > 1$ *then*

$$u - u_{Q_0} \in M^p .$$

and

$$\| u - u_{Q_0} \|_{M^p} \leq A \, [u]_{N^{(p,0)}}$$

where the constant A *depends only on* n *and* p.

REMARK. In the definition of the space $N^{(p,\lambda)}$ one can replace u_Q by a constant $\bar{u}_Q$ associated to Q and linearly to u. Thus one obtains a different space which is contained in $N^{(p,\lambda)}$. For,

$$\int_Q |u - u_Q| \, dx \leq 2 \int_Q |u - \bar{u}_Q| \, dx .$$

If, in particular, $\bar{u}_Q = 0$ then the hypothesis, in the theorem which corresponds to theorem 2.1, reduces to those of a well known theorem due to F. Riesz [8] whence the conclusion would be that not only $u \in M^p$ but also to L^p.

§ 3. A theorem of interpolation in the space $N^{(p,\lambda)}$.

DEFINITION 3.1. A linear operation T defined on functions in $\mathcal{F} = \mathcal{F}(Q_0)$ is said to be of strong type $N[p, (q, \mu)]$ if there exists a constant K such that

$$[Tu]_{N(q,\mu)} \leq K \| u \|_{L^p} \text{ for every } u \in \mathcal{F}; \tag{3.1}$$

the smallest of the constants K for which the inequality (3.1) holds is called the strong $N[p, (q, \mu)]$-norm.

Then we prove the following

THEOREM 3.1. *Let* $[p_i, q_i, \mu_i]$ *be real numbers such that* $p_i, q_i \geq 1$ $(i = 1, 2)$. *If* T *is a linear operation which is simultaneously of strong types* $N[p_i, (q_i, \mu_i)]$ *with respective norms* $K_i (i = 1, 2)$ *then* T *is of strong type* $N[p, (q, \mu)]$ *where*

$$\begin{cases} \dfrac{1}{p} = \dfrac{1-t}{p_1} + \dfrac{t}{p_2}, \quad \dfrac{1}{q} = \dfrac{1-t}{q_1} + \dfrac{t}{q_2} \\ \dfrac{\mu}{q} = (1-t)\dfrac{\mu_1}{q_1} + t\dfrac{\mu_2}{q_2} \quad \text{for} \quad 0 \leq t \leq 1 . \end{cases} \tag{3.2}$$

Moreover we have the inequality

$$[Tu]_{N(q,\mu)} \leq K_1^{1-t} K_2^t \| u \|_{L^p} \tag{3.3}$$

The theorem holds also in the limit cases $p_1 = +\infty$ *and* $\dfrac{1}{q_1} = \mu_1 = 0$.

This theorem is of the type of the Riesz-Thorin [12] and a theorem of Campanato and Murthy [4].

PROOF. We fix a $t \in [0, 1]$ and consequently the numbers p, q, μ defined by (3.2). We assume that $u \in \mathcal{F}$ with $\| u \|_{L^p} = 1$. We set

$$\frac{1}{p} = \alpha, \qquad \frac{1}{p_i} = \alpha_i; \qquad \frac{1}{q} = \beta, \qquad \frac{1}{q_i} = \beta_i,$$

and for complex z in the infinite strip

$$\Sigma(0, 1) = \{z = \xi + i\eta \in \mathbb{C}^1 ;\ 0 \leq \xi \leq 1,\ -\infty < \eta < +\infty\}$$

we define

$$\alpha(z) = (1 - z)\,\alpha_1 + z\alpha_2\,,$$

$$\beta(z) = (1 - z)\,\beta_1 + z\beta_2$$

$$\mu(z)\,\beta(z) = (1 - z)\,\mu_1\,\beta_1 + z\mu_2\,\beta_2\,.$$

We shall from now on consider only the class $\mathcal{F}$ of simple functions on Q_0.

For any $u \in \mathcal{F}$ with $\| u \|_{L_p} = 1$ we define the function $\tilde{u}$ depending on the complex parameter z by

$$\tilde{u}(y, z) = | u(y) |^{\alpha(z)\alpha^{-1}} e^{i \arg u(y)}$$

for $y \in Q_0$ and $z \in \Sigma(0, 1)$. It can be easily seen that for any $z \in \Sigma(0, 1)$ the function $\tilde{u}(y, z) \in \mathcal{F}$ and therefore $T\tilde{u}$ is defined.

Since

$$| \tilde{u}(y, z) | = | u(y) |^{[(1-\xi)\alpha_1 + \xi\alpha_2]\alpha^{-1}}$$

we have

$$\| \tilde{u}(y, i\eta) \|_{L_{p_1}} = \| \, | u |^{\alpha_1\alpha^{-1}} \|_{L_{p_1}} = \| u \|_{L_p}^{\alpha_1\alpha^{-1}} = 1$$

and analogously we have also

$$\| \tilde{u}(y, 1 + i\eta) \|_{L_{p_2}} = \| \, | u |^{\alpha_2\alpha^{-1}} \|_{L_{p_2}} = \| u \|_{L_p}^{\alpha_2\alpha^{-1}} = 1\,.$$

We note that these two relations are valid also when $\alpha_i = 0$. Let us fix a sistem $S \in \bar{S}$ of a finite number of subcubes Q_i of Q_0 as defined in § 2.

We shall denote by $\mathcal{F}_S$ the class of simple functions $v \in \mathcal{F}$ such that, denoting $\sup\limits_{Q_i} | v |$ by v_i, we have

$$\{\sum_i | v_i |^{q'} \, | Q_i |\}^{1/q'} = 1.$$

Now setting

$$\tilde{v}(y, z) = | v(y) |^{[1-\beta(z)]q'} e^{i \arg v(y)}$$

we obtain

$$| \tilde{v}(y, z) | = | v(y) |^{[1-(1-\xi)\beta_1 - \xi\beta_2]q'}$$

and since

$$\tilde{v}_i = | v_i |^{[1-(1-\xi)\beta_1 - \xi\beta_2]q'}$$

it follows that

$$\{\sum_i |\tilde{v}_i(y, z_k)|^{q'_k} |Q_i|\}^{1/q'_k} = \{\sum_i |v_i|^{(1-\beta_k)q'q'_k} |Q_i|\}^{1/q'_k} \tag{3.4}$$

$$= \{\sum_i |v_i|^{q'} |Q_i|\}^{\frac{1}{q'}(1-\beta_k)q'_k} = 1\,,$$

where $k = 1, 2$ and $z_1 = i\eta$, $z_2 = 1 + i\eta$.

For $z \in \Sigma\,(0, 1)$ and for a fixed system $S \in \overline{S}$ of a finite number of subcubes Q_i of Q_0 as in section 2 we introduce

$$\Phi(S, z) = \sum_i \int_{Q_i} [T\tilde{u} - (T\tilde{u})_{Q_i}]\, \tilde{v}\, dx\, |Q_i|^{-\mu(z)\beta(z)}$$

where $v \in \mathcal{F}_S$.

The function $\Phi(S, z)$ is holomorphic in the interior of $\Sigma\,(0, 1)$ and continuous and bounded in $\Sigma\,(0, 1)$. In fact, if c_j are the non zero values assumed by u and χ_j is the caracteristic function of the set $\{x \in Q_0\,;\ u(x) = c_j\}$ then

$$\tilde{u} = \sum_j e^{i \arg c_i} |c_i|^{\alpha(z)/\alpha} \chi_j$$

and

$$T\tilde{u} = \sum_j e^{i \arg c_j} |c_j|^{\alpha(z)/\alpha}\, T\chi_j\,,$$

$$(T\tilde{u})_{Q_i} = \sum_i e^{i \arg c_j} |c_j|^{\alpha(z)/\alpha} (T\chi_j)_{Q_i}\,.$$

Therefore $\Phi(S, z)$ can be expressed as a finite sum of exponentials of the form a^z with $a > 0$.

Using now the lemma 2.2 we get

$$|\Phi(S, i\eta)| \leq [T\tilde{u}]_{N(q_1, \mu_1)} \{\sum_i |\tilde{v}_i|^{q'_1} |Q_i|\}^{1/q'_1} \leq K_1 \tag{3.5}$$

and

$$|\Phi(S, 1 + i\eta)| \leq [T\tilde{u}]_{N(q_2, \mu_2)} \{\sum_i |\tilde{v}_i|^{q'_2} \cdot |Q_i|\}^{1/q'_2} \leq K_2\,. \tag{3.6}$$

Then by applying the theorem of three lines [12, Vol. II, p. 93] we conclude that

$$|\Phi(S, t)| \leq K_1^{1-t} K_2^t\,. \tag{3.7}$$

Then in view of the lemma 2.3

$$[T\tilde{u}]_{N^{(q,\mu)}} \leq K_1^{1-t} K_2^t \| u \|_{L^p}$$

which proves the required inequality (3.3).

In the limit cases where in $p_1 = +\infty$ or $\frac{1}{q_1} = \mu_1 = 0$ the proof can be carried over in a perfectly analogous way by making use, in the latter case, of the remark following the lemma 2.5. Taking into account (2.2) the semi-norm $[u]_{N^{(\infty,0)}}$ is to be substituted by $[u]_{\mathcal{L}^{(1,0)}}$. So, instead of (3.5), we have

$$\Phi(S, i\,\eta) \leq [T\tilde{u}]_{\mathcal{L}^{(1,0)}} \{\sum_i |\tilde{v}_i| \, |Q_i|\} \leq K_1 \tag{3.8}$$

since (3.4) holds good for $\beta_1 = \frac{1}{q_1} = 0$, $q_1' = 1$. (3.8) together with (3.6) gives (3.7) and so (3.3) follows.

§ 3. The spaces $\mathcal{L}_r^{(p,\lambda)}$.

In this section we consider subclasses of functions of $\mathcal{L}^{(p,\lambda)}$

DEFINITION 2.3. A function u is said to belong to the space $\mathcal{L}_r^{(q,\mu)}$ if $u \in \mathcal{L}^{(q,\mu)}$ and moreover setting, for any subcube Q of Q_0, $K(Q)$ to be the norm

$$[u]_{\mathcal{L}^{(q,\mu)}(Q)}$$

of the function u restricted to Q, there exists a number $L = L_u$ such that

$$\sum_i | K(Q_i) |^r \leq L$$

for any system $\{Q_i\} \equiv S$ of $\overline{S}$ introduced at the beginning of section 2.

LEMMA 3.1. *If* $u \in \mathcal{L}_r^{(q,\mu)}$ *and* λ *is a number such that* $\lambda \leq 1 - \frac{\mu}{nq}$ *and* $p = \frac{nq}{\mu}(1-\lambda) \geq r$ *then*

$$u \in N^{(p,\lambda)};$$

moreover

$$[u]_{N^{(p,\lambda)}} \leq [u]_{\mathcal{L}^{(q,\mu)}}.$$

PROOF. Since for any subcube Q of Q_0 we have

$$\int_Q | u - u_Q|^q \, dx \leq [K(Q)]^q \, | Q |^{1-\frac{\mu}{n}}$$

we obtain

$$\int_Q | u - u_Q | \, dx \leq K(Q) \, | Q |^{1-\frac{\mu}{nq}} .$$

Hence, for any $\{Q_i\} \equiv S$ of $\overline{S}$, we have

$$\sum_i \left| \int_{Q_i} | u - u_{Q_i} | \, dx \right|^p | Q_i |^{1-p-\lambda} \leq \sum_i | K(Q_i) |^p \, | Q_i |^{1-\frac{\mu p}{np}-\lambda}$$

$$\leq \sum_i | K(Q_i) |^r \, | K(Q_i) |^{p-r} \, | Q_0 |^{1-\frac{\mu p}{nq}-\lambda}$$

$$\leq | K(Q_0) |^p \cdot | Q_0 |^{1-\frac{\mu p}{nq}-\lambda} .$$

COROLLARY 3.1. *If* $u \in \mathcal{L}^{(q, n)} \equiv L^q(Q_0)$ *with* $q > 1$ *then* $u \in \mathcal{L}_q^{(1, n/q)}$ *and, hence* $u \in N^{(q, 0)}$ *with*

$$[u]_{N^{(q, 0)}} \leq 2 \, \| u \|_{L^q} .$$

In fact, by Holder's inequality we have that

$$\int_Q | u - u_Q | \, dx \leq \left(\int_Q | u - u_Q |^q \, dx \right)^{1/q} | Q |^{1-1/q}$$

$$\leq 2 \left(\int_Q | u |^q \, dx \right)^{1/q} | Q |^{1-1/q} .$$

Now since the function

$$K(Q) = 2 \left(\int_Q | u |^q \, dx \right)^{1/q}$$

satisfies the condition that $| K(Q) |^q$ is additive it follows that $u \in \mathcal{L}_q^{\left(1, \frac{n}{q}\right)}$.

DEFINITION 3.2. The space $H^1(Q_0)$ is the completion of the class $C^1(\overline{Q}_0)$ of once continuously differentiable functions on $\overline{Q}_0$ with respect to the norm

$$\| u \|_{H^1(Q_0)} = \| u \|_{L^2(Q_0)} + \| u_x \|_{L^2(Q_0)}$$

where u_x denotes the gradient of u.

LEMMA 3.2. *Any function* $u \in H^1(Q_0)$ *satisfying the condition that, for any subcube* $Q \subset Q_0$

$$\int_Q | u_x |^q \, dx \leq C^q \, | Q |^{1-\mu/n} \quad \textit{with } 1 < q < \mu \leq n$$

(C *being a positive constant independent of* $Q \subset Q_0$*) belongs to* $\mathcal{L}_{\tilde{q}}^{(1,\lambda)}$ *where* $\dfrac{1}{\tilde{q}} = \dfrac{1}{q} - \dfrac{1}{\mu}$ *and* $\lambda = \dfrac{n}{\tilde{q}}$.

As a consequence $u \in N^{(\tilde{q},0)}$ *and*

$$[u]_{N(\tilde{q},0)} \leq C \, | Q_0 |^{\left(1-\frac{\mu}{n}\right)\frac{1}{\tilde{q}}} .$$

PROOF. Applying Holder's inequality, the Poincaré inequality and using the fact that $\tilde{q} > q$ we obtain

$$\int_Q | u - u_Q | \, dx \leq \left(\int_Q | u - u_Q |^q \, dx \right)^{1/q} | Q |^{1-1/q}$$

$$\leq \left(\int_Q | u_x |^q \, dx \right)^{1/q} | Q |^{1-1/q+1/n}$$

$$\leq \left(\int_Q | u_x |^q \, dx \right)^{1/\tilde{q}} \left(\int_Q | u_x |^q \, dx \right)^{1/q-1/\tilde{q}} | Q |^{1-\frac{1}{q}+\frac{1}{n}}$$

$$\leq \left(\int_Q | u_x |^q \, dx \right)^{1/\tilde{q}} C^{q/\mu} \, | Q |^{1-\frac{1}{q}-\frac{1}{\mu}} .$$

Hence taking

$$K(Q) = C^{\frac{q}{\mu}}\left(\int_Q |u_x|^q\, dx\right)^{1/\tilde{q}}$$

we conclude that

$$u \in \mathcal{L}_{\tilde{q}}^{(1,\lambda)} \text{ and } [u]_{\mathcal{L}^{(1,\lambda)}} \leq C\,|Q_0|^{\left(1-\frac{\mu}{n}\right)\frac{1}{\tilde{q}}}.$$

Now, in view of lemma 3.1, it follows that $u \in N^{(\tilde{q},0)}$ and

$$[u]_{N^{(\tilde{q},0)}} \leq C\,|Q_0|^{\left(1-\frac{\mu}{n}\right)1/\tilde{q}}$$

and this completes the proof of the lemma.

We have now the following result which is an improvement of a result of Campanato [3] related to a well known result due to Morrey.

THEOREM 3.2. *Let $u \in H^1(Q_0)$ be such that for any subcube Q of Q_0*

$$\int_Q |u_x|^q\, dx \leq C^q\,|Q|^{1-\frac{\mu}{n}} \qquad 0 \leq \mu \leq n$$

with a constant C independent of Q. Then the following estimates hold for u:

(i) *If $q < \mu$ then $u - u_{Q_0} \in M^{\tilde{q}}$ where $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{1}{\mu}$ and*

$$\| u - u_{Q_0} \|_{M^{\tilde{q}}} \leq C\,|Q_0|^{\left(1-\frac{\mu}{n}\right)\frac{1}{\tilde{q}}};$$

(ii) *if $q = \mu$ then $u \in \mathcal{L}^{(1,0)}$ and*

$$\| u \|_{\mathcal{L}^{(1,0)}} \leq C;$$

(iii) *if $q > \mu$ then $u \in \mathcal{L}^{(1,\lambda)}$ where $\lambda = \frac{n}{\tilde{q}}$ and hence $u \in C_{0,\left(\frac{n}{\mu}-\frac{n}{q}\right)}$.*

PROOF. From the lemmas 3.2 and 3.1 together with the theorem 2.1 the assertion (i) follows. The assertion (ii) is a consequence of Poincaré inequality and the theorem 1.2 and finally, the assertion (iii) follows from the Poincaré inequality and the theorem 1.1 (see [1], [7]).

§ 4. Applications.

THEOREM 4.1. *Let T be a linear operation defined on the class $\mathcal{F}$ of (real valued) simple functions on Q_0 such that*

$$[Tu]_{\mathcal{L}^{(1,0)}} \leq K_1 \| u \|_{L^{p_1}},$$

$$[Tu]_{\mathcal{L}_r^{(q_2,\mu_2)}} \leq K_2 \| u \|_{L^{p_2}}$$

where $p_1, p_2, q_2 \geq 1$ and $r \leq \frac{nq_2}{\mu_2}$. If $p, q \geq 1$ are defined by

$$\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}, \qquad \frac{1}{q} = \frac{t}{q_2} - \frac{\mu_2}{n} \tag{4.1}$$

then

$$\| Tu - (Tu)_{Q_0} \|_{L^q} \leq \mathcal{K}\, K_1^{1-t} K_2^t \| u \|_{L^p} \text{ for } u \in \mathcal{F}$$

where $\mathcal{K}$ is a constant which is bounded if t is away from 0 and 1.

The theorem is valid also for $p_1 = +\infty$.

PROOF. From the remark following the lemma 2.5 we have $\mathcal{L}^{(1,0)} \equiv N^{(\infty,0)}$ and from the lemma 3.1 we have $\mathcal{L}_r^{(q_2,\mu_2)} \subset N^{(nq_2/\mu_2,0)}$. Hence the linear operation T is simultaneously of strong types $N[p_1,(\infty,0)]$ and $N[p_2,(nq_2/\mu_2,0)]$ and so by the theorem (3.1) it follows that T is of strong type $N[p,(q,0)]$ for p and q given by (4.1) for $0 \leq t \leq 1$. Now applying theorem 2.1 to $Tu \in N^{(q,0)}$ we see that

$$Tu - (Tu)_{Q_0} \in M^q$$

and

$$\| Tu - (Tu)_{Q_0} \|_{M^q} \leq A\, [u]_{N^{(q,0)}}$$

whence for $0 \leq t \leq 1$ and p, q defined by (4.1)

$$\| Tu - (Tu)_{Q_0} \|_{M^q} \leq A K_1^{1-t} K_2^t \| u \|_{L^p}.$$

Then using the theorem of Marcinkiewicz one obtains

$$\| Tu - (Tu)_{Q_0} \|_{L^q} \leq \mathcal{K}\, K_1^{1-t} K_2^t \| u \|_{L^p}$$

where $\mathcal{K}$ is a constant which is bounded for t in every closed subinterval of $(0, 1)$ and this completes the proof of the theorem.

The theorem 4.1 gives a proof of the theorem 3.1 of [11] when $\mu_2 = n$. In the case $\mu_2 < n$ the hypothesis that T is of weak type $[\alpha_2, (\beta_2, \mu_2)]$ in the theorem 3.1 of [11] will be replaced by the fact that T maps L^{α_2} into $\mathcal{L}_r^{(\beta_2, \mu_2)}$ where $r \leq \frac{n\beta_2}{\mu_2}$ introduced in this paper (see definition 2.3).

The theorem 4.2 of [11] follows from these preceeding considerations if the hypothesis (4.4) of [11] is replaced by the hypothesis that T transforms L^{α_2} into $\mathcal{L}_r^{(\beta_2, \mu_2)}$ with $r \leq \frac{n\beta_2}{\mu_2}$.

The theorem 4.3 of [11] subsists unaltered since we have, from the corollary 2.1 of this paper, that T is of strong type $N[\alpha_2, (\beta_2, 0)]$ and therefore the theorem 3.1 of this paper can be applied.

§ 5. Hölder continuous functions of strong type.

We have introduced, in the section 3, the function spaces $\mathcal{L}_r^{(p, \lambda)}$. In view of the theorem 1.1 of Campanato-Meyers in the case when $\lambda < 0$ we observe that the functions belonging to $\mathcal{L}_r^{(p, \lambda)}$ $(\lambda < 0)$ coincide with functions belonging to a subclass of Holder continuous functions and we call these Hölder continuous functions of strong type.

DEFINITION 5.1. A function u defined on a cube Q_0 is said to be Hölder continuous of strong type r with exponent $0 < \alpha < 1$ if the following two conditions are satisfied:

(i) u is Hölder continuous with exponent α in Q_0;

(ii) there exists a constant $L = L(u) > 0$ such that, for any system S of subcubes Q_i of $\bar{S}$ (see the beginning of § 2), one has

$$\sum_i |K(Q_i)|^r \leq L \tag{5.1}$$

where $K(Q)$ denotes the Hölder coefficient (with exponent α) of the restriction $u/_Q$ to the subcube Q of u.

We now prove the following

TEOREM 5.1. *Let Q_0 be a bounded cube in E^n and $0 < \alpha < 1$. A function u is Hölder continuous of strong type r with exponent α, where $r \leq \frac{n}{1-\alpha}$ if and only if u admits first derivatives (in the strong sense) which are functions belonging to $L^{\frac{n}{1-\alpha}}(Q_0)$.*

To this end, we shall make use of a criterion for a function to admit first derivatives (in the strong sense) which are functions belonging to L^p. This criterion is a consequence of a criterion due to F. Riesz [8] which we recall in the following.

LEMMA 6.1 [10]. *A necessary and sufficient condition in order that a function* $u \in C^1(Q_0)$ *has its derivative* u_{x_s} *in* $L^p(Q_0)$ *with*

$$\| u_{x_s} \|_{L^p(Q_0)} \leq L \tag{5.2}$$

is that for any system $S \in \overline{S}$ *(system of finite number of subcubes* Q_i *of* Q_0 *no two of which have a common interior point) we have*

$$\Sigma \left| \int_{\partial Q_i} u \, dx_1 \ldots \langle dx_s \rangle \ldots dx_n \right|^p \cdot | Q_i |^{1-p} \leq L^p . \tag{5.3}$$

The expression $dx_1 \ldots \langle dx_s \rangle \ldots dx_n$ means $dx_1 \ldots dx_{s-1} \, dx_{s+1} \ldots dx_n$

PROOF. Suppose (5.2) holds then, since

$$\int_{\partial Q_i} u \, dx_1 \ldots \langle dx_s \rangle \ldots dx_n = \int_{\partial Q_i} \frac{\partial u}{\partial x_s} dx$$

the expression in the left hand side of (5.3) can be majorized by $\| u_{x_s} \|^p_{L^p(Q_0)}$.

If, on the other hand, (5.2) holds then

$$\Sigma \left| \int_{Q_i} \frac{\partial u}{\partial x_s} dx \right|^p \cdot | Q_i |^{1-p} dx \leq L^p$$

and hence by the theorem of Riesz we obtain (5.3).

REMARK 5.1. The lemma 5.1 is valid also for functions which are continuous in Q_0 and admit derivatives in the strong sense.

In fact, if $\{u_m\}$ is a sequence of functions in $C^1(Q_0)$ which converges uniformly to u then for any $\varepsilon > 0$ we have, for $m > m_\varepsilon$

$$\Sigma \left| \int_{\partial Q_i} u_m \, dx_1 \ldots \langle dx_s \rangle \ldots dx_n \right|^p \cdot | Q_i |^{1-p} \leq (L + \varepsilon)^p$$

and hence

$$\| (u_m)_{x_s} \|_{L^p(Q_0)} \leq L + \varepsilon$$

from which it follows that u admits the derivatives u_{x_s} in the strong sense and is such that

$$\| u_{x_s} \|_{L^p(Q_0)} \leq L.$$

Proof of theorem 5.1. Let u be a Hölder continuous function of strong type r with exponent α in the cube Q_0. Since

$$\left| \int_{\partial Q_i} u \, dx_1 \dots \langle dx_s \rangle \dots dx_n \right| \leq K(Q_i) \cdot | Q_i |^{\frac{\alpha}{n} + \frac{n-1}{n}}$$

we have

$$\Sigma_i \left| \int_{\partial Q_i} u \, dx_1 \dots \langle dx_s \rangle \dots dx_n \right|^p | Q_i |^{1-p}$$

$$\leq \sum_i | K(Q_i) |^p \cdot | Q_i |^{(1-p)+\left(\frac{\alpha}{n} + \frac{n-1}{n}\right)p}$$

$$\leq K(Q_0)^{p-r} \sum_i | K(Q_i) |^r \leq K(Q_0)^{p-r} \cdot L$$

if

$$(1-p) + \left(\frac{\alpha}{n} + \frac{n-1}{n}\right) p = 0 ;$$

that is, if

$$\frac{1}{p} = \frac{1}{n} - \frac{\alpha}{n} .$$

Hence, from lemma 5.1, we deduce that

$$u_{x_i} \in L^p \quad \text{with} \quad p = \frac{n}{1-\alpha} .$$

If, conversely u is such that $u_x \in L^{\frac{n}{1-\alpha}}$ with $0 < \alpha < 1$ then we have, from Sobolev's lemma, that

$$K(Q) \leq C \sum_s \| u_{x_s} \|_{L^p(Q)}$$

where C does not depend on Q, and hence, for any $\{Q_i\} \equiv S \in \overline{S}$ we have

$$\sum_i | K(Q_i) |^p \leq C' \sum_s \int_{Q_0} | u_{x_s} |^p \, dx.$$

This proves the theorem completely.