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**Regularity at the boundary for solutions of linear parabolic differential equations**

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# REGULARITY AT THE BOUNDARY FOR SOLUTIONS OF LINEAR PARABOLIC DIFFERENTIAL EQUATIONS (*)

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SUMMARY: At the boundary solutions of linear parabolic differential equations are shown to be Holder continuous if the boundary data is Holder continuous. Moreover, this continuity of the solution is independent of hypothesis of continuity of the coefficients of the parabolic operator.

**1. Introduction.** Regularity at the boundary for solutions of linear parabolic differential equations has been obtained in the form of Schauder's estimates [2]. However, to obtain these estimates the coefficients of the operator were assumed to be Holder continuous and the boundary was assumed to be smooth. In this paper the continuity restrictions upon the coefficients will be removed in obtaining Holder continuity of the solution at the boundary for Holder continuous boundary data. Also, the boundary smoothness will be lightened. The principle method employed will be the use of the maximum principle. It should be emphasized that the Holder continuity obtained here is *only at the boundary.*

It is convenient now to state some notation and some assumptions that will be used throughout. Let $x = (x_1, \dots, x_n)$ denote a point in the $n$-dimensional Euclidean space $R^n$. Let $\Omega$ denote an open connected set in $R^n$, $D = \Omega \times (0, T]$, and $\partial D = \delta\Omega \times (0, T] \cup \bar{\Omega} \times \{0\}$. Suppose that the real valued function $u(x, t)$ is $C^2$ in $D$ with respect to the real variables $x_i$, $i = 1, \dots, n$, that $u(x, t)$ is $C'$ in $D$ with respect to the real variable $t$ and $x_i$, $i = 1, \dots, n$, and that $u(x, t)$ is continuous in $\bar{D} = D \cup \delta D$. Suppose

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that $u(x, t)$ is a bounded solution of

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t) u$$

$$- \frac{\partial u}{\partial t} = f(x, t), \quad (x, t) \text{ in } D, \tag{1}$$

$$u(x, t) = \varphi(x, t), \ (x, t) \text{ in } \partial D,$$

where $a_{ij}$, $b_i$, $c$, and $f$ are real valued measurable functions in $D$, $\varphi$ is a continuous real valued function in $\partial D$, and these functions satisfy the following inequalities uniformly over their respective domains of definition:

$$| \varphi | \leq M, \ -\gamma \leq c \leq 0, \ | b_i | \leq B,$$

$$|f| \leq F, \text{ and}$$

$$0 < \alpha \leq \sum_{i,j=1}^{n} a_{ij} \lambda_i \lambda_j \leq A \text{ for} \tag{2}$$

$$| \lambda |^2 = \sum_{i=1}^{n} \lambda_i^2 = 1 \ (\lambda_i \text{ real}).$$

The paper is divided quite naturally into three distinct parts by the three distinct parts of the boundary $\partial D$. First, the continuity of $u(x, t)$ will be studied at a point of $\Omega \times \{0\}$; then, at a point of $\delta\Omega \times (0, T]$; and finally, at a point of $\delta\Omega \times \{0\}$. In each case the construction of a new barrier function or the modification of a previously used barrier function will be necessary.

The author is indebted to Carlo Pucci for many helpful suggestions. Indeed, a portion of Pucci's treatment of the continuity at the boundary for solutions of elliptic equations [3] has proved useful here and with some modifications is reproduced below (see (16)-(31)).

2. **Continuity of $u(x, t)$ at a point of $\Omega \times \{0\}$.** Consider the following theorem.

THEOREM 1: Suppose that at $(x_0, 0)$ $(x_0 \in \Omega)$,

$$| \varphi(x, 0) - \varphi(x_0, 0) | \leq H | x - x_0 |^{\lambda}, \ 0 < \lambda \leq 2, \tag{3}$$

for all $x$ in $\bar{\Omega}$. Then, there exists a constant $K$ which depends only upon $M$, $\gamma$, $B$, $F$, $A$, $T$, $\lambda$, $n$, and

$$\delta = \min\ (\text{dist}\ (x_0\,, \delta\Omega), 1) \tag{4}$$

such that for $(x, t)$ in $D$,

$$| u(x, t) - \varphi(x_0\,, 0) | \leq K(| x - x_0 |^2 + t)^{\frac{\lambda}{2}} \tag{5}$$

PROOF: Consider the function

$$v(x, t) = | x - x_0 |^2 + \mu t, \quad \mu > 0. \tag{6}$$

As is well known [2], for $| x - x_0 | < \delta$ and $0 < t \leq T$,

$$Lv < -1 \tag{7}$$

when $\mu > 1 + 2nA + 2n\,B\delta$, which implies that $v(x, t)$ is a local barrier at the point $(x_0, 0)$. For $\lambda = 2$ and

$$\sigma = \max\left(\frac{2M}{\delta^2}\,,\ H,\ F + \gamma M\right), \tag{8}$$

the functions

$$w^{\pm}(x, t) = \sigma\, v(x, t) \pm (u(x, t) - \varphi(x_0\,, 0)) \tag{9}$$

satisfy

$$Lw^{\pm} < 0, \quad 0 < t \leq T, \quad | x - x_0 | < \delta, \tag{10}$$

and

$$w^{\pm} \geq 0 \tag{11}$$

for $| x - x_0 | = \delta$, $0 \leq t \leq T$ and $| x - x_0 | < \delta, t = 0$.

By the maximum principle,

$$w^{\pm} \geq 0, \ 0 \leq t \leq T, \ | x - x_0 | \leq \delta, \tag{12}$$

which implies that

$$| u(x, t) - \varphi(x_0\,, 0) | \leq \sigma\, v(x, t) \tag{13}$$

for $0 \leq t \leq T$ and $| x - x_0 | \leq \delta$. Hence, for $\lambda = 2$, (5) follows from (13).

Now, consider the case of $0 < \lambda < 2$. For all $\varepsilon > 0$, a function $g(\varepsilon) \geq 1$ will be found such that for $|x - x_0| \leq \delta$

$$H\,|x - x_0|^\lambda \leq \varepsilon + g(\varepsilon)\,\sigma\, v(x, 0). \tag{14}$$

Then, it will follow that for all $\varepsilon > 0$ the functions

$$w^{\pm}(x, t, \varepsilon) = \varepsilon + g(\varepsilon)\,\sigma\, v(x, t) \pm (u(x, t) - \varphi(x_0, 0)) \tag{15}$$

will satisfy (10), (11), (12), and the obvious modification of (13). The result (5) will then follow from a careful analysis of the function $\varepsilon + g(\varepsilon)\,\sigma\, v(x, t)$ as a function of $\varepsilon$.

Set $r = |x - x_0|$ and consider the inequality (14) written in terms of $r$ as

$$Hr^\lambda \leq \varepsilon + g(\varepsilon)\,\sigma\, r^2. \tag{16}$$

Solving for $g(\varepsilon)$,

$$g(\varepsilon) \geq H\,\sigma^{-1}\, r^{\lambda-2} - \varepsilon\,\sigma^{-1}\, r^{-2}. \tag{17}$$

For fixed $\varepsilon$ the function of $r$ on the right hand side of (17) assumes its maximum value for all $r > 0$ at the point

$$r = (2\,(2 - \lambda)^{-1}\, H^{-1}\, \varepsilon)^{\frac{1}{\lambda}}. \tag{18}$$

Substituting this value of $r$, it follows that

$$\sigma^{-1}\,\lambda\,(2 - \lambda)^{-1}\,(H\,2^{-1}\,(2 - \lambda))^{\frac{2}{\lambda}}\, \varepsilon^{\frac{\lambda-2}{\lambda}} = K_1\, \varepsilon^{\frac{\lambda-2}{\lambda}} \geq \tag{19}$$

$$\geq H\,\sigma^{-1}\, r^{\lambda-2} - \varepsilon\,\sigma^{-1}\, r^{-2}$$

for all $r > 0$. Hence for

$$g(\varepsilon) = \max\{1, K_1\, \varepsilon^{\frac{\lambda-2}{\lambda}}\}, \tag{20}$$

it follows that for every $\varepsilon > 0$, (16) is valid which implies by the maximum principle that

$$|u(x, t) - \varphi(x_0, 0)| \leq \varepsilon + g(\varepsilon)\,\sigma\, v(x, t) \tag{21}$$

for every $\varepsilon > 0$ and for $|x - x_0| \leq \delta$ and $0 \leq t \leq T$.

Consider the function

$$h(\varepsilon) = \varepsilon + g(\varepsilon)\,\sigma\, v(x, t). \tag{22}$$

For

$$(23) \qquad 0 < \varepsilon \leq K_1^{\frac{\lambda}{\lambda-2}},$$

$$(24) \qquad g(\varepsilon) = K_1 \, \varepsilon^{\frac{\lambda-2}{\lambda}}.$$

Restricting $\varepsilon$ to the interval $0 < \varepsilon \leq K_1^{\frac{\lambda}{\lambda-2}}$, it is desirable to minimize

$$(25) \qquad h(\varepsilon) = \varepsilon + v(x, t) \, \mathrm{K}_2 \, \varepsilon^{\frac{\lambda-2}{\lambda}}$$

as a function of $\varepsilon$ for fixed $(x, t)$. Considering the function for all $\varepsilon$, its minimum occurs at

$$(26) \qquad \varepsilon(x, t) = \left[\left(\frac{2-\lambda}{\lambda}\right) K_2 \, v(x, t)\right]^{\frac{\lambda}{2}} = K_3 \, [v(x, t)]^{\frac{\lambda}{2}}.$$

Consequently, for $(x, t)$ such that

$$(27) \qquad 0 < v(x, t) \leq (K_3^{-1} \, K_1^{\frac{\lambda}{\lambda-2}})^{\frac{2}{\lambda}},$$

it follows that

$$(28) \qquad 0 < \varepsilon(x, t) \leq K_1^{\frac{\lambda}{\lambda-2}}.$$

Thus, under the restrictions (23) and (27), it follows from (21) that

$$(29) \qquad | u(x, t) - \varphi(x_0, 0) | \leq h(\varepsilon(x, t)) = K_4 \, [v(x, t)]^{\frac{\lambda}{2}}.$$

Since

$$(30) \qquad | u(x, t) - \varphi(x_0, 0) | \leq 2M$$

for $(x, t)$ such that

$$(31) \qquad (K_3^{-1} \, K_1^{\frac{\lambda}{\lambda-2}})^{\frac{2}{\lambda}} \leq v(x, t),$$

the result (5) follows from a simple replacement of $K_4$ by a larger constant $K_5$.

REMARK: The restriction on $\lambda$ in (3) arises naturally out of the fact that the continuity cannot be better than Lipschitzian in $t$ for non-zero $c$

and $f$. For example, consider the bounded solution of

$$\frac{\partial^2 v}{\partial \zeta^2} - v - \frac{\partial v}{\partial t} = 1, \quad -\infty < \zeta < \infty, \ 0 < t,$$

$$v(\zeta, 0) = \begin{cases} 1, & -\infty < \zeta \leq -1, \\ \zeta^4, & -1 < \zeta < 1, \\ 1, & 1 \leq \zeta < \infty, \end{cases}$$

in a neighborhood of $(0, 0)$. The restriction on $\lambda$ can be removed by removing $c$ and $f$.

THEOREM 2: For $c(x, t) \equiv f(x, t) \equiv 0$ and

$$|\varphi(x, 0) - \varphi(x_0, 0)| \leq H |x - x_0|^{2+2\nu}, \ \nu > 0 \ (x_0 \in \Omega), \tag{33}$$

there exists a constant $K$ which depends only upon $M, B, A, T, \nu$, $n$ and $\delta$ such that for $(x, t)$ in $D$,

$$|u(x, t) - \varphi(x_0, 0)| \leq K(|x - x_0|^2 + t)^{1+\nu}. \tag{34}$$

PROOF: Consider

$$v(x, t) = (|x - x_0|^2 + \mu t)^{1+\nu}, \quad \mu > 0. \tag{35}$$

Now, for $|x - x_0| \leq \delta$ and $0 < t \leq T$,

$$\begin{aligned} Lv = {} & (1 + \nu)(|x - x_0|^2 + \mu t)^{\nu} \cdot \\ & \cdot \Big\{ (|x - x_0|^2 + \mu t)^{-1} 4\nu \sum_{i,j=1}^{n} a_{ij}(x_i - x_{i0})(x_j - x_{j0}) + \\ & + 2 \sum_{i=1}^{n} (b_i(x_i - x_{i0}) + a_{ii}) - \mu \Big\} \leq (1 + \nu)(|x - x_0|^2 + \mu t)^{\nu} \cdot \\ & \cdot \left\{ \frac{4\nu A |x - x_0|^2}{|x - x_0|^2 + \mu t} + 2n B\delta + 2nA - \mu \right\} < 0 \end{aligned} \tag{36}$$

when $\mu > 2nA + 2nB\delta + 4\nu A$. Hence, the result (34) follows from an argument similar to that of Theorem 1, (8) — (13).

3. **Continuity of $u(x, t)$ at a point of $\delta\Omega \times (0, T]$.** The bounded solution $v(\zeta, t)$ of

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial \zeta^2}, \quad 0 < \zeta < \infty, \quad 0 < t \leq t_0,$$

$$v(0, t) = H(t_0 - t)^\lambda, \quad 0 \leq t \leq t_0, \quad H > 0, \tag{37}$$

$$v(\zeta, 0) = H t_0^\lambda, \quad 0 \leq \zeta < \infty, \quad \lambda > 0,$$

is given by the formula

$$v(\zeta, t) = H t_0^\lambda - \int_0^t \frac{\partial M(\zeta, k(t-\tau))}{\partial \zeta} [H(t_0 - \tau)^\lambda - H t_0^\lambda] k \, d\tau, \tag{38}$$

where

$$M(\xi, \sigma) = \pi^{-\frac{1}{2}} \sigma^{-\frac{1}{2}} \exp\left\{\frac{-\xi^2}{4\sigma}\right\} \quad \sigma > 0. \tag{39}$$

LEMMA 1: For $0 \leq \zeta \leq \zeta_0 < 1$, there exist constants $K_1$, $K_2$, and $K_3$ which depend only upon $k, H, t_0$, $\lambda$ and $\zeta_0$ such that

$$v(\zeta, t_0) \leq \begin{cases} K_1 \zeta^{2\lambda}, & 0 < \lambda < \dfrac{1}{2}, \\ K_2 \zeta \, |\log \zeta|, & \lambda = \dfrac{1}{2}, \\ K_3 \zeta, & \lambda > \dfrac{1}{2}. \end{cases} \tag{40}$$

PROOF. The proof follows from elementary estimations of the formula (38) and is therefore omitted.

LEMMA 2; For $0 < \zeta < \infty$ and $0 < t < t_0$,

$$| v(\zeta, t_0) - v(\zeta, t) | \leq H(t_0 - t)^\lambda, \quad 0 < \lambda \leq 1. \tag{41}$$

PROOF: By the maximum principle, it follows that

$$\left| \frac{\partial v}{\partial t}(\zeta, t) \right| \leq \lambda H (t_0 - t)^{\lambda - 1}. \tag{42}$$

Hence, the result (41) follows from an elementary quadrature.

Since $v(\zeta, t)$ will be part of the barrier function used in this section, the following lemma is of interest.

LEMMA 3. If $0 < \lambda \leq 1$,

$$0 < k < 1, \quad \text{and} \quad 0 < k < \beta^2 (1 + 2\pi^{-\frac{1}{2}} t_0^{\frac{1}{2}} B)^{-2}, \tag{43}$$

then

$$\beta \frac{\partial^2 v}{\partial \zeta^2} + B \frac{\partial v}{\partial \zeta} \leq z(\zeta, t) \tag{44}$$

for $0 < \zeta < \infty$ and $0 \leq t \leq t_0$, where $\beta$ and $B$ are positive constants and where $z(\zeta, t)$ is the bounded solution of

$$\begin{aligned} &\frac{\partial z}{\partial t} = k \frac{\partial^2 z}{\partial \zeta^2}, \quad 0 < \zeta < \infty, \quad 0 < t \leq t_0, \\ &z(0, t) = -\lambda H (t_0 - t)^{\lambda - 1}, \quad 0 \leq t < t_0, \\ &z(\zeta, 0) = 0, \quad 0 < \zeta < \infty. \end{aligned} \tag{45}$$

PROOF: It follows from (38) that $\dfrac{\partial v}{\partial \zeta}$ and $\dfrac{\partial^2 v}{\partial \zeta^2}$ are bounded solutions of the heat equation with diffusivity $k$ for $0 < \zeta < \infty$ and $0 < t < t_0$. Moreover, both functions are equal to zero for $0 < \zeta < \infty$ and $t = 0$. Now, it can be shown [1, pp. 189-190] that

$$\frac{\partial v}{\partial \zeta}(0, t) = \int_0^t \frac{\lambda H (t_0 - \tau)^{\lambda - 1}}{\sqrt{\pi k (t - \tau)}} d\tau \tag{46}$$

and that

$$\frac{\partial^2 v}{\partial \zeta^2}(0, t) = -k^{-1} \lambda H (t_0 - t)^{\lambda - 1}. \tag{47}$$

Hence, it follows that

$$\begin{aligned} \beta \frac{\partial^2 v}{\partial \zeta^2}(0, t) + B \frac{\partial v}{\partial \zeta}(0, t) &\leq \lambda H (t_0 - t)^{\lambda - 1} \cdot \\ &\cdot \{2B\pi^{-\frac{1}{2}} t_0^{\frac{1}{2}} k^{-\frac{1}{2}} - \beta k^{-1}\} < -\lambda H (t_0 - t)^{\lambda - 1}, \end{aligned} \tag{43}$$

whenever $k$ satisfies the conditions in (43). For such a $k$, (44) follows from the maximum principle.

Consider now the main subject of this section.

THEOREM 3: Suppose that $\Omega$ is bounded with diameter $l$. Suppose that at $(x_0, t_0)$ $(x_0 \in \delta\Omega$ and $0 < t_0 \leq T)$,

$$(49) \qquad |\varphi(x, t) - \varphi(x_0, t_0)| \leq H(|x - x_0|^{\lambda_1} + (t_0 - t)^{\lambda_2}),$$

$$0 < \lambda_1 \leq 2, \quad 0 < \lambda_2 \leq 1,$$

for all $(x, t)$ in $\delta\Omega \times (0, t_0]$. Suppose also that $x_0$ is a point on $\delta\Omega$ such that there exists a sphere

$$(50) \qquad S = \{x : |x - \xi'| < 2\varrho\}$$

such that $\overline{S} \cap \overline{\Omega} = \{x_0\}$. Then, there exist constants $K_0, K_1, K_2, K_3$ and $K_4$ which depend only upon $M, \gamma, B, F, \alpha, A, t_0, \lambda_1, \lambda_2$, $n$ and $l$ such that for $\frac{t_0}{2} \leq t \leq t_0$, $|x - x_0| < e^{-1}$, and $(x, t)$ in $\Omega \times (0, t_0]$,

$$(51) \qquad |u(x, t) - \varphi(x_0, t_0)| \leq K_0 |x - x_0|^{\frac{\lambda_1}{2}} + \begin{cases} K_1 |x - x_0|^{2\lambda_2}, & 0 < \lambda_2 < \frac{1}{2}, \\ K_2 |x - x_0| \, |\log |x - x_0||, & \lambda_2 = \frac{1}{2} + \\ K_3 |x - x_0|, & \frac{1}{2} < \lambda_2 \leq 1, \end{cases}$$

$$+ K_4 (t_0 - t)^{\lambda_2}.$$

PROOF: Set

$$(52) \qquad \xi = \frac{1}{2}(x_0 + \xi')$$

and consider

$$(53) \qquad y(x) = e^{-\mu\varrho^2} - e^{-\mu|x - \xi|^2}, \quad \mu > 0.$$

By differentiation, it follows that

$$(54) \qquad Ly < -\mu e^{-\mu(l+\varrho)^2}, \ (x, t) \text{ in } \Omega \times (0, t_0],$$

when $\mu$ is chosen so that

$$(55) \qquad 4\alpha\varrho^2 \mu = 1 + 2nA + 2nB(l + \varrho).$$

Note that $y(x_0) = 0$ and $y(x) > 0$ for $x$ in $\bar{\Omega} - \{x_0\}$. Moreover, for

(56) $$\zeta = \left[\sum_{i=1}^{n} (x_i - \xi_i)^2\right]^{\frac{1}{2}} - \varrho,$$

it follows that for $(x, t)$ in $\Omega \times (0, t_0]$,

(57) $$Lv(\zeta, t) = \frac{\partial^2 v}{\partial \zeta^2}\left\{\sum_{i,j=1}^{n} a_{ij} \frac{(x_i - \xi_i)(x_j - \xi_j)}{|x - \xi|\,|x - \xi|} - k\right\} +$$

$$+ \frac{\partial v}{\partial \zeta}\left\{\sum_{i=1}^{n}\left(\frac{a_{ii}}{|x - \xi|} + \frac{b_i(x_i - \xi_i)}{|x - \xi|}\right) - \frac{1}{|x - \xi|}\sum_{i,j=1}^{n} a_{ij}\frac{(x_i - \xi_i)(x_j - \xi_j)}{|x - \xi|\,|x - \xi|}\right\} +$$

$$+ cv(\zeta, t) < z(\zeta, t) < 0$$

for $k$ such that

(58) $$0 < k < \frac{\alpha}{2}, \quad 0 < k < 1,$$

and

(59) $$0 < k < \alpha^2 4^{-1}\left(1 + 2\pi^{-\frac{1}{2}} t_0^{\frac{1}{2}}\left[\frac{nA}{\varrho} + \frac{nB(l + \varrho)}{\varrho}\right]\right)^{-2},$$

where $v(\zeta, t)$ is defined by (38) with $\lambda$ replaced by $\lambda_2$, $z(\zeta, t)$ is the bounded solution of (45) with the same replacement of $\lambda$, and the conditions on $k$ are those needed to apply Lemma 3.

Let

(60) $$\sigma = \min_{\substack{0 \le \zeta \le l \\ \frac{1}{2} t_0 \le t \le t_0}} |z(\zeta, t)| > 0,$$

and let

(61) $$\sigma_1 = \max\left(\frac{2^{1+\lambda_2} M}{H t_0^{\lambda_2}}, \frac{F + \gamma M}{\sigma}, 1\right).$$

Set

(62) $$w^{\pm}(x, t, \varepsilon) = \varepsilon + g_1(\varepsilon)\, y(x) + \sigma_1 v(\zeta, t) \pm (u(x, t) - \varphi(x_0, t_0)),$$

where for all $\varepsilon > 0$, $g_1(\varepsilon)$ is the obvious modification of $g(\varepsilon)$ defined by (24). From the previous paragraph and (61), it follows that

(63) $$Lw^{\pm} < 0, \; (x, t) \text{ in } \Omega \times \left(\frac{1}{2} t_0, t_0\right].$$

Next, for $\zeta = 0$ and $\frac{1}{2} t_0 \leq t \leq t_0$,

$$H (t_0 - t)^{\lambda_2} \leq \sigma_1 v (0, t), \tag{64}$$

and at $\zeta = 0$ and $t = \frac{1}{2} t_0$,

$$2M \leq \sigma_1 v \left(0, \frac{1}{2} t_0\right). \tag{65}$$

Since $\frac{\partial v}{\partial \zeta} > 0$, it follows that for $0 \leq \zeta \leq l$ and $\frac{1}{2} t_0 \leq t \leq t_0$,

$$H (t_0 - t)^{\lambda_2} \leq \sigma_1 v (\zeta, t) \tag{66}$$

and that

$$2M \leq \sigma_1 v \left(\zeta, \frac{1}{2} t_0\right). \tag{67}$$

Finally, from (53) it can be shown that there exist positive constants $\mu_1$ and $\mu_2$ such that for any $x$ in $\Omega$,

$$\mu_1 | x - x_0 |^2 \leq y (x) \leq \mu_2 | x - x_0 |. \tag{68}$$

Consequently, from the argument of Theorem 1, (16), ... , (19), it follows that for all $\varepsilon > 0$,

$$H | x - x_0 |^{\lambda_1} \leq \varepsilon + g_1 (\varepsilon) y (x), \quad x \text{ in } \bar{\Omega}. \tag{69}$$

Thus, from (66), (67), and (69), it follows that $w^{\pm} \geq 0$ for $(x, t)$ in $\partial\Omega \times \times \left(\frac{1}{2} t_0, t_0\right]$ and for $(x, t)$ in $\bar{\Omega} \times \left\{\frac{1}{2} t_0\right\}$. Hence, from (62) and the maximum principle, $w^{\pm} \geq 0$ in $\bar{\Omega} \times \left[\frac{1}{2} t_0, t_0\right]$; i. e., for $(x, t)$ in $\bar{\Omega} \times \left[\frac{1}{2} t_0, t_0\right]$,

$$| u (x, t) - \varphi (x_0, t_0) | \leq \varepsilon + g_1 (\varepsilon) y (x) + \sigma_1 v (\zeta, t). \tag{70}$$

From $\varepsilon + g_1 (\varepsilon) y (x)$, the first term $K_0 | x - x_0 |^{\frac{\lambda_1}{2}}$ in the result (51) follows from a similar argument to that of Theorem 1, (22), ... , (31). Writing

$$v (\zeta, t) = [v (\zeta, t) - v (\zeta, t_0)] + v (\zeta, t_0), \tag{71}$$

the term $K_4 (t_0 - t)^{\lambda_2}$ in (51) follows from Lemma 2. Finally, estimating $v(\zeta, t_0)$ by Lemma 1, the second term on the right hand side of equation (51) follows from the monotonicity of the functions $\zeta^{2\lambda_2}$, $\zeta \mid \log \zeta \mid$ $(0 < \zeta < e^{-1})$, and $\zeta$ and the fact that

$$\zeta = \left[\sum_{i=1}^{n} (x_i - \xi_i)^2\right]^{\frac{1}{2}} - \varrho \leq \mid x - x_0 \mid . \tag{72}$$

REMARK : In the case of $n = 1$, the continuity obtained is simply that of the heat equation expressed in Lemmas 1 and 2. Also, for the case of $\varphi(x, t) \equiv \varphi(t)$ for $(x, t)$ in $\partial\Omega \times (0, t_0]$, the term $K_0 \mid x - x_0 \mid^{\frac{\lambda_1}{2}}$ is eliminated from the result (51) leaving the continuity terms arising from the heat equation.

4. **Continuity of $u(x, t)$ at a point of $\partial\Omega \times \{0\}$.** The statement of the following theorem will essentially complete the present discussion of the continuity at the boundary.

THEOREM 4 : Retaining the hypothesis of Theorem 3 concerning $\Omega$ and $x_0 \in \delta\Omega$, suppose that at $(x_0, 0)$,

$$\mid \varphi(x, t) - \varphi(x_0, 0) \mid \leq H(\mid x - x_0 \mid^{\lambda_1} + t^{\lambda_2}),\ 0 < \lambda_1 \leq 2,\ 0 < \lambda_2 , \tag{73}$$

for all $(x, t)$ in $\delta\Omega \times (0, T] \cup \bar{\Omega} \times \{0\}$. Then, there exists a constant $K$ which depends only upon $M$, $\gamma$, $B$, $F$, $\alpha$, $A$, $\lambda_1$, $\lambda_2$, $n$ and $l$ such that for $(x, t)$ in $D$,

$$\mid u(x, t) - \varphi(x_0, 0) \mid \leq K(\mid x - x_0 \mid^{\frac{\lambda_1}{2}} + t^{\lambda_2}). \tag{74}$$

PROOF : The proof is similar to those of Theorem 1 and 3, where in this case

$$w^{\pm}(x, t, \varepsilon) = \varepsilon + g_2(\varepsilon)\, \sigma_2\, y(x) + Ht^{\lambda_2} \pm (u(x, t) - \varphi(x_0, 0)), \tag{75}$$

$g_2(\varepsilon)$ is an obvious modification of $g(\varepsilon)$ defined by (20), and $\sigma_2$ is a sufficiently large positive constant.

REMARK : For the case $n = 1$, $\lambda_1$ is not divided by 2 since $x^{\lambda_1} + t^{\lambda_2}$ is a local barrier at the origin, if $\lambda_1 < 1$ or $\lambda_2 < 1$.

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