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AN EIGENFUNCTION EXPANSION METHOD FOR PROBLEMS WITH OVERSPECIFIED DATA (*)

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1. Introduction. We consider problems such as those arising in partial differential equations where separation of variables is possible, an eigenfunction expansion of solutions exists, and each solution may be represented by its sequence of generalized Fourier coefficients. Likewise, data given on one or more data surfaces may be expanded in terms of the eigenfunctions and each data set may also be represented by its sequence of Fourier coefficients. In this paper we consider problems with approximate and overspecified data.

In Section 2 such problems are considered in an abstract setting, the analysis here already having been reduced to consideration of sequences of Fourier coefficients. It turns out that certain of the Fourier coefficients of the solution should be obtained from one data set and certain should be obtained from another, depending upon the degree and measure of accuracy claimed for each data set. The interesting result here is that for this particular approximate solution the error bound is simultaneously « almost best possible » with respect to every possible norm used to measure the error.

In Section 3 we apply these results to some problems of analytic continuation on a disc or annulus. Analytic continuation provides a particularly good illustration because of the extremely simple nature of the eigenfunction expansion (Taylor or Laurent series). Also, these results are interesting in their own right, one of them being a close analogue of Hadamard's three circle theorem.

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The present method is particularly useful in the study of problems in partial differential equations which are not well posed in the sense of Hadamard. For large classes of such problems continuous dependence on data can be restored by restricting attention to the class of solutions satisfying a prescribed bound. See Pucci [3] and John [1] for two of the early papers in this line. The prescribed bound then induces a problem with overspecification of data. The methods of the present paper, in less abstract and general a form, were introduced in [2] where they were applied to three of the classical improperly posed problems, harmonic continuation, backward solution of the heat equation, and the Cauchy problem for Laplace's equation. Certain of the present analytic continuation results were also alluded to there.

2. **The abstract method.** Consider the linear space of real (complex) sequences $X = \{x_n\}$, $n \geq 0$, for which the I quadratic norms

$$\| X \|_i = \left(\sum_{n=0}^{\infty} | x_n B_{in} |^2 \right)^{\frac{1}{2}}, \quad i = 1, \ldots, I, \tag{2.1}$$

are all finite, where $\{B_{in}\}$, $n \geq 0$, are given real (complex) sequences, $i = 1, \ldots, I$. Let $\| X \|$ be any other norm (non-negative, sub-additive real function would be sufficient) defined on this space.

We consider first the problem of maximizing $\| X \|$ with respect to the quadratic constraints

$$\| X \|_i \leq m_i, \qquad i = 1, \ldots, I. \tag{2.2}$$

The following method reduces (except for a factor of I) the maximization with respect to I constraints to I separate maximizations with respect to one constraint. Decompose sequences X in the following fashion:

$$X = \sum_{i=1}^{I} X^i \tag{2.3}$$

where the sequence X^i has its nth term x^i_n given by

$$\begin{aligned} x^i_n &= x_n \quad \text{when} \quad \max_{j<i} \frac{B_{jn}}{m_j} < \frac{B_{in}}{m_i} = \max_{j=1,\ldots,I} \frac{B_{jn}}{m_j}, \\ x^i_n &= 0 \quad \text{otherwise.} \end{aligned} \tag{2.4}$$

Notice that with this decomposition, if an X^i satisfies the single constraint $\| X^i \|_i \leq m_i$, then it also satisfies the other constraints, $\| X^i \|_k \leq m_k$, $k = 1, \dots, I$. Notice also that the sum of one or more of the X^i has no larger a $\| \ \|_k$ norm than X itself.

LEMMA 1. *Let N denote the supremum of $\| X \|$ with respect to the I constraints* (2.2). *Let M_i denote the supremum of $\| X^i \|$ with respect to the single constraint $\| X^i \|_i \leq m_i$. Then*

$$M_i \leq M \leq M_1 + \dots + M_I, \qquad i = 1, \dots, I. \tag{2.5}$$

PROOF: We have pointed out that the set of X^i satisfying the single constraint $\| X^i \|_i \leq m_i$ is a subset of the set of X satisfying all I constraints; thus, $M_i \leq M$. Suppose now that X satisfies all I constraints. By the subadditivity of $\| \ \|$ we have $\| X \| \leq \Sigma \| X^i \|$. Then $\| X^i \|_i \leq \| X \|_i \leq m_i$; thus, $\| X^i \| \leq M_i$, which completes the proof.

We next consider the problem of determining an approximation to an unknown sequence when several different approximate data sequences, with varying estimates of accuracy, are specified. Suppose there exists at least one sequence X satisfying

$$\| X - G_i \|_i \leq m_i, \qquad i = 1, \dots, I, \tag{2.6}$$

where the data sequences $G_1, \dots, G_I$ are given. We wish to find a sequence approximating every X satisfying (2.6). Letting $\sum_{i=1}^{I} (G_i)^i$ be our approximation, we have the following error bound.

LEMMA 2. *If X satisfies* (2.6), *then*

$$\left\| X - \sum_{i=1}^{I} (G_i)^i \right\| \leq M_1 + \dots + M_I. \tag{2.7}$$

PROOF.

$$\left\| X - \sum_{i=1}^{I} (G_i)^i \right\| = \left\| \sum_{i=1}^{I} (X - G_i)^i \right\| \leq \sum_{i=1}^{I} \| (X - G_i)^i \|.$$

But, since $\| (X - G_i)^i \|_i \leq \| X - G_i \|_i \leq m_i$, we have $\| (X - G_i)^i \| \leq M_i$, which completes the proof.

We point out that the error bound $M_1 + \dots + M_I$ appearing in (2.5) is « best possible to within a factor of I » in the following sense; for every

X (and every $\varepsilon > 0$) there exist Y such that $\| X - Y \|_i \leq m_i$, $i = 1, \dots, I$, and yet $\| X - Y \| + \varepsilon \geq M \geq \max_i M_i \geq (M_1 + \dots + M_I)/I$. In other words, the approximation $\sum_{i=1}^{I} (G_i)^i$ is « almost best possible », independently of the particular norm $\| \quad \|$ used to measure the error.

Clearly the constraints m_i cannot be arbitrarily small for given data sequences G_i. In fact, the m_i must satisfy the following a posteriori compatibility conditions with respect to the approximation $\sum_{i=1}^{I} (G_i)^i$.

LEMMA 3. *Suppose X satisfying* (2.6) *exists. Then*

$$\left\| G_k - \sum_{i=1}^{I} (G_i)^i \right\|_k \leq I \, m_k, \quad k = 1, \dots, I.$$

PROOF.

$$\left\| G_k - \sum_{i=1}^{I} (G_i)^i \right\|_k = \left\| \sum_{\substack{i=1 \\ i \neq k}}^{I} (G_k - G_i)^i \right\|_k \leq$$

$$\leq \| \sum_{i \neq k} (G_k - X)^i \|_k + \| \sum_{i \neq k} (X - G_i)^i \|_k .$$

But,
$$\| \sum_{i \neq k} (G_k - X)^i \|_k \leq \| G_k - X \|_k \leq m_k .$$

Also, since $(X - G_i)^i$ satisfies the single constraint

$$\| (X - G_i)^i \|_i \leq \| X - G_i \|_i \leq m_i ,$$

it also satisfies $\| (X - G_i)^i \|_k \leq m_k$. This completes the proof.

This lemma implies that if there exist X satisfying (2.6), then the approximation $\sum_{i-1}^{I} (G_i)^i$ itself must almost satisfy (2.6).

The hypotheses may be relaxed and the method extended somewhat. It was pointed out that the « norm » $\| \quad \|$ need only be a non-negative, sub-additive real function. Also, the « norms » $\| \quad \|_i$ need only be semi-norms, with some of the terms B_{in} being zero; this in fact is quite useful in applications. The weighted l_2 norms may be replaced by weighted l_p norms, $1 \leq p \leq \infty$,

$$\| X \|_i = \left(\sum_{n=0}^{\infty} | x_n B_{in} |^p \right)^{\frac{1}{p}},$$

and the results remain unchanged. Rather than sequences $\{x_n\}$ we can consider functions $x(t)$ of the real variable t; such an extension is useful when considering Fourier integrals rather than Fourier series; see Section 11 of [2] for example.

3. Analytic continuation on an annulus. Let M, N be integers, not necessarily positive, $-\infty < M \leq N < \infty$. For functions $f(z)$ analytic and single-valued on an annulus we divide the Laurent expansion into low, medium, and high order parts;

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n = \sum_{-\infty}^{M-1} a_n z^n + \sum_{M}^{N} a_n z^n + \sum_{N+1}^{\infty} a_n z^n \tag{3.1}$$

$$= f_{-\infty}^{M-1}(z) + f_M^N(z) + f_{N+1}^{\infty}(z).$$

Likewise, for L_2 complex functions $g(\theta)$ defined only on a single circle, $\{|z| = a\}$ say, we make the corresponding orthogonal decomposition of the Fourier expansion;

$$g(\theta) = \sum_{-\infty}^{\infty} A_n (a^n e^{-in\theta}) \tag{3.2}$$

$$= g_{-\infty}^{M-1}(\theta) + g_M^N(\theta) + g_{N+1}^{\infty}(\theta).$$

Moreover, we extend g formally as a function of z by means of its Laurent expansion;

$$g(z) = \sum_{-\infty}^{\infty} A_n z^n \tag{3.3}$$

$$= g_{-\infty}^{M-1}(z) + g_M^N(z) + g_{N+1}^{\infty}(z).$$

This perhaps converges nowhere outside the original circle; however, $g_{-\infty}^{M-1}(z)$ converges for $|z| > a$, $g_M^N(z)$ converges everywhere, and $g_{N+1}^{\infty}(z)$ converges for $0 < |z| < a$. Denote the uniform norm and the L_2 norm on the circle of radius r as follows:

$$|f|_r = \sup_{\theta} |f(re^{i\theta})|, \tag{3.4}$$

$$\|f\|_r = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta\right)^{\frac{1}{2}} = \left(\sum_{-\infty}^{\infty} |a_n r^n|^2\right)^{\frac{1}{2}}.$$

Let $[\alpha]$ denote the greatest integer such that $[\alpha] \leq \alpha$.

Corresponding to Lemma 1 we have the following result, similar to Hadamard's three circle theorem. This result provides a stability bound for the problem of analytic continuation on an annulus when approximate data is given on the inner and outer circle, or when appoximate data is given only on the inner circle and a prescribed bound is imposed on the outer circle.

THEOREM 1. *Suppose $f(z)$ is analytic and single-valued in $\{a < |z| < 1\}$ and*

$$\|f\|_a \leq m = a^\alpha, \qquad \|f\|_1 \leq 1. \tag{3.5}$$

Then

$$|f|_r \leq a^\alpha \left(\frac{r}{a}\right)^{[\alpha]} \left[1 - \left(\frac{a}{r}\right)^2\right]^{-\frac{1}{2}} + r^{[\alpha+1]}[1 - r^2]^{-\frac{1}{2}}, \quad a < r < 1, \tag{3.6}$$

this bound being best possible to within a factor of two. Moreover, simplification yields

$$|f|_r \leq m^{\frac{\log r}{\log a}} \left\{\left[1 - \left(\frac{a}{r}\right)^2\right]^{-\frac{1}{2}} + [1 - r^2]^{-\frac{1}{2}}\right\}, \quad a < r < 1. \tag{3.7}$$

PROOF. We wish to maximize $|f|_r$ with respect to the quadratic constraints

$$\begin{aligned} &(a) \quad \left(\sum_{-\infty}^{\infty} |a_n a^n|^2\right)^{\frac{1}{2}} \leq m = a^\alpha, \\ &(b) \quad \left(\sum_{-\infty}^{\infty} |a_n 1^n|^2\right)^{\frac{1}{2}} \leq 1. \end{aligned} \tag{3.8}$$

Clearly $f, a_n |f|_r, \|f\|_a, a^n, a^\alpha, \|f\|_1, 1, 1$ here correspond to $X, x_n, \|X\|, \|X\|_1, B_{1n}, m_1 \|X\|_2, B_{2n}, m_2$ of Lemma 1. Then the decomposition $f = f_{-\infty}^{[\alpha]} + f_{[\alpha+1]}^{\infty}$ here corresponds to the decomposition $X = X_1 + X_2$ there. The supremum of $|f|_r$ with respect to (3.8), the supremum of $|f_{-\infty}^{[\alpha]}|_r$ with respect to (3.8 a), and the supremum of $|f_{[\alpha+1]}^{\infty}|_r$ with respect to (3.8 b) then correspond to M, M_1, M_2 respectively.

Fortunately, the maximization of $|f_{-\infty}^{[\alpha]}|_r$ with respect to (3.8 a) may be carried out exactly. The constraints (3.8) are invariant with respect to rotations; hence, it is sufficient to maximize $|f_{-\infty}^{[\alpha]}|$ at the single point r

on the real axis. The Schwarz inequality then gives the exact maximum;

$$|f_{-\infty}^{[a]}(r)| = \left|\sum_{-\infty}^{[a]} a_n r^n\right| \leq \left(\sum_{-\infty}^{[a]} |a_n a^n|^2\right)^{\frac{1}{2}} \left(\sum_{-\infty}^{[a]} \left(\frac{r}{a}\right)^{2n}\right)^{\frac{1}{2}} \leq$$

$$\leq a^a \left(\frac{r}{a}\right)^{[a]} \left(\sum_{0}^{\infty} \left(\frac{a}{r}\right)^{2n}\right)^{\frac{1}{2}} = a^a \left(\frac{r}{a}\right)^{[a]} \left[1 - \left(\frac{a}{r}\right)^2\right]^{-\frac{1}{2}}.$$

We likewise obtain the exact maximum for $|f_{[a+1]}^{\infty}|_r$ with respect to (3.8b);

$$|f_{[a+1]}^{\infty}(r)| = \left|\sum_{[a+1]}^{\infty} a_n r^n\right| \leq \left(\sum_{[a+1]}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \left(\sum_{[a+1]}^{\infty} r^{2n}\right)^{\frac{1}{2}} \leq$$

$$\leq 1 \left(\sum_{[a+1]}^{\infty} r^{2n}\right)^{\frac{1}{2}} = r^{[a+1]} [1 - r^2]^{-\frac{1}{2}}.$$

Therefore (3.6), including the fact that the bound is best possible to within a factor of two, follows from Lemma 1. Finally, (3.7) follows from (3.6) and the proof is completed.

Corresponding to Lemma 2 we have the following result.

THEOREM 2. *Suppose $f(z)$ is analytic and single-valued in $\{a < |z| < 1\}$ and*

$$\|f - g\|_a \leq m = a^a, \quad \|f - h\|_1 \leq 1, \tag{3.9}$$

where the L_2 data functions $g(\theta)$ and $h(\theta)$ are given on the circles $\{|z| = a\}$ and $\{|z| = 1\}$ respectively. Then $|f - (g_{-\infty}^{[a]} + h_{[a+1]}^{\infty})|_r$ satisfies the bound satisfied by $|f|_r$ in (3.6) *and* (3.7).

The problem of analytic continuation with a prescribed bound imposed on the outer circle is of course taken care of by setting $h = 0$. We mention, without writing them down, that there exist a posteriori compatibility conditions corresponding to Lemma 3.

Instead of functions analytic on an annulus we may consider functions analytic on the whole unit disc. Then, dividing the Taylor series into a low and a high order part yields the following result.

THEOREM 3. *Suppose $f(z)$ is analytic and single-valued on $\{|z| < 1\}$ and*

$$\|f\|_a \leq m = a^a, \qquad \|f\|_1 \leq 1. \tag{3.10}$$

Then

$$(3.11)\qquad |f|_r \leq a^\alpha \left(\frac{r}{a}\right)^{[\alpha]} \left[\sum_0^{[\alpha]} \left(\frac{a}{r}\right)^{2n}\right]^{\frac{1}{2}} + r^{[\alpha+1]}(1-r^2)^{-\frac{1}{2}}, \quad r < 1,$$

this bound being best possible to within a factor of two. Moreover, simplification yields

$$(3.12)\qquad |f|_r \leq m^{\frac{\log r}{\log a}} \left\{\left[\sum_0^{[\alpha]} \left(\frac{a}{r}\right)^{2n}\right]^{\frac{1}{2}} + (1-r^2)^{-\frac{1}{2}}\right\}, \quad a \leq r < 1.$$

THEOREM 4. *Suppose $f(z)$ is analytic and single-valued on $\{|z| < 1\}$ and*

$$(3.13)\qquad \|f - g\|_a \leq m = a^\alpha, \qquad \|f - h\|_1 \leq 1,$$

where the L_2 data functions $g(\theta)$ and $h(\theta)$ are given on the circles $\{|z| = a\}$ and $\{|z| = 1\}$ respectively. Then $|f - (g_0^{[\alpha]} + h_{[\alpha+1]}^\infty)|_r$ satisfies the bound satisfied by $|f|_r$ in (3.11) and (3.12).

To further illustrate the method we consider the problem of analytic continuation on the annulus from approximate data on an intermediate circle $\{|z| = b\}$, $0 < a < b < 1$, with prescribed bounds imposed on the inner and outer circles. Or, we may consider approximate data given on all three circles. The following bound is obtained by dividing the Laurent series into low, medium, and high order parts.

THEOREM 5. *Suppose $f(z)$ is analytic and single-valued in the annulus $\{a < |z| < 1\}$ and*

$$(3.14)\qquad \|f\|_a \leq M, \quad \|f\|_b \leq m, \quad \|f\|_1 \leq 1.$$

Let β and γ be defined by $m/1 = (b/1)^\beta$, $M/m = (a/b)^\gamma$, and assume that $\gamma < \beta$. Then

$$(3.15)\qquad |f|_r \leq M\left[\sum_{-\infty}^{[\gamma]} \left(\frac{r}{a}\right)^{2n}\right]^{\frac{1}{2}} + m\left[\sum_{[\gamma+1]}^{[\beta]} \left(\frac{r}{b}\right)^{2n}\right]^{\frac{1}{2}} + 1\left[\sum_{\beta+1}^{\infty} r^{2n}\right]^{\frac{1}{2}}$$

for $a < r < 1$, this bound being best possible to within a factor of three.

We make no attempt to simplify the bound (3.15) above. Also, if it turns out that $\gamma \geq \beta$, we merely dispense with the information on the intermediate circle and apply Theorem 1.

THEOREM 6. *Suppose $f(z)$ is analytic and single-valued in the annulus $\{a < |z| < 1\}$ and*

$$\|f - e\|_a \leq M, \quad \|f - g\|_b \leq m, \quad \|f - h\|_1 \leq 1, \tag{3.16}$$

where the L_2 data functions $e(\theta)$, $g(\theta)$, and $h(\theta)$ are given on the circles of radius a, b, and 1 respectively. Then $|f - (e_{-\infty}^{[\gamma]} + g_{[\gamma+1]}^{[\beta]} + h_{[\beta+1]}^{\infty})|_r$ satisfies the bound satisfied by $|f|_r$ in (3.15).

Again, the problem with prescribed bounds imposed on the inner and outer circles is taken care of by setting $e = h = 0$. In this case, evaluation of the approximation function $g_{[\gamma+1]}^{[\beta]}(z)$ requires the evaluation of only a finite number of Fourier coefficients.

We point out that the uniform norm $|f|_r$ could be replaced by the uniform norm of a derivative of f or combinations thereof. Likewise, $\|f\|_r$ could be replaced by the L_2 norm of a derivative or integral of f; see theorems 3 and 4 of [2] for example. The method and results would be completely analogous; the bounds obtained would only be somewhat more complicated.

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REFERENCES

[1] JOHN, F., *Continuous dependence on data for solutions of partial differential equations with a prescribed bound.* Communications on Pure and Applied Mathematics **13**, 551-585 (1956).

[2] MILLER, K., *Three circle theorems in partial differential equations and applications to improperly posed problems.* Archive for Rational Mechanics and Analysis, **16**, 126-154 (1964).

[3] PUCCI, C., *Sui problemi di Cauchy non ben posti.* Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. (8) **18**, 473-477 (1955).