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# INTEGRAL REPRESENTATION AND ISOMORPHISM PROPERTIES OF SOME CLASSES OF FUNCTIONS

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## Introduction.

It appeared from conversations the three authors had in Moscow, May 1963, that each of them had a way of defining « Sobolev spaces of order 0 » (see precise definitions in the text); but it was not completely obvious that the definitions were equivalent. In this paper we present the three main ways of defining these spaces, together with their main properties and we prove also that the various definitions of the Banach spaces introduced coincide (up to an equivalence of the norm).

Chapter I (S. M. Nikolsky) uses the theory of approximation and constructive theory of functions, Chapter II (J. L. Lions) uses the theory of interpolation of Banach spaces and Chapter III (L. I. Lizorkin) uses trace spaces with fractionnal derivatives.

For various values of the parameters, some of the spaces introduced here were already considered by a number of mathematicians; we refer to the bibliography. We note also that this paper has direct connections with previous works of the authors and of Besov (see for instance the references [1], [2], [3], [4]).

## Preliminaries

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be points of the $n$-dimensional Euclidean space $R^n$, $|x| = \sqrt{x_1^2 + \dots + x_n^2}$. A generalized function $f(x)$ over the space $S$ (of infinitely differentiable functions, that decrease with their derivatives faster then any power of $|x|$ for $|x| \to \infty$) will be called $S'$-distribution

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and we shall write $f \in S'$. Let us consider for $r > 0$ the function

(1) $$G_r(x) = \frac{|x|^{\frac{r-n}{2}}}{2^{\frac{n+r-2}{2}} \pi^{\frac{n}{2}} \Gamma\left(\frac{r}{2}\right)} K_{\frac{n-r}{2}}(|x|)$$

where $K_\nu(t)$ is the Mcdonald's function of order $\nu$.

The following convolution

(2) $$G_r * f,$$

where $f \in S'$, makes sense ([9], II, p. 104).

The kernel $G_r$ decreases exponentially at infinity and therefore the convolution (2) may be written in the form

(3) $$(G_r * f)(x) = \int_{R^n} G_r(x-y) f(y)\, dy$$

for functions $f \in L_p(R^n)$, $1 \leq p \leq \infty$. The convolution $G_r * f$ will be caledl the Bessel integral of order $r$ of $f$ and we shall write

(4) $$G_r * f \equiv f_{(r)} \equiv \mathcal{J}_r f.$$

It is known, that the operation $\mathcal{J}_r$ transforms the space $S'$ onto itself in a one-to-one way and bicontinuously. Hence every $S'$ distribution $f$ can be represented in the form:

$$f = \mathcal{J}_r \varphi, \quad \varphi \in S'.$$

It is natural to call $\varphi$ a Bessel derivative of order $r$ of $f$ and write

(5) $$f^{(r)} = (\mathcal{J}_r)^{-1} f = \mathcal{J}_{-r} f = \varphi.$$

If we put $\mathcal{J}_0 f = f_{(0)} = f^{(0)}$, the operation $\mathcal{J}_r$ becomes well defined for all real $r$ and $f \in S'$. It posseses the group property

(6) $$\mathcal{J}_{r_1}(\mathcal{J}_{r_2}) = \mathcal{J}_{r_2}(\mathcal{J}_{r_1}) = \mathcal{J}_{r_1+r_2}.$$

We recall also, that for negative $r$ the operation $\mathcal{J}_r$ can be written as a convolution of $f$ with a distribution of $S'$ (see [9]), in particular, for $r = -2k$, $K$ integer, we have

$$\mathcal{J}_{-2k} f = (-\Delta + 1)^k f$$

where $\widetilde{f}$ is the Laplace operator. Denoting by $\widetilde{f}$ the Fourier transform of $f$:

$$\widetilde{f}(\lambda) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} f(x)\, e^{-i\lambda x}\, dx$$

we have

$$\widetilde{\mathcal{I}_r f} = (1 + |\lambda|^2)^{\frac{r}{2}} \widetilde{f}.$$

## CHAPTER I

## THE METHOD OF APPROXIMATION THEORY

We want to show here that the methods used in [1a], [4b] give the possibility of defining the spaces $B^r_{p,q}$ or $H^r_p$ also for $r = 0$, as Banach space, the elements of which are $S'$ distributions. The Bessel differentiation of order $r$ transforms $B^r_{p,q}(H^r_p)$ in a part of $S$, which can be identified with $B^0_{p,q}(H^0)_p$. This transformation

$$f = \mathcal{I}_r \varphi$$

is one-to-one. We put by definition $\varphi \in B^0_{p,q}(H^0_p)$ for every $f \in B^r_{p,q}(H^r_p)$ and we set

$$\|\varphi\|_{B^0_{p,q}} = \|f\|_{B^r_{p,q}} \; (\|\varphi\|_{H^0_p} = \|f\|_{H^r_p}).$$

This definition will be correct, if it turns out that the spaces defined in this way do not depend on $r$, i. e that the norms $\|\mathcal{I}_{r_1}\varphi\|_{B^r_{p,q}}$ and $\|\mathcal{I}_r \varphi\|_{B^{r_2}_{p,q}}$ are equivalent. In virtue of the group property (see (6) of the Introduction) of the operator $\mathcal{I}_r$ it is sufficient for this, to show the isomorphism of spaces $B^{r_1}_{p,q}$ and $B^{r_2}_{p,q}$ under the operation $\mathcal{I}_{r_2-r_1}$.

This is done in this chapter by means of approximation theory.

I. *Classes* $H^r_p$.

Let $R^n$ be $n$ dimensional real space of points $x = (x_1, \dots, x_n)$. We shall write $x = (x_j, x'_j)$, $x'_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$. Then it is possible to write $f(x) = f(x_j, x'_j)$ for a function $f(x)$ defined on $R^n$. We put:

$$|f|_{L_p(R_n)} \equiv \|f\|_p = \left\{ \int_{R^n} |f(x)|^p\, dx \right\}^{\frac{1}{p}}, \qquad (1 \leq p \leq \infty).$$

We denote by

$$\Delta_{h, x_j} f(x) = f(x_j + h, x'_j) - f(x_j, x'_j)$$

the first difference of the function $f$ in point $x$ in the direction $x_j$ with the step $h$ and by

$$\Delta^k_{h, x_j} f(x) = \Delta^{k-1}_{h, x_j} f(x) \qquad (k = 2, 3, \dots)$$

the difference of order $k$.

Let $r > 0$ and $r = \bar{r} + r'$, where $\bar{r}$ is an integer and $0 < r' \leq 1$; $1 \leq p \leq \infty$. Let also $k \geq 2$ be an integer. By definition [4a].

$$f \in H^r_p(R_n) = H^r_p, \quad \text{if}$$

1) $f \in L_p(R_n) = L_p$,

2) there are derivatives in Sobolev's sense $f^{(\bar{r})}_{r_j}$ in $L_p\,(j = 1, \dots, n)$ satisfying the inequalities

$$\| \Delta^k_{h, x_j} \cdot f^{(\bar{r})}_{x_j} \|_p \leq M \, | h |^{r'}, \qquad (j = 1, \dots, n). \tag{3}$$

Here $M$ does not depend on $h$.

We also write

$$\|f\|_{H^r_p} = \|f\|_p + M^r_f \tag{4}$$

where $M^r_f$ denotes the least constant $M$ in inequalities (3) for a given function $f$. The definition of classes $H^r_p$ depends on $k \geq 2$ unessentially. It is known that there are constants $C_1$, $C_2$ depending only on integers $k_1$, $k_2 \geq 2$, for which

$$C_1 \|f\|^{(k_1)}_{H^r_p} \leq \|f\|^{(k_2)}_{H^r_p} \leq C_2 \|f\|^{(k_1)}_{H^r_p}.$$

Here the symbol $\|\cdot\|^{(k)}$ shows that the norm (4) is defined for a given $k$.

We shall use the definition of the classes $H^r_p$ only for $k = 2$ or $k = 4$.

A function $g_\nu(z) = g_\nu(z_1, \dots, z_n)$ of the complex variables $z = (z_1, \dots, z_n)$ is said to be of exponential type of degree $\nu > 0$ (in $z_1, \dots, z_n$) if it satisfies the following conditions:

1) $g_\nu(z)$ is an entire function of $z_1, \dots, z_n$;

2) For every $\varepsilon > 0$ there exists a constant $A_\varepsilon$ such that

$$| g_\nu(z) | \leq A_\varepsilon \, l^{(\nu+\varepsilon) \sum_1^n | z_j |}$$

We shall use the following approximation theorem (see [48]) where $H^r_p = H^{(r, \dots, r)}_p(R_n)$;

THEOREM 1. *If $f \in H_p^r$, then*

$$f(x) = \sum_0^\infty Q_s(x),\ x \in R_n, \tag{5}$$

*where the series converges in $L_p$ and where the $Q_s$ are entire functions of exponential type of degrees $2^s$ $(s = 0, 1, 2, \ldots)$ satisfying the inequalities*

$$\| Q_s \|_p \leq \frac{C \| f \|_{H_p^r}}{2^{sr}} \quad (s = 0, 1, 2, \ldots)$$

*where the constant $C$ does not depend on $f$. Conversely, if the function $f$ is expanded in series (5) where the $Q_s$ are functions of exponential type of degrees $2^s$, which satisfy the inequalities*

$$\| Q_s \| \leq \frac{M}{2^{sr}} \tag{6}$$

*then $f \in H_p^r$ and*

$$\| f \|_{H_p^r} \leq cM,$$

*the constant $c$ not depending on $M$.*

As usual we call the quantity

$$E_\nu(f)_p = \inf_{g_\nu} \| f - g_\nu \|_p \tag{7}$$

the best approximation of the function $f$ by functions of exponential type of given degree $\nu$.

We start from the class $H_p^1$. Every function $f$ of this class defines the unique function

$$\varphi = \mathcal{I}_{-1}(f) \tag{8}$$

which is generally speaking a distribution; we denote by $H_p^0$ the set of functions $\varphi$ corresponding by (8) to all $f \in H_p^1$; $H_p^0$ can be considered as a Banach space when provided with the norm

$$\| \varphi \|_{H_p^0} = \| \mathcal{I}_1(\varphi) \|_{H_p^1} \tag{9}$$

We prove

THEOREM 2. *The mapping*

$$\varphi \to f = \mathcal{I}_r \varphi \tag{10}$$

*is one to one from* $H_p^0$ *onto* $H_p^r$. *There are constants* $c_1$ *and* $c_2$, *not depending on* $f$ (*or* $\varphi$) *such that*

$$c_1 \| f \|_{H_p^r} \leq \| \varphi \|_{H_p^0} \leq c_2 \| f \|_{H_p^r} . \tag{11}$$

The proof of this theorem is essentially based on the following lemma.

LEMMA I. *Let* $\alpha > 0$ *and* $r + \alpha > 0$. *Then the operation* (10) *is one to one from* $H_\varrho^\alpha$ *onto* $H_\varrho^{(r+\alpha)}$ *and*

$$c_1 \| f \|_{H_\varrho^{(r+\alpha)}} \leq \| \varphi \|_{H_\varrho^r} \leq c_2 \| f \|_{H_\varrho^{r+\alpha}} , \tag{12}$$

$$\varphi = f^{(\alpha)} = \mathcal{I}_{-\alpha}(f) ,$$

*where* $c_1$ *and* $c_2$ *depend only on* $r, \alpha, \varrho$.

In the following we shall write $<<$ instead $\leq c$, where $c$ is a constant that may depend on $r, \alpha, \varrho$, but must not depend on the considered functions $f, \varphi, \ldots$

Theorem I follows from lemma I directly. Indeed, if $f \in H_p^r$, then we have by (12)

$$\| f \|_{H_p^r} << \| f^{(r-1)} \|_{H_p^1} << \| f \|_{H_p^r} ,$$

which implies (II), if we take into account the following relations

$$\| f^{(r-1)} \|_{H_p'} = \| f^{(r)} \|_{H_p^0} = \| \varphi \|_{H_p^0} .$$

From theorem 1 we obtain the:

COROLLARY 1.
*The mapping*:

$$\varphi \rightarrow \mathcal{I}_{r_2 - r_1} \varphi = f \qquad (r_1 , r_2 \geq 0)$$

*is an isomorphism from* $H_\varrho^{r_1}$ *onto* $H_\varrho^{r_2}$.

To prove Lemma I it is sufficient to prove the following two particular cases of it.

LEMMA 2. *If* $r, \alpha > 0$ *and* $\varphi \in H_p^\alpha$, *then*

$$f = \varphi_{(r)} \in H_p^{r+\alpha} \text{ and } \tag{13}$$

$$\| f \|_{H_p^{r+\alpha}} \leq c \| \varphi \|_{H_p^\alpha} ,$$

*where* $c$ *does not depend on* $\varphi$.

LEMMA 3. *If $r$, $\alpha > 0$ and $f \in H_p^{r+\alpha}$ then*

(14)
$$\varphi = f^{(\alpha)} \in H_p^{(r)} \text{ and}$$
$$\| \varphi \|_{H_p^r} \leq c \, \| f \|_{H_p^{r+\alpha}},$$

*where $c$ does not depend on $f$.*

Note that the integral

$$f(x) = \int_0^{2\pi} K_r(x-u)\,\varphi(u)\,du \qquad \left(\int_0^{2\pi} \varphi(u)\,du = 0\right),$$

where $\mathcal{K}_r(u)$ is the Weyl kernel (see [11], [9], [8], corresponds to our operation $\mathcal{I}_r \varphi$ in the periodic case. Many cases of Lemmas 1 and 2 where proved by Hardy and Littlewood [7], A. Zygmund [10] and Y. Ogievetsky [5].

It is possible to prove that the kernel $G_r(u)$ satisfies the inequality

(15)
$$\left| \frac{\partial^s}{\partial u_j^s} G_r(u) \right| \leq c \, \frac{e^{-|u|}}{|u|^{n+s-r}}, \qquad (s = 0, 1, 2, \ldots)$$

where the constant $c$ depends only on $r$, $s$. Let us begin with the following auxiliary lemma

LEMMA 4. *For $0 < r - s \leq 1$*

(16)
$$\mathcal{I} \equiv \int \left| \Delta_{h,x_j}^2 \frac{\partial^s G_r(u)}{\partial u_j^s} \right| du \leq c \, |h|^{r-s} \qquad (j = 1, \ldots, n;\ s = 0, 1, \ldots)$$

*$c$ depending only on $r$, $s$.*

PROOF. We make the proof for $j = 1$, for the other values of $j$ it is analogous. It is possible to take $h > 0$, without loss of generality. Consider the sets

$$E_{i_1, \ldots, i_n}^h \subset R_n \qquad (i_k = 0, 1;\ k = 1, \ldots, n)$$

of points $u = (u_1, \ldots, u_n)$, where

$$0 \leq u_k < h \quad \text{if } i_k = 0$$
$$h \leq u_k < \infty \quad \text{if } i_k = 1,$$

Let

(17)
$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2$$
$$\mathcal{I}_1 = \int\limits_{|u_1|<3h} \left| \Delta_{hx_1}^2 \frac{\partial^s G_r(u)}{\partial u_1^s} \right| du,$$
$$\mathcal{I}_2 = \int\limits_{|u_1|>3h} \left| \Delta_{hx_1}^2 \frac{\partial^s G_r(u)}{\partial u_1^s} \right| du.$$

Putting $u = (u, u')$, $u' = (u_2, \dots, u_n) \in R'$, we shall have

$$\mathcal{I}_1 << \int\limits_{-h}^{5h} du_1 \int\limits_{R'} \frac{e^{-|u|}}{|u|^{n+s-r}} du' + 2 \int\limits_{-2h}^{4h} du_1 \int\limits_{R'} \dots du' +$$
$$+ \int\limits_{-3h}^{3h} du_1 \int\limits_{R'} \dots du' << \int\limits_0^{5h} du_1 \int\limits_0^{\infty} du_2 \dots \int\limits_0^{\infty} \frac{e^{-|u|}}{|u|^{n+s-r}} du_n =$$
$$= \sum_{i_k=0,1} \int\limits_{E^{5h}_{0, i_2, \dots, i_n}} \frac{e^{-|u|}}{|u|^{n+s-r}} du.$$

Each of the integrals, entering in the last sum is after a suitable change of variables the integral over a set of the kind

$$E^{5h}_{\underbrace{0,\dots,0}_{m},\,\underbrace{1,\dots,1}_{n-m}} \qquad (m \geq 1).$$

Let $n - m > 0$. By introducing the polar coordinates for the variables $u_1, \dots u_m$ and $u_{m+1}, \dots u_n$ separately we obtain $\left(\text{putting } \varrho^2 = \sum_{j=1}^{m} u_j^2,\ \varrho'^2 = \sum_{m+1}^{n} u_j^2\right)$

$$\int\limits_{e'_h} \frac{e^{-|u|}}{|u|^{n+s-r}} du = \int\limits_0^{5\sqrt{m}\,h} \varrho^{m-1} d\varrho \int\limits_{5h}^{\infty} \varrho'^{n-m-1} \frac{e^{-(\varrho^2+\varrho'^2)^{1/2}}}{(\varrho^2+\varrho'^2)^{\frac{n+s-r}{2}}} d\varrho' \leq$$

$$\leq \int\limits_0^{5\sqrt{m}\,h} d\varrho \int\limits_{5h}^{\infty} \frac{e^{-\varrho'}}{\varrho'^{2+s-r}} d\varrho' << h \int\limits_{5h}^{1} \varrho'^{r-s-2} d\varrho' +$$
$$+ h \int\limits_1^{\infty} \varrho'^{r-s-2} e^{-\varrho'} d\varrho' << h\,(h^{r-s-1} + 1) << h^{r-s} + h.$$

And if $n - m = 0$, then

$$\int_{e'_h} \frac{e^{-|u|}}{|u|^{n+r-r}} du = \int_0^{5\sqrt{h}\,h} \varrho^{h-1} \frac{e^{-\sqrt{\varrho^2+\varrho'^2}}}{(\sqrt{\varrho^2+\varrho'^2})^{n+s-r}} d\varrho <<$$

$$<< \int_0^{5\sqrt{h}\,h} \varrho^{r-s-1} d\varrho << h^{r-s} .$$

Therefore

(18) $$\mathcal{I}_1 << h^{r-s} + h.$$

Now we proceed to estimate $\mathcal{I}_2$. We have

$$\mathcal{I}_2 = \int_{|u_1|>3h} \left| \Delta^2_{hx_1} \frac{\partial^s G_r(u)}{\partial u_1^s} \right| du =$$

$$= \int_{|u_1|>3h} \left| \int_0^h \int_0^h \frac{\partial^{s+2} G_r(u_1+t+\tau, u')}{\partial^{s+2} u_1} dt\, d\tau \right| du <<$$

$$<< h^2 \int_{-\infty}^{-3h} du_1 \int \frac{e^{-\sqrt{(u_1+2h)^2+|u'|^2}}}{((u_1+2h)^2+|u'|^2)^{\frac{h+s+2-r}{2}}} du' +$$

$$+ \int_{3h}^{\infty} du_1 \int_{R'} \frac{e^{-\sqrt{u_1^2+|u'|^2}}}{(\sqrt{u_1^2+|u'|^2})^{\frac{n+s+2-r}{2}}} du <<$$

$$<< h^2 \int_h^\infty du_1 \int_0^\infty du_2 \dots \int_0^\infty du_n \frac{e^{-|u|}}{|u|^{n+s+2-r}} =$$

$$= h^2 \Sigma \int_{E_{1, i_2, \dots, i_n}} \frac{e^{-|u|}}{|u|^{n+2+s+r}} du .$$

Each of the integrals of the last sum by means of renumbering of $u_2, \dots u_n$ is reduced to the integral

$$A = h^2 \int_{e''_n} \frac{e^{-|u|}}{|u|^{n+s+2-r}} du$$

where

$$e''_n = E^h_{\underbrace{1, \dots, 1}_{m}, \underbrace{0, \dots, 0}_{n-m}} \qquad (m > 1).$$

We obtain, introducing polar coordinates for $u_1, \dots, u_m$ and $u_{m+1}, \dots, u_n$

$$\left(\varrho^2 = \sum_{j=1}^{m} u_j^2,\ \varrho'^2 = \sum_{j=m+1}^{n} u_j^2\right)$$

$$A = h^2 \int_h^\infty \varrho^{m-1} d\varrho \int_0^{h\sqrt{n-m}} \varrho'^{\,n-m-1} \frac{e^{-\sqrt{\varrho^2+\varrho'^2}}}{(\varrho^2+\varrho'^2)^{\frac{n+s+2-r}{2}}} d\varrho' <<$$

$$<< h^3 \int_h^\infty \varrho^{r-4-s} e^{-\varrho} d\varrho << h^3 (h^{r-s-3} + 1) << h^{r-s} + h^3 .$$

And if $m = n$, then

$$A = h^2 \int_h^\infty \varrho^{n-1} \frac{e^{-\varrho}}{\varrho^{n+s+2-r}} d\varrho = h^2 \int_h^\infty \varrho^{r-s-3} e^{-\varrho} d\varrho << h^2 (h^{r-s-2} + 1) << h^{r-s}.$$

Therefore

(19) $$\mathcal{I}_2 \leq h^{r-s} + h^3 .$$

If $r - s \leq 1$, then for $|h| \leq 1$ from (17) (18) and (19) it follows $\mathcal{I} << h^{r-s}$.

On the other hand if $|h| \geq 1$, then we have, obviously

$$\mathcal{I} \leq 4 \left| \int \frac{\partial^s}{\partial u_j^s} G_r(u) \right| du << \int e^{-|u|} \frac{1}{|u|^{n+s-r}} du \leq c\,|h|$$

Thus lemma 4 is proved.

PROOF of LEMMA 2.

Let

$$f(x) = \int G_r(x-u)\,\varphi(u)\,du \tag{20}$$

and $\varphi(u) \in H_p^{(\alpha)}$.

Then

$$\Delta^4_{hx_1} f(x) = \Delta^2_{hx_1}\int \Delta^2_{hx_1} G_r(x-u)\,\varphi(u)\,du = \tag{21}$$

$$= \Delta^2_{hx_1}\int \Delta^2_{hx_1} G_r(t)\,\varphi(x-t)\,dt =$$

$$= \int \Delta^2_{hx_1} G_r(t)\,\Delta^2_{hx_1}\varphi(x-t)\,dt$$

Set $r = \bar{r} + r'$, $\alpha = \bar{\alpha} + \alpha'$, $r + \alpha = \varrho = \bar{\varrho} + \varrho'$, where $\bar{r}, \bar{\alpha}, \bar{\varrho}$ are integers and $0 < r', \alpha', \varrho' \leq 1$.

Take now the partial derivatives of both sides of (21):

$$\Delta^4_{hx_1}\frac{\partial^{\bar{\varrho}} f}{\partial x_1^{\bar{\varrho}}}(x) = \frac{\partial^{\bar{\varrho}-\bar{\alpha}}}{\partial x_1^{\bar{\varrho}-\bar{\alpha}}}\int \Delta^2_{hx_1} G_r(t)\,\Delta^2_{hx_1}\frac{\partial^{\bar{\alpha}}}{\partial x_1^{\bar{\alpha}}}\varphi(x-t)\,dt =$$

$$= \frac{\partial^{\bar{\varrho}-\bar{\alpha}}}{\partial x_1^{\bar{\varrho}-\bar{\alpha}}}\int \Delta^2_{hx_1} G_r(x-u)\,\Delta^2_{hx_1}\frac{\partial^{\bar{\alpha}}\varphi}{\partial x_1^{\bar{\alpha}}}(u)\,du =$$

$$= \int \Delta^2_{hx_1}\frac{\partial^{\bar{\varrho}-\bar{\alpha}}}{\partial x_1^{\bar{\varrho}-\bar{\alpha}}} G_r(x-u)\,\Delta^2_{hx_1}\frac{\partial^{\bar{\alpha}}}{\partial x^{\bar{\alpha}}}\varphi(u)\,du$$

$$= \int \Delta^2_{hx_1}\frac{\partial^{\bar{\varrho}-\bar{\alpha}}}{\partial x_1^{\bar{\varrho}-\bar{\alpha}}} G_r(u)\,\Delta^2_{hx_1}\frac{\partial^{\bar{\alpha}}}{\partial x_1^{\bar{\alpha}}}\varphi(x-u)\,du.$$

Applying the generalized Minkovsky inequality and taking into account that

$$r - (\bar{\varrho} - \bar{\alpha}) = (r + \bar{\alpha}) - \bar{\varrho} < r + \alpha - \bar{\varrho} = \varrho - \bar{\varrho} = \varrho' \leq 1$$

i.e

$$r - (\bar{\varrho} - \bar{\alpha}) < 1$$

we obtain by lemma 4 (where one must take $s = \bar{\varrho} - \bar{\alpha}$).

$$\left\| \Delta^4_{hx_1}\frac{\partial^{\bar{\varrho}} f(x)}{\partial x_1^{\bar{\varrho}}}\right\|_p \leq \int \left\| \Delta^2_{hx_1}\frac{\partial^{\bar{\varrho}-\bar{\alpha}}}{\partial x_1^{\bar{\varrho}-\bar{\alpha}}} G_p(u)\right\| \left\| \Delta^2_{hx_1}\frac{\partial^{\bar{\alpha}}\varphi(x-u)}{\partial x_1^{\bar{\alpha}}}\right\|_p du =$$

$$= \left\| \Delta^2_{hx_1} \frac{\partial^{\bar{a}} \varphi}{\partial x_1^{\bar{a}}} \right\|_p h^{r-(\bar{\varrho}-\bar{a})} \leq$$

$$\leq c \, \| \varphi \|_{H_p^{(a)}} h^{a'} h^{r-(\bar{\varrho}-\bar{a})} = c \, \| \varphi \|_{H_p^{(a)}} h^{\varrho'} .$$

It may be proved analogously, that

$$\left\| \Delta^4_{hx_j} \frac{\partial^{\bar{\varrho}} f}{\partial x_j^{\bar{\varrho}}} \right\|_p \leq c \, \| \varphi \|_{H_p^{(a)}} h^{\varrho'} .$$

From (15) and (20) it follows easily

$$\| f \|_p \leq c_1 \| \varphi \|_p .$$

The last two inequalities imply $f \in H_p^{\varrho} = H_p^{r+a}$ and inequality (13) holds true. Lemma 2 is proved. Lemma 3 will be proved in § 3.

Lemma 5. *If $\varphi \in L_p$, then $f = \varphi_{(r)} \in H_p^r$ and*

$$\| f \|_{H_p^r} \leq c \, \| \varphi \|_p \tag{22}$$

*where c does not depend on $\varphi$.*

Proof. If $r = 1, 2, \ldots$ is an integer, then, as it is known

$$f \in W_p^r \quad \text{and} \quad f \in W_p^r \rightarrow H_p^r \ (^1) \quad \text{and}$$

$$\| f \|_{H_p^r} \leq c_1 \| f \|_{W_p^r} \leq c_2 \| \varphi \|_p ,$$

The lemma is proved.

If $r > 0$ is not an integer, then as in the preceding considerations we have

$$\Delta^2_{hx_1} f^{(\bar{r})}_{x_j} (x) = \int \Delta^2_{hx_1} G^{(\bar{r})}_r (u) \, \varphi (x - u) \, du .$$

---

(1) If $E, E_1$ are Banach spaces, $E \subset E_1$ and $\| x \|_{E_1} \leq c \| x \|_E$, where $c$ does not depend on $x$, then we write $E \rightarrow E_1$.

Therefore by lemma 4

$$\| \Delta^2_{hx_1} f^{(\bar{r})}_{xj}(x) \|_p \leq \int | \Delta^2_{hx_1} G^{(\bar{r})}_r(u) | \, \| \varphi \|_p \, du \leq h^{r-\bar{r}} \| \varphi \|_p = h^{r'} \| \varphi \|_p ,$$

and the lemma is proved.

COROLLARY. *Conditions of lemma* 5 *imply the existence of a constant* $c$ *which does not depend on* $\varphi$ *and* $\nu \geq 1$ *such that*

$$E_\nu(f)_p \leq \frac{c \| \varphi \|_p}{\nu^{r'}} \quad (\nu \geq 1) \tag{23}$$

Indeed, according to theorem I for the class $H^r_p$, there exists a constant $c_1$ such that

$$E_\nu(f)_p \leq c_1 \frac{\| f \|_{H^r_p}}{\nu^r}, \qquad \nu \geq 1 . \tag{24}$$

Inequality (23) follows from (22), (24).

LEMMA 6. *If* $\varphi \in L_p$, *then there is a constant* $c$ *which does not depend on* $\varphi$ *and* $v$ *such that*

$$E_\nu(f)_p \leq \frac{c E_\nu(\varphi)_p}{\nu^{r'}} . \tag{25}$$

PROOF. Let $g_\nu$ be a function of exponential type of degree $\nu$ such that

$$\| \varphi - g_\nu \|_p = E_\nu(\varphi)_p = \inf_{g_\nu} \| f - g_\nu \|_p .$$

It is known that such a function exists. Evidently $g_\nu \in L_p$ and

$$g_\nu(x) = \mathcal{I}_r(g_\nu) = \int G_r(u) \, g_\nu(x-u) \, du$$

is also a function of exponential type of degree $\nu$, which belongs to $L_p$.

On another hand we have

$$f(x) - g_\nu(x) = \int G_r(x-u) [\varphi(u) - g_\nu(u)] \, du.$$

Therefore, using (23), where one should substitute $f - g_\nu$, $\varphi - g_\nu$ for $f$, $\varphi$ respectively, we get

$$E_\nu(f)_p = E_\nu(f - g_\nu)_p \leq \frac{c_1}{\nu^r} \| \varphi - g_\nu \|_p = c_1 \frac{1}{\nu^r} E_\nu(\varphi)_p$$

and (25) is proved.

Lemmas 5, 6 in the one dimensional periodic case are known in many cases (see [4a] [5]).

## 2. Analogue of Bernstein inequality.

LEMMA 7 (1). *Let a function $\psi_\nu(t)$ be of period $2\nu$ in each of the variables $t_j$ and defined by the equality*

$$\psi(t) = \psi_\nu(t) = (\sqrt{1 + |t|^2})^\alpha \quad (\alpha > 0\,;\ 0 \leq t_j \leq \nu) \qquad (j = 1, \dots, n)$$

*Then its Fourier series*

$$\psi_\nu(t) = \sum_k c_k e^{i \frac{\pi}{\nu} kt} \quad (k = (k_1, \dots, k_n)\,;\ \ k_j = 0, \pm 1, \pm 2, \dots) \tag{1}$$

*converges absolutely and the following inequality*

$$\sum_k |c_k| \leq \chi (1 + n\nu^2)^{\frac{\alpha}{2}} \tag{2}$$

*is true, where $\chi$ does not depend on $\nu > 0$.*

PROOF. It is possible to reduce the proof of this theorem to the known absolute convergence theorems of trigonometric series (see [11], [6], [3]; [8]). However, it can be done only with some restrictions on $\alpha$. Therefore, we prove this lemma by means of direct estimates of Fourier coefficients. For the sake of simplicity of writing we consider the case $n = 2$. We have

$$c_{km} = \frac{1}{(2\nu)^2} \int_{-\nu}^{\nu} \int_{-\nu}^{\nu} \psi(\lambda, \mu)\, e^{-i\left(\frac{k\pi}{\nu}\lambda + \frac{m\pi}{\nu}\mu\right)} d\lambda\, d\mu =$$

$$= \frac{b}{\nu^2} \int_0^\nu \int_0^\nu \psi(\lambda, \mu) \cos \frac{k\pi}{\nu} \lambda \,\cos \frac{m\pi}{\nu} \mu\, d\lambda\, d\mu$$

---

(1) This lemma was proved by P. I. Lisorkin.

where $b = 4$ for $k \neq 0,\ m \neq 0$; $b = 2$ for $k = 0,\ m = 0$ or $k \neq 0,\ m \neq 0$, $b = 1$ for $k = m = 0$. Integrating by parts we obtain the equalities

$$c_{km} = \frac{4\nu^2}{k^2 m^2 \pi^4}\left[\cos m\pi \cos k\pi\, \psi_{\lambda,\mu}(\nu,\nu) - \cos m\pi \int_0^\nu \psi_{\lambda^2,\mu}(\lambda,\nu) \cos\frac{k\pi}{\nu}\lambda\, d\lambda\right.$$

$$\left. - \cos k\pi \int_0^\nu \psi_{\lambda\mu^2}(\nu,\mu)\cos\frac{m\pi}{\nu}\mu\, d\mu + \int_0^\nu\int_0^\nu \psi_{\lambda^2\mu^2} \cos\frac{k\pi\lambda}{\nu}\cos\frac{m\pi\mu}{\nu}\, d\lambda\, d\mu\right],$$

$$k = 0,\ m \neq 0;$$

$$c_{k0} = \frac{1}{k^2\pi^2}\left[\cos k\pi \int_0^\nu \psi_\lambda(\nu,\mu)\, d\mu - \int_0^\nu\int_0^\nu \psi_{\lambda\lambda}(\lambda,\mu)\cos\frac{k\pi}{\nu}\lambda\, d\lambda\, d\mu\right],\ k \neq 0,$$

$$c_{0m} = \frac{1}{m^2\pi^2}\left[\cos m\pi \int_0^\nu \psi_\mu(\lambda,\nu)\, d\lambda - \int_0^\nu\int_0^\nu \psi_{\mu\mu}(\lambda,\mu)\cos\frac{m\pi\mu}{\nu}\, d\lambda\, d\mu\right],\ m \neq 0.$$

Hence by simple calculations we get

$$|c_{km}| \leq \frac{4\nu^2}{k^2 m^2 \pi^4}\left(|\psi_{\lambda\mu}(\nu,\nu)| + \int_0^\nu |\psi_{\lambda^2\mu}(\lambda,\nu)|\, d\lambda + \int_0^\nu |\psi_{\lambda\mu^2}(\nu,\mu)|\, d\mu + \right.$$

$$\left. + \int_0^\nu\int_0^\nu |\varphi_{\lambda^2\mu^2}(\lambda,\mu)|\, d\lambda\, d\mu\right) \leq \frac{\chi}{k^2 m^2}(1 + 2\nu^2)^{\alpha/2};$$

$$|c_{k0}| \leq \frac{\chi}{k^2}(1 + 2\nu^2)^{\alpha/2},\ k \neq 0;\ |c_{0m}| \leq \frac{\chi}{m^2}(1 + 2\nu^2)^{\alpha/2},$$

$$|c_{00}| = \frac{1}{\nu^2}\int_0^\nu\int_0^\nu \psi(\lambda,\mu)\, d\lambda\, d\mu \leq (1 + 2\nu^2)^{\alpha/2},$$

where the constant $\chi$ does not depend on $\nu$ and may be calculated explicitely. The convergence of the series $\sum_k \sum_m |c_{km}|$ and also the inequality

$$\sum_k \sum_m |c_{km}| \leq \chi (1 + 2\nu^2)^{\frac{\alpha}{2}}$$

follow from these estimates.
The considerations are analogous for arbitrary $n$ and thus the lemma is proved.

THEOREM 2. (*Analogous of the Bernstein's inequality*).
*There is a constant $\mathcal{A}$ depending on $\alpha$ such that for every entire function of exponential type $g_\nu(x) \in L_\varrho$ of degree $\nu$, one has*

$$\text{(3)} \qquad \| g_\nu^{(\alpha)} \|_p \leq \mathcal{A} (1 + n\nu^2)^{\alpha/2} \| g_\nu \|_p$$

*and $g_\nu^{(\alpha)}$ is of exponential type of degree $\nu$ ($\alpha > 0$, $\nu > 0$).*

PROOF. Consider first the case $1 \leq p \leq 2$. Then from $g_\nu \in L_p$ it follows $g_\nu \in L_2$ (see [$4_a$], I. 10) and by the Paley-Wiener theorem there is a function $\mu(x) \in L_2(\Delta_\nu)$ where

$$\Delta_\nu = \{-\nu \leq x_j \leq \nu\}$$

such that $\tilde{g}_\nu = \mu$, i. e.

$$g_\nu(x) = \frac{1}{(2\pi)^{n/2}} \int_{\Delta_\nu} \mu(t)\, e^{ixt}\, dt\,.$$

On another hand, according to (9) of the introduction

$$\text{(5)} \qquad g_\nu^{(\alpha)}(x) = \mathcal{I}_{-\alpha}\, g_\nu = \overline{(1 + |t|^2)^{\alpha/2}\, \tilde{g}_\nu} =$$

$$= \int_{\Delta_\nu} \mu(t)\,(1 + |t|^2)^{\alpha/2}\, e^{ixt}\, dt = \int_{\Delta_\nu} \mu(t)\, \psi_\nu(t)\, e^{ixt}\, dt$$

where $\psi_\nu(t)$ is the periodic function with period $2\nu$ in $t_j$ ($j = 1, \ldots,$) which coincides with $(1+|t|^2)^{\alpha/2}$ on $\Delta_\nu$. Let (1) be its Fourier series. Then

$$g_\nu^{(\alpha)}(x)=\int_{\Delta_\nu}\mu(t)\sum_k c_k\, e^{i\left(\frac{\pi}{\nu}k+x\right)t}\,dt=$$

$$=\sum_k c_k\int_{\Delta_\nu}\mu(t)\, e^{i\left(\frac{\pi}{\nu}k+x\right)t}\,dt$$

and consequently

$$\|g_\nu^{(\alpha)}\|_p\leq\sum_k|c_k|\left\|g_\nu\left(\frac{\pi}{\nu}k+x\right)\right\|_p\leq$$

$$\leq\sum_k|c_k|\,\|g_\nu\|_p\leq\chi\,(1+n\nu^2)^{\alpha/2}\,\|g_\nu\|_p\,.$$

The last inequality is written on the ground of Lemma 7. Note that $\mu(t)=0$ for $t\notin\Delta_\nu$. Hence from the Paley-Wiener theorem and (4) it follows that $g_\nu^{(\alpha)}$ is an entire function of degree $\nu$. This proof is analogous to the corresponding one given by P. Civin [6], who proved an inequality of type (2) under some other conditions.

For $p>2$ the function $\mu(t)$ in inequality (4) is in general a distribution and the proof must be changed.

So let $2<p\leq\infty$. Instead of the classical Paley-Wiener theorem we may use its generalization [9]. It says that the Fourier transform of the entire function of degree $\leq\nu$ with polynomial growth on $E_n$, is a $S'$ distribution with support in $\Delta_\nu$. Instead of (5), we write

$$\widetilde{g_\nu^{(\alpha)}}=(1+|\lambda|^2)^{\alpha/2}\,\widetilde{g_\nu}\,,\ \text{and the support of }\widetilde{g_\nu}\subset\Delta_\nu\,.$$

However we cannot substitute the multiplier $(1+|\lambda|^2)^{r/2}$ by the periodical continuation of $(1+|\lambda|^2)^{r/2}$ from $\Delta_\nu$ since the corresponding multiplying operator in $S'$ is not defined. To overcome this difficulty we proceed as follows. First we extend our function $(1+|\lambda|^2)^{\alpha/2}$ from $\Delta_\nu$ to $\Delta_{\nu+\varepsilon}$, $\varepsilon>0$ and then from $\Delta_{\nu+\varepsilon}$ to $R_n$ with period $2(\nu+\varepsilon)$ so as we obtain a function $\mu_\varepsilon(\lambda)\in c^\infty(R_n)$.

Then

$$\mu_\varepsilon(\lambda)=\sum c_k^\varepsilon\, e^{i\frac{\pi k\lambda}{\nu+\varepsilon}}\,.$$

It is possible to differentiate this series as many times as we please and all the obtained series converge uniformly. Using the continuity of the multiplying operator [9], we write

(6) $$\tilde{g}_\nu^{(\alpha)} = (\Sigma\, c_k^\varepsilon\, e^{\frac{i\pi k\lambda}{\nu+\varepsilon}})\, \tilde{g}_\nu = \widetilde{\Sigma\, c_k^\varepsilon\, g_\nu\left(x + \frac{\pi k}{\nu+\varepsilon}\right)}.$$

Since the Fourier transform maps $S'$ into $S'$ continuously, it follows from (6) that

(7) $$g_\nu^{(\alpha)}(x) = \Sigma\, c_k^\varepsilon\, g_\nu\left(x + \frac{\pi k}{\nu+\varepsilon}\right)$$

where the equality is understood in the sense of $S'$. However using the boundedness of $g_\nu$ and the absolute convergence of the series $\sum_k |c_k^\varepsilon|$, we can conclude that the series in (7) converges uniformly and equality (7) is the usual one. It is possible to construct the $\mu_\varepsilon(\lambda)$ in such a way that

$$\lim_{\varepsilon\to 0} \sum_k |c_k^\varepsilon| = \sum_k |c_k|.$$

It follows from equality (7) that

$$\| g_\nu^{(\alpha)} \|_p \leq \| g_\nu \|_p \sum_k |c_k^\varepsilon|$$

and since $\varepsilon$ is arbitrary, using lemma 5, we obtain the theorem.

### 3. Proof of Lemma 3.

Note that

$$f_k \in L_p, \; f_k^{(\alpha)} \in L_p \qquad (\alpha > 0, \; k = 1, 2, \ldots)$$

$$\| f_k - f \|_p \to 0, \; \| f_k^{(\alpha)} - \varphi \|_p \to 0 \qquad (k \to \infty)$$

imply

$$f^{(\alpha)} = \varphi.$$

Let $f \in H^{r+\alpha}$. Then (see the approximation theorem 1 § 1)

(1) $$f = \sum_0^\infty Q_s$$

where $Q_s$ are entire function of exponential type of degree $2^s$ for which

$$\| Q_s \|_\varrho \leq \frac{c \| f \|_{H_p^{r+\alpha}}}{2^{s(r+\alpha)}}$$

From (1) it follows that

$$f^{(\alpha)} = \Sigma\, Q_s^{(\alpha)} \,, \tag{2}$$

where (see (3) § 2)

$$\| Q_s^{(\alpha)} \|_p = c 2^{s\alpha} \| Q_s \|_p \leq c_r \| f \|_{H_p^{r+\alpha}} \frac{1}{2^{sr}} \,, \tag{3}$$

and $Q_s^{(\alpha)}$ are functions of exponential type of degree $2^s$. From (2) and (3) in accordance with the same approximation theorem it follows that $f^{(\alpha)} \in H_p^r$ and inequality (14) § 1 of lemma 3 is proved.

## 4. Classes $B_{p,q}^r$.

Let $1 \leq q < \infty$, $r > 0$, $1 \leq \varrho \leq \infty$. By definition *a function f, defined on $B_n$, belongs to the class $B_{p,q}^r$ if the following norm*

$$\| f \|_{B_{p,q}^r} = \| f \|_p + \left\{ \sum_1^\infty a^{kqr} E_{a^k}^q (f)_p \right\} \frac{1}{q} \qquad (a > 1)$$

*is finite* (see O. V. Besov) [1]).

We define the class $B_{p,q}^0$ analogously to $H_p^0$ as the set of distribution $\varphi$ for which

$$\varphi_1 = \mathcal{I}_1 \varphi = f \in B_{p,q}^1 \,.$$

We put

$$\| \varphi \|_{B_{p,q}^0} = \| \varphi_1 \|_{B_{p,q}^1} \,.$$

All what we said about classes $H_p^r$ is true also for classes $B_{p,q}^r$. In particular the analogous of theorem 1 and Lemma 1 are true where one must change $H$ in $B$. To get it, it is sufficient to prove Lemmas 2 and 3 (where we change $H$ in $B$).

Proof of lemma 2. Let $r$, $\alpha > 0$ and $\varphi \in B_{p,q}^r$. Then, using (15) for $s = 0$ we obtain

$$\| \varphi_r \|_\varrho << \| \varphi \|_\varrho$$

and

$$\| \varphi_r \| B_{p,\theta}^{(r+\alpha)} = \| \varphi_r \|_p + \left\{ \sum_{k=1}^{\infty} a^{k\theta(r+\alpha)} E_{a^k} (\varphi_p)_p \right\}^{1/\theta} <<$$

(lemma 6)

$$<< \| \varphi \|_p + \left\{ \sum_{k=1}^{\infty} a^{k\theta(r+\alpha)} \frac{1}{a^{k\theta\alpha}} E_{a^k}^{\theta} (\varphi)_p \right\}^{1/\theta} =$$

$$= \| \varphi \|_p + \left\{ \sum_{k=1}^{\infty} a^{k\theta r} E_{a^k}^{\theta} (\varphi)_p \right\}^{1/\theta} = \| \varphi \|_{B_{p,\theta}^{(\alpha)}} .$$

Proof of lemma 3. Let $r, \alpha > 0$ and $f \in B_{p,q}^{r+\alpha}$. Let $g_{a^k}(x)$ be the entire function of degree $a^k$, which gives the best approximation of $f$ of order $a^k$:

$$E_{a^k}(f) = \| f - g_{a^k} \|_p \to 0 , \qquad k \to \infty .$$

Then

$$f = g_{a^0} + \sum_{1}^{\infty} (g_{a^k} - g_{a^{k-1}}) \tag{1}$$

and

$$\varphi = f^{(r)} = g_{a^0}^{(r)} + \sum_{1}^{\infty} (g_{a^k} - g_{a^{k-1}})^{(r)}$$

in the sense of $L_p$ convergence.

The convergence of the last series in the $L_p$ norm will be seen below. We have

$$E_{a^k}(\varphi) \leq \left\| f^{(r)} - g_{a^0}^{(r)} - \sum_{\mu=1}^{\infty} (g_{a^\mu} - g_{a^{\mu-1}})^{(r)} \right\|_p <<$$

$$<< \sum_{\mu=k}^{\infty} \| (g_{a^\mu} - g_{a^{\mu-1}})^{(r)} \|_p << \quad \text{(inequality (3) of § 3)}$$

$$<< \sum_{\mu=k}^{\infty} a^{\mu r} \| g_{a^\mu} - g_{a^{\mu-1}} \|_p << \sum_{\mu=k}^{\infty} a^{\mu r} ( \| g_{a^\mu} - f \|_p + \| f - g_{a^{\mu-1}} \|_p ) <<$$

$$<< \sum_{\mu=k}^{\infty} a^{\mu r} (E_{a^\mu}(f)_p + E_{a^{\mu-1}}(f)_p) \leq 2 \sum_{\mu=k}^{\infty} a^{\mu r} E_{a^{\mu-1}}(f)_p .$$

Therefore

$$\| \varphi \|_{B^{(\alpha)}_{p,\theta}} = \| \varphi \|_p + \left\{ \sum_1^{\infty} a^{k\theta\alpha} E^{\theta}_{a^k} (\varphi)_p \right\}^{1/\theta} <<$$

$$<< \| \varphi \|_p + \left\{ \sum_1^{\infty} a^{k\theta\alpha} \left( \sum_{\mu=k}^{\infty} a^{\mu r} E_{a^{\mu}-1} (f)_p \right)^{\theta} \right\}^{1/\theta}.$$

We take a number $\delta \, (0 < \delta < \alpha)$ and put $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Then

$$\| \varphi \|_{B^{(\alpha)}_{p,\theta}} << \| \varphi \|_p + \left\{ \sum_1^{\infty} a^{k\theta\alpha} \left[ \sum_{\mu=k}^{\infty} a^{\mu(\delta-\alpha)} a^{\mu(r+\alpha-\delta)} E_{a^{\mu}-1} (f)_p \right]^{\theta} \right\}^{1/\theta} <<$$

$$<< \| \varphi \|_p + \left\{ \sum_1^{\infty} a^{k\theta\alpha} \left( \sum_{\mu=k}^{\infty} a^{\mu(\delta-\alpha)\theta'} \right)^{\theta/\theta'} \sum_{\mu=k}^{\infty} a^{\mu(r+\alpha-\delta)\theta} E^{\theta}_{2^{\mu}-1} (f)_p \right\}^{1/\theta} <<$$

$$<< \| \varphi \|_p + \left\{ \sum_1^{\infty} a^{k\theta\alpha} a^{k(\delta-\alpha)\theta} \sum_{\mu=k}^{\infty} a^{\mu(r+\alpha-\delta)\theta} E^{\theta}_{a^{\mu}-1} (f)_p \right\}^{1/\theta} <<$$

$$<< \| \varphi \|_p + \left\{ \sum_{\mu=1}^{\infty} \sum_{k=1}^{\mu} a^{k\delta\theta} a^{\mu(r+\alpha-\delta)\theta} E^{\theta}_{a^{\mu}-1} (f)_p \right\}^{1/\theta} <<$$

$$<< \| \varphi \|_p + \left\{ \sum_{\mu=1}^{\infty} a^{\mu\delta\theta} a^{\mu(r+\alpha-\delta)\theta} E^{\theta}_{a^{\mu}-1} (f)_p \right\}^{1/\theta} <<$$

$$<< \| \varphi \|_p + \left\{ \sum_{\mu=1}^{\infty} a^{\mu(r+\alpha)} E^{\theta}_{a^{\mu}-1} (f)_p \right\}^{1/\theta} <<$$

$$<< \| \varphi \|_p + E_1 (f)_p + \left\{ \sum_{\mu=1}^{\infty} a^{\mu(r+\alpha)} E^{\theta}_{a^{\mu}} (f)_p \right\}^{1/\theta} <<$$

$$<< \| f \|_p + \left\{ \sum_{\mu=1}^{\infty} a^{\mu(r+\alpha)} E^{\theta}_{a^{\mu}} (f)_p \right\}^{1/\theta} = \| f \|_{B^{(r+\alpha)}_{p,\theta}}.$$

The last inequality follows from the relations

$$E_1 (f)_p \leq \| f \|_p \qquad \text{and} \qquad B^{r+\alpha}_{p,q} \rightarrow H^{r+\alpha}_p$$

(see [1a]):

$$\| \varphi \|_p \leq \| \varphi \|_{H^{\alpha}_p} << \text{(see lemma 1)} <<$$

$$<< \| f \|_{H^{r+\alpha}_p} << \| f \|_{B^{r+\alpha}_{p,q}}.$$

§ 5. Let $g \subset R_n$ be an open set, $\Gamma$ its boundary and $g_\delta$ $(\delta > 0)$ the set of points $x$ with distance to $\Gamma$ greater than $\delta$. Let also $r > 0$ and as before $r = \bar{r} + r'$, where $\bar{r}$ is integer and $0 < r' \leq 1$. By definition $f \in H_p^r(g)$ if $f \in W_p^{\bar{r}}(g)$ and for every partial derivative $\mathcal{D}^{\bar{r}} f$ of order $\bar{r}$ the following inequality is fulfilled

$$(1) \qquad \| \mathcal{D}^{\bar{r}} f(x+h) - 2\mathcal{D}^{\bar{r}} f(x) + \mathcal{D}^{\bar{r}} f(x-h) \|_{L_p(g_{|h|})} \leq M |h|$$

$$|h|^2 = \sum_{i=1}^{h} h_i^2$$

where the constant $M$ does not depend on $h = (h_1, \dots, h_n)$.

We put

$$\|f\|_{H_p^r(g)} = \|f\|_{L_p(g)} + M_f$$

where $M_f$ is the least constant $M$ in (1).

It was given in § 1 another definition of $H_p^r(g)$, $g = R_n$. Both definitions are equivalent for $g = R_n$ (see § 6).

Analogously $f \in B_{p,q}^r(g)$, if there is a finite norm

$$\|f\|_{B_{p,q}^r(g)} = \|f\|_{L_p(g)} + \Sigma \left\{ \int_g \left[ \int_g \frac{\mathcal{D}^{\bar{r}} f(x+y) - 2\mathcal{D}^{\bar{r}} f(x) + \mathcal{D}^{\bar{r}} f(x-y)}{|x-y|^{n+p\alpha}} \, dx \right]^{q/p} dy \right\}^{1/q}.$$

O. V. Besov showed that if the boundary $\Gamma$ of $G$ satisfies a Lipschitz condition, then for every $f \in B_{p,q}^r(g)$, it can be constructed its continuation $\bar{f} \in B_{p,q}^r(R_n)$ such that

$$\|\bar{f}\|_{B_{p,q}^r(R_n)} \leq c \|f\|_{B_{p,q}^r(g)}$$

where the constant $c$ does not depend on $p, r$. The corresponding continuation theorem for the classical classes $W_p^r(g)$ $(r = 1, 2, \dots)$ was proved by Calderon [3]. The continuation theorem for the classes $W_p^r H^{(\alpha)}(g)$ for $g$ with sufficiently smooth boundary was proved in [4c]. Note that after the mentioned Besov's result, it is possible to say that for $r = 0, 1, 2, \dots$ and $0 < \alpha < 1$ the classes $W_p^r H^{(r)}(g)$ and $H_p^{(r+\alpha)}(g)$ are equivalent. (1)

Let now $f \in B_{p,q}^r(g)$ and $\bar{f} \in B_{p,q}^r(R_n)$ be its continuation on $R_n$. We can write it as follows

$$\bar{f} = G_r * \varphi$$

where $\varphi \in B_{p,q}^0(R_n)$ is defined uniquely. Thus we have on $g$

$$f(x) = G_r * \varphi .$$

---

(1) Added in proof. See also a forthcoming paper by Adams, Aronszajn and K. T. Smith.

## §. 6. Equivalence of the two definitions of $H_p^r(R_n)$.

Let as before $r = \bar{r} + r'$ where $\bar{r}$ is an integer and $0 < r' \leq 1$. Let also $f \in H_p^r(R_n) = H_p^r$ according to the definition given in § 1. Then by approximation theorem I (see § I)

$$f(x) = \sum_{s=0}^{\infty} Q_s(x)$$

where $Q_s$ are entire function of exponential type of degree $2^s$ and

$$\| Q_s \|_p \leq \frac{c}{2^{sr}}$$

$$\varphi = \mathcal{D}^{\bar{r}} f = \sum_{s=0}^{\infty} \mathcal{D}^{\bar{r}} Q_s = \sum_{s=0}^{\infty} q_s$$

where $\mathcal{D}^{\bar{r}} f$ is a partial derivative of $f$ of order $\bar{r}$.
According to the generalized Bernstein inequality [4e]

$$\| g_s \|_p \leq 2^{s\bar{r}} \frac{c}{2^{sr}} \leq \frac{c}{2^{sr'}}$$

Putting

$$\Delta_h^2 \varphi(x) = \varphi(x+h) - 2\varphi(x) + \varphi(x-h), \qquad h = (h_1, \dots, h_n),$$

we have

$$\Delta_h^2 \varphi(x) = \sum_0^{N-1} \Delta_h^2 q_s + \sum_N^{\infty} \Delta_h^2 q_s = \mathcal{I}_1 + \mathcal{I}_2, \tag{1}$$

where the integer $N$ satisfies inequalities

$$\frac{1}{N} \leq |h| < \frac{1}{N-1}.$$

Evidently

$$\| \mathcal{I}_2 \|_p \leq 4 \sum_{N+1}^{\infty} \| q_s \|_p << \frac{1}{2^{Np}} << |h|^{p'}. \tag{2}$$

Further

$$\Delta_h^2 q_s = |h|^2 \int_0^1 \int_0^1 \frac{\partial^2 q_s}{\partial h^2} [x + (u + v - 1) h] \, du \, dv$$

where $\frac{\partial^2 q}{\partial h^2}$ is a second derivative of $q$, taken in the direction of the vector $h = (h_1, \dots, h_n)$

Therefore

$$\| \Delta_h^2 q_s \|_p \leq | h |^2 \int_0^1 \int_0^1 \left\| \frac{\partial^2 q_s}{\partial h^2} \right\|_p du\, dv << | h |^2 2^{2s} 2^{e/r'} = | h |^2 2^{(2-r')s}$$

and

$$\mathcal{I}_1 << \sum_0^{N-1} \| \Delta_h^2 q_s \|_p << | h |^2 2^{(2-r')N} << | h |^{r'} . \tag{3}$$

From (1), (2), (3), it follows

$$\| \Delta_h^2 f^{(\bar{r})} \|_p \leq | h |^{r'}$$

Moreover

$$\| \Delta_h^k f^{(\bar{r})} \|_e \leq | h |^{r'}$$

## CHAPTER II

## SPACES $B_{p,q}^r(R^n)$ AS INTERPOLATION SPACES

### 1. Some known results on interpolation spaces.

1.1. Let $A_0$ and $A_1$ be two Banach spaces, contained in a vector topological space $\mathcal{A}$, the injection $A_i \rightarrow \mathcal{A}$ being continuous. We denote [23] by $S(p_0, \xi_0, A_0; p_1, \xi_1, A_1)$, $\xi_0 > 0$, $\xi_1 < 0$, $1 \leq p_i \leq \infty$, the vector space spanned in $A_0 + A_1$ (1) by

$$a = \int_{-\infty}^{+\infty} u(t)\, dt, \tag{1.1}$$

when $u$ varies, subject to conditions

$$\exp(\xi_i t)\, u \in L_{p_i}(A_i)\,(^2), \qquad i = 0, 1. \tag{1.2}$$

Provided with the norm

$$\| a \|_S = \inf. [\max_{i=0,1} ( \| \exp (\xi_i t) u \|_{L_{p_i}(A_i)})] \int_{-\infty}^{+\infty} u(t)\, dt = a,$$

it is a Banach space, called [23] *space of means*.

We shall set

(1.3) $$S(q, \eta, A_0 ; q, \eta - 1, A_1) = S(q, \eta ; A_0, A_1), \quad 0 < \eta < 1, \quad 1 \leq q \leq \infty.$$

It is easy to check that

$$S(q, \xi_0, A_0 ; q, \xi_1, A_1) = S\left(q, \frac{\xi_0}{\xi_0 - \xi_1} ; A_0, A_1\right).$$

Cf. important complement in [25].

1.2. *Reiteration property.*

Roughly speaking, the reiteration property says that a space of means of two spaces of means is again a space of means (with different parameters of course). Actually there is even more. Let us recall some definitions first. A Banach space $A$ is called an *intermediate space « between » $A_0$ and $A_1$ if*

(1.4) $$A_0 \cap A_1 \subset A \subset A_0 + A_1 .$$

An intermediate space is *of class $\mathcal{K}_\theta (A_0, A_1)$ if*

(1.5) $$S(1, \theta ; A_0, A_1) \subset A \subset S(\infty, \theta ; A_0, A_1), \quad 0 < \theta < 1.$$

*If $Y_i$, $i = 0, 1$, is an intermediate space of class $\mathcal{K}_{\theta_i}(A_0, A_1)$, then*

(1.6) $$S(q, \eta ; Y_0, Y_1) = S(q, (1 - \eta)\theta_0 + \eta\theta_1 ; A_0, A_1)$$

with equivalent norms.

It follows from the definition that $S(q, \eta ; A_0, A_1)$ is of class $\mathcal{K}_\eta(A_0, A_1)$.

1.3. *Interpolation property* [23].

Let $B_0$, $B_1$ be a second couple of Banach spaces, with properties similar to $A_0$, $A_1$. Let $\pi$ be a continuous linear mapping from $A_i$ to $B_i$, $i = 0, 1$ (i. e. for instance $\pi_i$ is a continuous linear mapping from $A_i \to B_i$ such that $\pi_0 = \pi_1$ on $A_0 \cap A_1$ and $\pi = \pi_i$; then, for every $p_i$, $\xi_i$, *$\pi$ is a continuous linear mapping from* $S(p_0, \xi_0, A_0; p_1, \xi_1, A_1) \to S(p_0, \xi_0, B_0; p_1, \xi_1, B_1)$.

This is the interpolation property for continuous linear mappings.

1.4. *Duality property* [23].

In general, if $X$ is a Banach space, we denote by $X'$ the dual space of $X$, provided with the dual norm. Then, *if* $q \neq \infty$, *one has*:

$$(1.7) \qquad (S(q, \theta; A_0, A_1))' = S(q', \theta; A_0', A_1'), \quad 1/q + 1/q' = 1.$$

Since $S(q, \theta; A_0, A_1) = S(q, 1-\theta; A_1, A_0)$, it follows from (1.7) that

$$(1.8) \qquad (S(q, \theta; A_0, A_1))' = S(q', 1-\theta; A_1', A_0').$$

1.5. *Trace spaces*.

We extract the following particular case from [20].

We consider functions $t \to v(t)$, $t > 0$, such that

$$(1.9) \qquad t^\alpha v \in L_q(0, \infty; A_0),$$

$$(1.10) \qquad t^\alpha v'' \in L_q(0, \infty; A_1)$$

(here $v''$ denotes the second derivative in $t$ of $v$ considered as a vector valued distribution on $]0, \infty[$, [18]]; we assume that $0 < 1/q + \alpha < 2$; then $v(0)$ is meaningful and spans, when $v$ varies subject to conditions (1.9) (1.10), *a trace space*, denoted by $T_0^2(q, \alpha; A_0, A_1)$.

This space is a Banach space when provided with the norm

$$\| a \|_T = \inf. [\max (\| t^\alpha v \|_{L_q A_0}, \| t^\alpha v'' \|_{L_q A_1})].$$

It is proved in [23] (see also [15]) that

$$T_0^2(q, \alpha ; A_0, A_1) = S\left(q, \frac{1}{2}\left(\frac{1}{q} + \alpha\right); A_0, A_1\right), \tag{1.11}$$

with equivalent norms.

1.6. *Complex spaces.*

We shall also use, in several places, the *complex spaces* $[A_0, A_1] = [A_0; A_1, \delta(\theta)]$ (cfr. [13], [14], [19], and also [18]). The complex spaces have the interpolation property. The space $[A_0, A_1]_\theta$ is of class $\mathcal{K}_\theta(A_0, A_1)$ (cf. [23]).

## 2. Spaces $B^r_{p,q}$.

Let $W^m_p$ be the Sobolev space [30] on $R^n$ of functions $u$ such that $D^\alpha u \in L_p(R^n)$ for every $|\alpha| \leq m$; provided with the norm

$$\Big(\sum_{|\alpha|\leq m} \| D^\alpha u \|^p_{L_p}\Big)^{1/p},$$

it is a Banach space (and a Hilbert one if $p = 2$).

We shall always assume that $1 < p < \infty$.

We define $W^{-m}_p$ by duality:

$$W^{-m}_p = (W^m_{p'})', \ 1/p + 1/p' = 1.$$

We can now set the

DEFINITION 2.1. Let $r$ be any real number, $\geq 0$ or $< 0$. Let $m$ be an integer such that $m > |r|$. We define *algebraically* (i. e. for the moment we do not put a norm on this space) $B^r_{p,q}$ by

$$B^r_{p,q} = S(q, \eta ; W^m_p, W^{-m}_p), \ (1 - 2\eta)\, m = r. \tag{2.1}$$

We have first to check:

PROPOSITION 2.1. *The space $B^r_{p,q}$ does not depend on $m$ (provided $m > |r|$); moreover all the norms of the spaces $S(q, \eta ; W^m_p, W^{-m}_p)$ are equivalent, when $m$ varies satisfying $(1 - 2\eta)\, m = r$.*

PROOF.

This proposition is a consequence of the reiteration property. Indeed, it is known [13], [19] that

$$[W_p^M, W_p^{-M}]_\theta = W_p^{(1-2\theta)M} \qquad \text{if } (1-2\theta)M \text{ is an integer.} \tag{2.2}$$

Consequently, let $M$ be given satisfying $M > m$. Then applying (2.2) with

$$(1-2\theta_0)M = m, \ (1-2\theta_1)M = -m,$$

we see that $W_p^m$ (resp. $W_p^{-m}$) is of class $\mathcal{K}_{\theta_0}(W_p^M, W_p^{-M})$ (resp. $\mathcal{K}_{\theta_1}(W_p^M, W_p^{-M})$) and (1.6) gives

$$S(q, \eta; W_p^m, W_p^{-m}) = S(q, (1-\eta)\theta_0 + \eta\theta_1; W_p^M, W_p^{-M}),$$

with equivalent norms, and since

$$(1 - 2(1-\eta)\theta_0 - 2\eta\theta_1)M = (1-2\eta)m,$$

the result follows.

We now *choose* a norm on $B_{p,q}^r$ by taking $m = [|r|] + 1$, where $[|r|] =$ integer part of $|r|$ and defining $B_{p,q}^r$ by (2.1), *with the norm of the space of means.*

*In particular*

$$B_{p,q}^0 = S\left(q, \frac{1}{2}; W_p^1, W_p^{-1}\right) \qquad \text{(with the norm of } S\text{).} \tag{2.3}$$

REMARK 2.1. We still have to prove that the spaces $B_{p,q}^r$ just defined coincide with the ones introduced in [1] and in Chapter 1.

## 3. Interpolation properties of spaces $B_{p,q}^r$.

3.1. We shall prove first

THEOREM 3.1. *Let $r_0$ and $r_1$ be arbitrary real numbers. Then*

$$S(q, \eta; B_{p,q_0}^{r_0}, B_{p,q_1}^{r_1}) = B_{p,q}^{(1-\eta)r_0+\eta r_1}$$

*with equivalent norms.*

PROOF.

We choose $m$ (integer) such that $|r_i| < m$; then

$$B^{r_i}_{p,q_i} = S(q_i, \theta_i, W^m_p, W^{-m}_p), (1 - 2\theta_i) m = r_i,$$

with equivalent norms.

Therefore, using (1.6), we have

$$S(q, \eta; B^{r_0}_{p,q_1}) = S(q, (1-\eta)\theta_0 + \eta\theta_1; W^m_p, W^{-m}_p)$$

hence the result follows.

3.2. Let $\mathcal{F}$ be the Fourier transform, $\mathcal{F}^{-1}$ its inverse, and

$$J_\varrho = \mathcal{F}^{-1}((1 + |\xi|^2)^{-\varrho/2}\mathcal{F}).$$

We define

$$\mathcal{H}^r_p = \{u \mid J_{-r} u \in L_p(R^n)\}, \tag{3.1}$$

provided with the norm $\| J_{-r} u \|_{L_p}$.
We have [13], [14], [19]:

$$\mathcal{H}^r_p = [W^m_p, W^{-m}_p]_\eta, \quad (1 - 2\eta) m = r, \tag{3.2}$$

with equivalent norms. (One has $\mathcal{H}^r_p = W^r_p$ if $r$ is an integer).
Consequently, $\mathcal{H}^r_p$ is of class $\mathcal{K}^p_2(W^m_p, W^{-m}_p)$ and the reiteration theorem gives:

THEOREM 3.2. *Let $r_0$ and $r_1$ be arbitrary real numbers. One has*

$$S(q, \eta; \mathcal{H}^{r_0}_p, \mathcal{H}^{r_1}_p) = B^{(1-\eta)r_0+\eta r_1}_{p\,q} \tag{3.3}$$

*with equivalent norms.*

REMARK 3.1

One has also (same proof), for instance

$$S(q, \eta; B^{r_0}_{p}, \mathcal{H}^{r_1}_p) = B^{(1-\eta)r_0+\eta r_1}_{p,q}.$$

## 4. Identity of spaces $B^r_{p,q}$ with spaces previously introduced.

4.1. *The case* $r > 0$.

Let $m$ be an integer $> r$. We apply (3.3) with $r_0 = m$, $r_1 = 0$, $(1 - \eta) m = r$. It comes

$$B^r_{p,q} = S(q, \eta ; W_p^m, L_p). \tag{4.1}$$

But then the constructive characterisation of $S(q, \eta ; W_p^m, L_p)$ which is given in [23] Chap. VII, § 2, shows the identity of $B^r_{p,q}$ with spaces introduced in [1]. More precisely, define the translations group $G_i(t)$ by

$$G_i(t) f(x) = f(x_1, \dots, x_{i-1}, x_i + t, x_{i+1}, \dots, x_n).$$

Let us set

$$r = \bar{r} + \xi, \quad 0 < \xi \leq 1$$

We consider two cases:

*first case:* $0 < \xi < 1$.

Then « $u \in B^r_{p,q}$ » is equivalent to the following conditions:

$$u \in W_p^{\bar{r}}, \tag{4.2}$$

$$\begin{cases} \text{for every } \alpha, |\alpha| = \bar{r} \text{ and every } i = 1, \dots, n, \text{ one has} \\ t^{-\xi-1/q} (G_i(t) D^\alpha u - D^\alpha u) \in L_q(0, \infty ; L_p). \end{cases} \tag{4.3}$$

The norm in $B^r_{p\,q}$ is equivalent to

$$\left( \| u \|^p_{W_p^{\bar{r}}} + \sum_{i=1}^{i=n} \sum_{|k|=m} \| t^{-\xi-1/q} (G_i(t) D^\alpha u - D^\alpha u) \|^p_{L_q(0, \infty ; L_p)} \right)^{1/p}.$$

*Second case:* $\xi = 1$.

Then « $u \in B^r_{p,q}$ » is equivalent to the following conditions:

(4.2) unchanged,

$$\begin{cases} \text{for every } \alpha, |\alpha| = \bar{r}, \text{ and every } i = 1, \dots, n, \text{ one has} \\ t^{-1-1/q} (G_i(2t) - 2G_i(t) + 1) D^\alpha u \in L_q(0, \infty ; L_p). \end{cases} \tag{4.3}$$

The norm in $B^r_{p,q}$ is equivalent to

$$\left(\|u\|^p_{W^r_p} + \sum_{i=1}^{i=n} \sum_{|\alpha|=m} \|t^{-1-1/q}(G_i(2t) - 2G_i(t) + I)D^\alpha u\|^p_{L_q(0,\infty;L_p)}\right)^{1/p}.$$

REMARK 4.1

For the identity between $B^r_{p,q}$ and the spaces defined by approximations properties (using entire functions of exponential type) cf. Chapter 1 and also, for a different method, [26].

REMARK 4.2

According to the equivalence between trace spaces and spaces of means, the $B^r_{p,q}$ are also *trace spaces*. Using [31] they can also appear as trace spaces of harmonic (or meta-harmonic) functions. This gives the equivalence of the $B^r_{p,q}$, $r > 0$, with the spaces defined in Chapter 3. (Cf. also [17] [17 bis]). This property is extended to every $r$ and also to *fractionnal derivatives* by Lizorkin, in Chapter 3. (For other results on trace of functions defined by properties of fractionnal derivatives cf. also [16]).

Another, more particular, trace property is given in Section 7 below.

4.2. *The case* $r \leq 0$.

It is obvious from the definition that $J_\varrho$ is an isomorphism from $\mathcal{H}^r_p$ onto $\mathcal{H}^{r+\varrho}_p$ for every $r$; using this remark with $r = r_i$, $i = 0, 1$, using the interpolation property and (3.3), we get

THEOREM 4.1. *For every* $r$ *and* $\varrho$, $J_\varrho$ *is an isomorphism from* $B^r_{p,q}$ onto $B^{r+s}_{p,q}$, $1 < p < \infty$, $1 \leq q \leq \infty$.

Since :

a) spaces $B^r_{p,q}$ defined by interpolation coincide, when $r > 0$ (with equivalent norms) with similar spaces defined in Chapters 1 and 3,

b) spaces defined in Chapters 1 and 3 also have the analogous property than the one of Theorem 4.1;

it follows that *all the spaces* $B^r_{p,q}$ *introduced in this paper coincide* (with equivalent norms) *for every* $r$.

## 5. Duality.

THEOREM 5.1. *We assume that* $1 < q < \infty$. *Then*

$$(B^r_{p,q})' = B^{-r}_{p',q'}, \quad 1/p + 1/p' = 1/q + 1/q' = 1 \tag{5.1}$$

*with equivalent norms.*

Proof.

$B^r_{p,q} = S(q, \eta; W^m_p, W^{-m}_p)$ if $(1-2\eta)\, m = r$, by definition; using 1.4, (5.1) follows immediately.

So we have in particular proved :

$$(B^0_{p,q})' = B^0_{p',q'}$$

## 6. Complex spaces between $B^r_{p,q}$.

Let us prove first:

Theorem 6.1. *One has*

$$[B^{r_0}_{p,q}, B^{r_1}_{p,q}]_\theta = B^{(1-\theta)r_0+\theta r_1}_{p,q}, \quad 1 < p < \infty, 1 < q < \infty, \tag{6.1}$$

*with equivalent norms.*

Proof.

We choose $m > | r_i |$ such that $B^{r_i}_{p,q} = S(q, \theta_i; W^m_p, W^{-m}_p), (1-2\theta_i)\, m = r_1$, and then we apply [21] (where it is essentially proved that complex spaces between spaces of means are spaces of means).

We are now going to consider spaces $B^r_{p,q}$ where $p = q$; we simplify the notation by setting

$$B^r_{p,q} = B^r_p. \tag{6.2}$$

We prove now

Theorem 6.2 *One has*

$$[B^r_{p_0}, B^r_{p_1}]_\theta = B^r_p, \text{ with equivalent norms,} \tag{6.2}$$

*where*

$$1 < p_i < \infty, 1/p = (1-\theta)/p_0 + \theta/p_1.$$

Proof.

1) It is enough to prove (6.2) for $r \geq 1$; indeed, since (Theorem 4.1) $J_s$ is an isomorphism from $B^r_p$ onto $B^{r+s}_p$, it is an isomorphism also from $[B^r_{p_0}, B^r_{p_1}]_\theta$ onto $[B^{r+s}_{p_0}, B^{r+s}_{p_1}]_\theta$; we choose $s$ such that $r + s \geq 1$; then, if (6.2) is proved for $r \geq 1$, it will be proved for $r + s$ and then for $r$, $r$ arbitrary.

2) Let us define $\Omega = \{x, t \mid t > 0, x \in R^n\}$; we denote by $\gamma$ the « trace operator » on the hyperplane $t = 0$, i.e. the operator: $v \to v(..., 0)$; it is well known (cf. for instance [22]) that the operator

$$\{-\Delta + 1, \gamma\} : v \to \{-\Delta v + v, \gamma v\}, \Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2},$$

is an isomorphism from $W_p^m(\Omega)$ (resp. $W_p^1(\Omega)$) $(1 < p < \infty)$ onto $W_p^{m-2}(\Omega) \times$ $\times B_p^{m-1/p}$ (resp. $W_p^{-1}(\Omega) \times B_p^{1-1/p}$). We now interpolate, using the complex spaces. We set

$$[W_p^m(\Omega), L_p(\Omega)]_\eta = \mathcal{H}_p^s(\Omega), \ (1-\eta)\, m = s.$$

One can check that

$$[W_p^m(\Omega), W_q^1(\Omega)]_\theta = \mathcal{H}_p^s\, \Omega), \quad \text{if} \quad (1-\theta)\, m + \theta = s,$$

and

$$[W_p^{m-2}(\Omega), W_p^{-1}(\Omega)]_\theta = \mathcal{H}_p^{s-2}(\Omega),$$

and by theorem 6.1.

$$[B_p^{m-1/p}, B_p^{1-1/p}]_\theta = B_p^{s-1/p}$$

Consequently

(6.3) $\{-\Delta + 1, \gamma\}$ *is an isomorphism from* $\mathcal{H}_p^s(\Omega)$ *onto* $\mathcal{H}_p^{s-2}(\Omega) \times B_p^{s-1/p}$;

*for every* $s \geq 1$, and for every $p, 1 < p < \infty$.

We use this result for $(s_0, p_0)$ and $(s_1, p_1)$, where

$$s_i = 1/p_i + r, \ r \text{ fixed } \geq 1.$$

Using again complex interpolation, we obtain

(6.4) $\{-\Delta + 1, \gamma\}$ *is an isomorphism from* $[\mathcal{H}_{p_0}^{s_0}(\Omega), \mathcal{H}_{p_1}^{s_1}(\Omega)]_\theta$

*onto*

$$[\mathcal{H}_{p_0}^{s_0-2}(\Omega), \mathcal{H}_{p_1}^{s_1-2}(\Omega)]_\theta \times [B_{p_0}^r, B_{p_1}^r]_\theta$$

But using a result of [14], we have (3)

$$[\mathcal{H}_{p_0}^{s_0}(\Omega), \mathcal{H}_{p_1}^{s_1}(\Omega)]_\theta = \mathcal{H}_p^{(1-\theta)s_0+\theta s_1}(\Omega)\, \mathcal{H}_p^{r+1/p}(\Omega),$$

where $p$ is given as in Theorem 6.2, and analogous result with $s_i - 2$ instead of $s_i$. Threfore (6.4) gives

(6.5) $\begin{cases} \{-\Delta + 1, \gamma\} \textit{ is an isomorphism from } \mathcal{H}_p^{r+1/p}(\Omega) \textit{ onto} \\ \mathcal{H}_p^{r+1/p-2}(\Omega) \times [B_{p_0}^r, B_{p_1}^r]_\theta . \end{cases}$

By comparison of this result with (6.3) (where we take $s = r + 1/p$), we obtain the desired result.

Remark 6.1

Theorem 6.2 gives *an extension of the classical Riesz-Thorin theorem to spaces* $B_p^r$, $r$ fixed. It would be interesting to obtain also an extension of the classical convexity inequalities.

Remark 6.2

Similar reasoning to the one used in proving Theorem 6.2 has been used in [22].

Remark 6.3

A more general result has been recently proved (by an entirely different method) by P. Grisvard, namely:

$$[B_{p_0, q_0}^{r_0}, B_{p_1, q_1}^{r_1}]_\theta = B_{p_\theta, q_\theta}^{r_\theta}, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

$$r_\theta = (1-\theta) r_0 + \theta r_1, \quad 1 < p_i < \infty, \quad 1 < q_i \leq \infty, \quad r_\theta = (1-\theta) r_0 + \theta r_1$$

Cf. Grisvard [17].

## 7. A trace theorem.

We consider again the open set $\Omega = \{x, t \mid t > 0, x \in R^n\}$. We *define*

$$B_{p,q}^r(\Omega) = S(q, \eta\,; W_p^m(\Omega), W_p^{-m}(\Omega)), \quad (1-2\eta)\, m = r \tag{7.1}$$

(again, this definition does not depend on $m$, and gives *equivalent* norms when $m$ varies, subject to $(1-2\eta)\, m = r$).

In particular we shall use

$$B_{p,p}^r(\Omega) = B_p^r(\Omega).$$

If $h \in L_p(\Omega)$ and satisfies

$$-\Delta h + h = 0 \text{ in } \Omega, \tag{7.2}$$

one can define ([22]) the trace $\gamma h$ on the hyperplane $t = 0$ and

$$\gamma h = u \in B_p^{-1/p}. \tag{7.3}$$

Reciprocally [22] if $u \in B_p^{-1/p}$, there exists $h$, unique, satisfying (7.2) and (7.3). We prove now

THEOREM 7.1. *A necessary and sufficient condition for $u$ to be in $B_p^0$ is that there exists $h$ such that* (7.2) *holds, together with*

(7.4) $$h \in B_p^{1/p}(\Omega)$$

(7.5) $$\gamma h = u.$$

*Function $h$ is unique and the norms $\| u \|_{B_p^0}$ and $\| h \|_{B_p^{1/p}(\Omega)}$ are equivalent.*

PROOF.

1) Let $h$ be given satisfying (7.2) and (7.4). Then by definition

$$h \in B_p^{1/p}(\Omega) = S(p, \eta\,;\, W_p^1(\Omega), L_p(\Omega)) \quad (1 - \eta = 1/p)$$

and this space is contained in

$$S(p, \eta\,;\, L_p(0, \infty\,;\, W_p^1), \quad L_p(0, \infty\,;\, L_p))$$

and by [23] this last space coincides (with equivalent norms) with

$$L_p(0, \infty\,;\, S(p, \eta\,;\, W_p^1, L_p)) = L_p(0, \infty\,;\, B_p^{1/p}).$$

Therefore

(7.6) $$h \in L_p(0, \infty\,;\, B_p^{1/p}).$$

Since $\Delta_x = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2$ is a continuous linear mapping from $B_p^{1/p}$ into $B_p^{1/p-2}$, then

(7.7) $$\frac{\partial^2 h}{\partial t^2} = h - \Delta_x h \in L_p(0, \infty\,;\, B_p^{1/p-2}).$$

Therefore (cf. 1.5)

$$h(0) = \gamma h \in T_0^2(p, 0\,;\, B_p^{1/p}, B_p^{1/p-2}) = S(p, 1/2p\,;\, B_p^{1/p}, B_p^{1/p-2})$$

and by theorem 3.1, this space is identical with $B_p^0$. Consequently $\gamma h = u \in B_p^0$ and the mapping $h \longrightarrow u$ is continuous from the subspace of $B_p^{1/p}(\Omega)$ of functions satisfying (7.2) into $B_p^0$.

2) Let us set:

$$A_0 = \{h \mid h \in W_p^1(\Omega), \ -\Delta h + h = 0, \text{ norm of } W_p^1(\Omega)\},$$

$$A_1 = \{h \mid h \in L_p(\Omega), \ -\Delta h + h = 0, \text{ norm of } L_p(\Omega)\}.$$

We recalled that $u \longrightarrow h$ ($h$ solution of (7.2), (7.5)) is an isomorphism from $B_p^{1-1/p}$ (resp. $B_p^{-1/p}$) onto $A_0$ (resp. $A_1$). Then it is also an isomorphism from $S(p, \eta\,;\, B_p^{1-1/p}, B_p^{-1/p})$ onto $S(p, \eta\,;\, A_0, A_1)$. But if $h \in S(p, \eta\,;\, A_0, A_1)$ then $-\Delta h + h = 0$ and $h \in S(p, \eta\,;\, W_p^1(\Omega), L_p(\Omega)) = B_p^{1-\eta}(\Omega)$, hence the result follows by choosing $1 - \eta = 1/p$.

## Chapter III

## THE CHARACTERIZATION OF THE SPACES $B^r_{p,q}(R^p)$ BY MEANS OF TRACES OF METHAHARMONIC FUNCTIONS.

As it is mentioned in the Introduction, the most exhausting exposition of the theory of the spaces $B^r_{p,q}$ for $r > 0$ was given in [1]. This exposition was based on the theory of approximation. However for $p = q$ these spaces were considered a bit earlier by E. Gagliardo, L. N. Slobodetzky, A. A. Vasharin and at the same time by J. L. Lions, P. I. Lizorkin, S. V. Uspensky [32] as the spaces of boundary values of functions from the corresponding « weight » classes (see [1] for references). For $p = q$ such considerations with weight classes were accomplished by O. V. Besov [33]. Finally the weight classes of methaharmonic functions were considered in [34]. Thus, we may say that the following theorem was in essence proved in the mentioned works.

Theorem A. *The necessary and sufficient condition for the function $f(x)$, defined on $R^n$, to belong to $B^r_{p,q}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $r > 0$, is that the quantity*

$$M^r_{p,q}(f) = \left(\int_0^\infty t^{q(l-r)}\left[\int_{R^n}\left(\sum_{l_1+\cdots+l_n=l}\left|\frac{\partial^l F(x,t)}{\partial x_1^{l_1}\ldots\partial x_n^{l_n}\,\partial t}\right|^p + |F|^p\right)dx\right]^{q/p}\frac{dr}{t}\right)^{1/q}$$

*is finite, where $F(x_1, \ldots, x_n, t)$ is the methaharmonic continuation of $f(x)$ in the semispace $R^+_{n+1} = \{x \in R_n, t > 0\}$, $t = r + 1$. The quantities $M^r_{p,q}(f)$ and $\|f\|_{B^r_{p,q}}$ are equivalent, i. e. there are constants $C_1$ and $C_2$ which do not depend on $f$ such that*

$$c_1 \|f\|_{B^r_{p,q}} \leq M^r_{p,q}(f) \leq c_2 \|f\|_{B^r_{p,q}} .$$

In this chapter we substitute the quantity $M^r_{p,q}$ by a simpler but equivalent one, namely:

$$\left\{ \int_{R_n}^{\infty} t^{q(l-r)} \left[ \int_{R_n} \left| \frac{\partial^l F}{\partial t^l} \right|^p dx \right] \frac{dt}{t} \right\}^{1/q}$$

and with its help we develop the theory of the spaces $B^r_{p,q}$ for arbitrary real $r$. Naturally we are in need of a notion of methaharmonic continuation of $S'$-distribution.

## 1. Methaharmonic continuation of $S'$-distribution into the semispace $R^+_{n+1}$.

We form the convolution of the distribution $f(x) \in S'$ with the Poisson kernel for the methaharmonic operator

$$(*) \qquad P_t(x) = \frac{2t}{(2\pi)^{\frac{n+1}{2}}} \frac{1}{(|x|^2 + t^2)^{\frac{n+1}{4}}} \mathcal{K}_{\frac{n+1}{2}} (\sqrt{|x|^2 + t^2}) .$$

One may consider $P_t(x)$ as an abstract function of the parameter $t \geq 0$ with values in $S'$ (in particular $P_0(x) = \delta(x)$). It is known (see for instance [34)] that for $f \in L_p(E_n)$ this convolution can be written in the form

$$P_t * f = \int_{R_n} P_t(x-y) f(y)\, dy = F(x, t)$$

and is a methaharmonic continuation of the function $f(x)$ in $R^+_{n+1}$, i. e.

$$(1) \qquad -\Delta F(x, t) + F(x, t) = 0, \quad F(x, 0) \overset{L_p}{=} f.$$

Here we shall consider the convolution $P_t * f$ in the framework of the theory of distributions [5]. We have

$$P_t * f \in S' \qquad \text{for every} \qquad t > 0.$$

Using the rule of differentiation of the convolution and that of differentiation of the abstract function with respect to the parameter, we find that the convolution $P_t * f$ is a generalized solution of the methaharmonic equation (1) for $t > 0$ and therefore is a function. We shall denote

$$P_t * f = F(x, t), t > 0.$$

Since $P_t(x) \longrightarrow \delta(x)$ when $t \longrightarrow 0$, it follows from the continuity of the convolution, that

$$F(x, t) \overset{S'}{\longrightarrow} f \quad \text{for} \quad t \longrightarrow 0.$$

The function $F(x, t)$ will be called the methaharmonic continuation of $f$ in $R^+_{n+1}$ in what follows.

## 2. Bessel and Liouville integrals (derivatives) of the methaharmonic continuation of $S'$-distribution and connection between them.

For $f \in S'$ we take the Bessel integral $f_{(r)}$ of order $r > 0$ and then consider the methaharmonic continuation $P_t * f_{(r)}$ of the latter.

THEOREM 1. For $r > 0$

$$(2) \qquad P_t * f_{(r)} = \frac{1}{\Gamma(r)} \int_0^\infty F(x, t + \tau)\, \tau^{r-1}\, d\tau,$$

*where* $F(x, t) = P_t * f$.

PROOF. Using the property of convolution we write

$$(3) \qquad P_t * f_{(r)} = P_t * (G_r * f) = G_r * (P_t * f) = G_r * F(x, t) = F_{(r)}(x, r).$$

The operator of convolution with $P_t$ possesses a semi-group property, i. e. for $t, \tau > 0$ we have

$$(4) \qquad P_\tau * F(x, t) = F(x, t + \tau).$$

Multiplying this relation by $\dfrac{\tau^{r-1}}{\Gamma(r)}$ and integrating it with respect to $\tau$, we obtain

$$(5) \quad \frac{1}{\Gamma(r)} \int_0^\infty F(x, t + \tau)\, \tau^{r-1}\, d\tau = \left(\frac{1}{\Gamma(r)} \int_0^\infty P_\tau\, \tau^{r-1}\, d\tau\right) * F(x, t) = G_r * F(x, t)\ (^1).$$

From (3) and (5) we get (2).

The relation (5) itself is important for us and we formulate it separately.

---

(1) We use the formula 6.596,3 [36].

LEMMA 1. *The Liouville integration of the methaharmonic continuation of $S'$-distribution with respect to $t$ and the Bessel integration of it along the hyperplane $t =$ const give the same result,* i. e.

$$\frac{1}{\Gamma(r)}\int_t^\infty \frac{F(x,\xi)}{(\xi-t)^{1-r}}\,d\xi = G_r * F(x,t) \equiv F_{(r)}(x,t).$$

Lemma 1 can be generalized to negative values of $r$. Indeed, the right hand side of equality (5) is defined for negative $r$. But the left hand side of (5) can be independently defined for $r < 0$ as the Liouville derivative or order $(-r)$ of the function $F(x,t)$ with respect to $t$. We denote this derivative by $\dfrac{\vartheta^\alpha F(x,t)}{\vartheta t^\alpha}$, $\alpha = |r|$ and we put (by definition)

$$\frac{\vartheta}{\vartheta t}F(x,t) = (-1)^{[\alpha]+1}\frac{\partial^{[\alpha]+1}}{\partial t^{[\alpha]+1}}\frac{1}{\Gamma([\alpha]+1-\alpha)}\int_t^\infty \frac{F(x,\xi)\,d\xi}{(\xi-t)^{\alpha-[\alpha]}}. \tag{6}$$

If we denote the Liouville integration of $F(x,t)$ with respect to $t$ by $T_{(r)}F(x,t)$, $r > 0$ and if we write for $r < 0$

$$T_{(r)}F = \frac{\vartheta^\alpha}{\vartheta t^\alpha}F(x,t), \quad \alpha = -r,$$

and for $r = 0$

$$T_{(0)}F = F$$

then the operation $T_{(r)}$ becomes well defined for all real $r$ (on the methaharmonic continuations of $S'$-distributions). It is easely checked that $T_{(r)}$ possesses the group property

$$T_{(r_1)}T_{(r_2)} = T_{(r_2)}T_{(r_1)} = T_{(r_1+r_2)}. \tag{7}$$

In what follows we shall use the symbol $\dfrac{\vartheta_r F}{\vartheta t^r}$ also for $r \leq 0$, i. e. we put

$$\frac{\vartheta^r}{\vartheta t^r}F = T_{(r)}F.$$

We note also, that formula (6) can be written in the form

$$\frac{\vartheta^\alpha}{\vartheta t^\alpha}F(x,t) = (-1)^l\frac{\partial^l}{\partial t^l}\frac{1}{\Gamma(l-\alpha)}\int_t^\infty \frac{F(x,\xi)}{(\xi-t)^{1-(l-\alpha)}}\,d\xi, \quad l > [\alpha]. \tag{8}$$

For integer $\alpha > 0$ we obtain from (6)

$$\frac{\vartheta^\alpha}{\vartheta t^\alpha} F(x, t) = (-1)^\alpha \frac{\partial^\alpha F}{\partial t^\alpha}(x, t). \tag{9}$$

The mentioned generalization of Lemma 1 can now be formulated as follows.

THEOREM 2.

$$T_r F(x, t) = F_{(r)}(x, t) = P_t * f_r(x), \qquad -\infty < r < \infty. \tag{10}$$

REMARK. We recall, that the operator $T_{(r)}$ acts on $F(x, t)$ as a function of $t$ and $F_{(r)}(x, t) = \mathcal{I}_r F(x, t)$, where the operator $\mathcal{I}_r$ acts on $F(x, t)$ as a function of $x$.

PROOF. For $r > 0$, Theorem 2 follows from Theorem 1 and Lemma 1.

For $r = 0$ it is valid since $T_{(0)}$ and $\mathcal{I}_0$ are reduced to the identity operator. Let $r$ be negative. We set $-r = \alpha$ and take an even number $2k \geq [\alpha] + 1$. From (8) we have

$$\frac{\vartheta^\alpha F(x, t)}{\vartheta t^\alpha} = \frac{\partial^{2k}}{\partial t^{2k}} \frac{1}{\Gamma(2k - \alpha)} \int_t^\infty \frac{F(x, \xi)}{(\xi - t)^{1-(2k-\alpha)}} d\xi =$$

$$= \frac{1}{\Gamma(2k - \alpha)} \int_t^\infty \frac{\partial^{2k} F(x, \xi)}{\partial \xi^{2k}} \frac{d\xi}{(\xi - t)^{1-(2k-\alpha)}} = \frac{1}{\Gamma(2k - \alpha)} \int_t^\infty \frac{(-\Delta + 1)^k F(x, \xi)}{(\xi - t)^{1-(2k-\alpha)}} d\xi =$$

$$= \frac{1}{\Gamma(2k - \alpha)} \int_t^\infty \frac{F^{(2k)}(x, \xi)\, d\xi}{(\xi - t)^{1-(2k-\alpha)}}.$$

The above expression is equal to $F^{(2k)}_{(2k-\alpha)}$, i. e. to $F^\alpha$. This proves the first equality in (10). The equality

$$F_{(r)}(x, t) = P_t * f_{(r)}.$$

can be proved in the same way as (3) [for $r \leq 0$, $G_r$ means the analytic continuation of the kernel $G_r$, $r > 0$ (see [35], I, p. 48). This equality states in particular that the function $F_{(r)}(x, t)$ is methaharmonic in $E^+_{n+1}$ and has boundary values on $E_n$ which coincide with $f_{(r)}(x)$. Theorem 2 is proved.

In conclusion we reproduce an estimate of Liouville integral which will be used in section 3 below. Generalizing an estimate given by Hardy and Littlewood, Kober has proved the following (see [37], p. 199, th 2):

$$Let\, f(t) \in L_q(0, \infty),\ 1 \leq q \leq \infty,\ \eta > \frac{1}{q'}\ \alpha > 0,$$

$$h_{\eta,\alpha}(z) = \frac{z^\eta}{\Gamma(\alpha)} \int_z^\infty (t-z)^{\alpha-1}\, t^{-\eta-\alpha} f(t)\, dt$$

*Then*

$$\| h_{\eta,\alpha} \|_{L_q(0,\infty)} \leq k \| f \|_{L_q(0,\infty)},\ k = \frac{\Gamma(\alpha)\,\Gamma\left(\frac{1}{q}+\eta\right)}{\Gamma\left(\frac{1}{q}+\eta+\alpha\right)}.$$

Hence, substituting $t^{-\eta-\alpha} f(t) = g(t)$, $\eta = \sigma - \frac{1}{q}$, we obtain

$$\text{(11 a)}\qquad \left\{ \int_0^\infty \left| z^\sigma \int_z^\infty \frac{g(t)\, dt}{(t-z)^{1-\alpha}} \right|^q \frac{dz}{z} \right\}^{1/q} \leq k \left\{ \int_0^\infty \left| z^{\sigma+\alpha} g(z) \right|^q \frac{dz}{z} \right\}^{1/q},\ 1 \leq q < \infty$$

$$\text{(11 b)}\qquad \underset{z \in (0,\infty)}{\text{ess. sup.}} \left| z^\sigma \int_z^\infty \frac{g(t)\, dt}{(t-z)^{1-\alpha}} \right| \leq k\, \underset{z \in (0,\infty)}{\text{ess. sup.}} \left| z^{\sigma+\alpha} g(z) \right|,\ q = \infty.$$

Relation (11) will be called in what follows the H. L. K. — inequality (Hardy-Littlewood-Kober).

## 3. Renormalization of the spaces $B^r_{p,q}(R_n)$.

THEOREM 3. *If $F(x,t)$ is the methaharmonic continuation of the $S'$-distribution $f(x)$ with finite integral $M^r_{p,q}(f)$ then*

$$\text{(12)}\qquad c_1 M^r_{p,q}(f) \leq \left\{ \int_0^\infty t^{q(l-r)} \left[ \int_{R_n} \left| \frac{\partial^l F_{(x,t)}}{\partial t^l} \right|^p dx \right]^{q/p} \frac{dt}{t} \right\}^{1/q} \leq M^r_{p,q}(f) \Bigg|$$

*where $1 < p < \infty$, $1 \leq q \leq \infty$ and the constant $c_1$ does not depend on $f$.*

Proof. The right hand side inequality in (12) is evident. To prove the left hand side we use the Fourier transform with respecto to $x$ (for fixed $t$). We write

$$\tilde{g}(x, t) = \frac{1}{(2\pi)^{n/2}} \int\limits_{R_n} g(x, t)\, e^{i\lambda x}\, d\lambda,\ \lambda x = \lambda_1 x_1 + \dots + \lambda_n x_n .$$

Then we have

$$\frac{\widetilde{\partial^l F(x, t)}}{\partial x_1^{l_1} \dots \partial x_n^{l_n} \partial t^{l_{n+1}}} = (i\lambda_1)^{l_1} \dots (i\lambda_n)^{l_n} \frac{\widetilde{\partial^{l_{n+1}} F(x, t)}}{\partial t^{l_{n+1}}} =$$

$$= \frac{(i\lambda_1)^{l_1} \dots (i\lambda_n)^{l_n}}{(1 + |\lambda|^2)} (1 + |\lambda|^2)^{l'/2} \frac{\widetilde{\partial^{l_{n+1}} F(x, t)}}{\partial t^{l_{n+1}}},\ l_1 + \dots + l_n = l',\ l' + l_{n+1} = l.$$

We denote

$$\frac{\partial^{l_{n+1}}}{\partial t^{l_{n+1}}} F(x, t) = v(x, t).$$

By definition of Bessel derivative and by theorem 2 we get

$$(1 + |\lambda|^2)^{l'/2}\, \tilde{v}(x, t) = \widetilde{v^{(l')}}(x, t) - \frac{\partial^{l'}}{\partial t^{l'}} v(x, t) = (-1)^{l'} \frac{\widetilde{\partial^l F(x, t)}}{\partial t^l} .$$

Since the function

$$\frac{(i\lambda_1)^{l_1} \dots (i\lambda_n)^{l_n}}{(i + |\lambda|^2)^{l'/2}}$$

is a multiplicator of $(L_p, L_p)$ - type, $1 < p < \infty$, it follows from Michlin's theorem that

$$\int\limits_{R^n} \left| \frac{\partial^l F(x, t)}{\partial x_1^{l_1} \dots \partial x_n^{l_n} \partial t^{l_{n+1}}} \right|^p dx \leq c \int\limits_{R_n} \left| \frac{\partial^l F(x, t)}{\partial t^l} \right|^p dx \tag{13}$$

For an analogous reason the inequality

$$\int\limits_{R_n} | F(x, t) |^p\, dx \leq c \int\limits_{R_n} \left| \frac{\partial^l F(x, t)}{\partial t^l} \right|^p dx \tag{14}$$

is also valid.

By (13) and (14) we obtain

$$(15)\qquad \left\{\int\limits_{R_n}\left(\sum_{\Sigma l_i = l}\left|\frac{\partial^l F(x,t)}{\partial x_1^{l_1}\dots\partial t^{l_{n+1}}}\right|^p + |F|^p\right)dx\right\}^{1/p} \leq c\left\{\int\limits_{R_n}\left|\frac{\partial^l F(x,t)}{\partial t^l}\right|^p dx\right\}^{1/p}$$

Raising this inequality to the power $q$, multiplying it by $t^{q(l-r)-1}$ and integrating with respect to $t$ from $0$ to $\infty$ we obtain the desired inequality for $1 \leq q < \infty$. For $q = \infty$ we multiply (15) by $t^{l-r}$ and then take supremum. The theorem is proved.

Theorems A and 3 enable us to proceed as follows.

*Preliminary definition.* The $S'$-distribution $f$ belongs to the space $B^r_{p,q}(R_n)$, $1 < p < \infty$, $1 \leq q \leq \infty$, $r > 0$, if the following integral

$$(16)\qquad \left\{\int^{\infty} t^{(l-r)q}\left(\int\limits_{R_n}\left|\frac{\partial^l F(x,t)}{\partial t^l}\right|^p dx\right)^{q/p}\frac{dt}{t}\right\}^{1/q},\ l = [p] + 1,$$

*is finite, where $F(x,t)$ is the methaharmonic continuation of $f$ in $R_{n+1}$.* Expression (16) may be taken as the norm and then $B^r_{p,q}(R)$ becomes a Banach space.

Expression (16) will be used for extending the theory of spaces $B^r_{p,q}(R_n)$ for $r \leq 0$. We use it in order to give a direct definition of $B^r_{p,q}$. It should be emphasized that this definition is equivalent to the previous one by theorem $A$.

For the mentioned extension a lemma is of importance, which will be proved in next section.

## 4. Main lemma.

Let $F(x,t)$ be the methaharmonic continuation of a certain $S'$-distribution $f$ and let $\dfrac{\vartheta^\gamma F}{\vartheta t^\gamma}$ be the Liouville derivative (for $\gamma > 0$) or Liouville integral (for $\gamma < 0$) of $F$. Then, an arbitrary real number $r$ is fixed.

MAIN LEMMA. *If for a certain $\gamma > r$ the quantity*

$$Q^{r,\gamma}_{p,q}(F) \equiv Q^{r,\gamma}_{p,q} = \left\{\int\limits_0^\infty t^{(\gamma-r)q}\left(\int\limits_{R_n}\left|\frac{\partial^\gamma}{\partial t^\gamma}F(x,t)\right|^p dx\right)^{q/p}\frac{dt}{t}\right\}^{1/q}, 1 \leq p \leq \infty, 1 \leq q \leq \infty$$

*is finite, then for every $\gamma' > r$ we have*

$$0 < A \leq \frac{Q_{p,q}^{r,\gamma'}}{Q_{p,q}^{r,\gamma}} \leq B < \infty \tag{17}$$

*where the constants $A$ and $B$ do not depend on $F(x, t)$.*

**Proof.** Without loss of generality it is possible to take $\gamma' < \gamma$. Denoting $\frac{\vartheta^\gamma F}{\vartheta t^\gamma} = v$, we have

$$\frac{\vartheta^{\gamma'}}{\vartheta t^{\gamma'}} F(x, t) = T_{(\gamma-\gamma')}\left(\frac{\vartheta^\gamma F}{\vartheta t^\gamma}\right) = \frac{1}{\Gamma(\gamma - \gamma')} \int_t^\infty v(x, \tau) \frac{d\tau}{(\tau - t)^{1-(\gamma-\gamma')}},$$

$$\int_0^\infty t^{(\gamma'-r)q} \left\{ \int_{R_n} \left| \frac{\vartheta^{\gamma'}}{\vartheta t^{\gamma'}} F \right|^p dx \right\}^{q/p} \frac{dt}{t} =$$

$$= \frac{1}{(\Gamma(\gamma - \gamma'))^q} \int_0^\infty t^{(\gamma'-r)q} \left\{ \int_{R_n} \left| \int_t^\infty \frac{v(x, t)\, d\tau}{(\tau - t)^{1-(\gamma-\gamma')}} \right|^p dx \right\}^{p/q} \frac{dt}{t} \leq$$

$$\leq \frac{1}{[\Gamma(\gamma - \gamma')]^q} \int_0^\infty t^{(\gamma'-r)q} \left\{ \int_t^\infty \frac{d\tau}{(\tau - t)^{1-(\gamma-\gamma')}} \left[ \int_{R_n} | v(x, \tau) |^p dx \right]^{1/p} \right\}^q \frac{dt}{t} \leq$$ (by inequality (11))

$$\leq \left[ \frac{\Gamma\left(\frac{1}{q} + \gamma' - r\right)}{\Gamma\left(\frac{1}{q} + \gamma - r\right)} \right]^p \int_0^\infty t^{(\gamma-r)q} \left\{ \int_{R_n} | v |^p dx \right\}^{q/p} \frac{dt}{t}.$$

Hence

$$Q_{p,q}^{r,\gamma'} \leq B Q_{p,q}^{r,\gamma}, \quad 1 \leq p \leq \infty, \quad 1 \leq q < \infty,$$

where $B = \dfrac{\Gamma\left(\frac{1}{q} + \gamma' - r\right)}{\Gamma\left(\frac{1}{q} + \gamma - r\right)}$.

The converse estimate is more difficult. Since the Liouville differentiation (integration) and positive translation with respect to $t$ may be commuted we can write

$$\frac{\vartheta^\gamma}{\vartheta t^\gamma} F(x, t + \tau) = \frac{\vartheta^{\gamma-\gamma'}}{\vartheta \tau^{\gamma-\gamma'}} \left[ \frac{\vartheta^{\gamma'}}{\vartheta t^{\gamma'}} F(x, t + \tau) \right] \tag{19}$$

Using the semi-group property of the operator of methaharmonic continuation (see (3)) we write (denoting $P_t(x) = P(x, t)$)

$$\frac{\vartheta^{\gamma'}}{\vartheta t^{\gamma'}} (F(x, t+\tau) = \int_{R_n} P(x-y, \tau) \frac{\vartheta^{\gamma'} F(y, t)}{\vartheta t^{\gamma'}} \, dy \, .$$

Hence, using (19), we obtain

$$\text{(20)} \qquad \frac{\vartheta^{\gamma}}{\vartheta t^{\gamma}} F(x, t+\tau) = \int_{R_n} \frac{\vartheta^{\gamma-\gamma'}}{\vartheta \tau^{\gamma-\gamma'}} P(x-y, \tau) \frac{\vartheta^{\gamma'}}{\vartheta t^{\gamma'}} F(y, t) \, dy.$$

It is this equality that will give us the possibility to prove the converse estimate.

Let's evaluate first the kernel $\frac{\vartheta^{\gamma-\gamma'}}{\vartheta \tau^{\gamma-\gamma'}} P(x-y, \tau)$. If we put $\gamma - \gamma' = \alpha$, $|x-y| = a$, the corresponding estimate may be written in the form

$$\text{(21)} \qquad \left| \frac{\vartheta_\alpha}{\vartheta t^\alpha} P(x-y, \tau) \right| \leq d \, \frac{\mathcal{K}_{\frac{n+\alpha}{2}}\left(\sqrt{\frac{a^2+\tau^2}{2}}\right)}{\left(\sqrt{\frac{a^2+\tau^2}{2}}\right)^{\frac{n+\alpha}{2}}} \, .$$

To prove (21) it is sufficient to use a representation of $P(x, t)$ by formula (*) and the properties of function $\mathcal{K}_\nu(t)$. We omit the details.

Now we shall give an auxiliary construction that will play an essential role in the following calculations. Consider the function

$$\text{(22)} \qquad g(t, \tau) = \begin{cases} 2^\mu (t+\tau)^\lambda \tau^\mu , & 0 < \tau < t \\ 2^\sigma (t+\tau)^\omega t^\sigma , & \tau \geq t \end{cases}$$

where the parameters $\lambda$, $\mu$, $\omega$, $\sigma$ will be specified later. For $\mu > -1$, $\sigma > -1$, $\lambda + \mu = \omega + \sigma = \beta$, elementary calculation gives

$$\text{(23a)} \qquad \int_0^\infty \int_0^\infty g(t, \tau) \, v(t+\tau) \, dt \, d\tau = \int_0^\infty t^{\beta+1} v(t) \, dt$$

$$\text{(23b)} \qquad \operatorname*{ess.\,sup.}_{t,\, \tau > 0} g(t, \tau) \, v(t+\tau) = \operatorname*{ess.\,sup.}_{t>0} t^\beta v(t)$$

By (23*a*) we obtain for $\beta = (\gamma - r)\, q - 2$

$$(24)\quad \int_0^\infty\int_0^\infty g(t,\tau)\left\{\int_{R_n}\left|\frac{\vartheta^\gamma F}{\vartheta t^\gamma}(x,t+\tau)\right|^p dx\right\}^{q/p} dt\, d\tau = \int_0^\infty t^{(\gamma-r)q}\left\{\int_{R_n}\left|\frac{\vartheta^\gamma F}{\vartheta t^\gamma}(x,t)\right|^p dx\right\}^{q/p}\frac{dt}{t}.$$

We are now able to prove the converse estimate. Using (20), (24) and Minkowski's inequality we get

$$\int_0^\infty t^{(\gamma-r)q}\left\{\int_{R_n}\left|\frac{\vartheta^\gamma}{\vartheta t^\gamma}F(x,t)\right|^p dx\right\}^{q/p}\frac{dt}{t} =$$

$$= \int_0^\infty\int_0^\infty g(t,\tau)\left\{\int_{R_n}\left|\int_{R_n}\frac{\vartheta^\alpha}{\vartheta\tau^\alpha}P(x-y,\tau)\frac{\vartheta^{\gamma'}}{\vartheta t^{\gamma'}}F(y,t)\,dy\right|^p dx\right\}^{q/p} dt\, d\tau \le$$

(25)

$$\le \int_0^\infty\int_0^\infty g(t,\tau)\left\{\int_{R_n}\left|\frac{\vartheta^\alpha}{\vartheta\tau^\alpha}P(z,\tau)\right| d\tau\left[\int_{R_n}\left|\frac{\vartheta^{\gamma'}}{\vartheta t^{\gamma'}}F(x-z,t)\right|^p dx\right]^{1/p}\right\}^q dr\, d\tau =$$

$$= \int_0^\infty\left\|\frac{\vartheta^{\gamma'}}{\vartheta t^{\gamma'}}F(\cdot\,,t)\right\|_p^q dt\int_0^\infty g(t,\tau)\left\{\int_{R_n}\left|\frac{\vartheta^\alpha}{\vartheta\tau^\alpha}P(z,\tau)\right| dz\right\}^q d\tau.$$

The integral in curly brackets satisfies

$$\int_{R^n}\left|\frac{\vartheta^\alpha}{\vartheta\tau^\alpha}P(z,\tau)\right| dz \le c\,\frac{1}{\tau^\alpha}\,e^{-\tau/2}$$

where the constant $c > 0$ (depending on $n$, $\alpha$) is finite.

Therefore, setting $\sigma = \mu = (\gamma - r)\, q$, we have

$$\int_0^\infty g(\tau,\tau)\left\{\int_{R_n}\left|\frac{\vartheta^\alpha}{\vartheta\tau^\alpha}P(z,\tau)\right| dz\right\}^q d\tau \le c\int_0^\infty g(t,\tau)\frac{e^{-q\frac{\tau}{2}}}{\tau^{\alpha q}}\,d\tau =$$

$$= c2^\mu\int_0^t (t+\tau)^{-2}\,\tau^{(\gamma-r)q}\frac{e^{-q\frac{\tau}{2}}}{\tau^{\alpha q}}\,d\tau + c2^\mu\int_t^\infty (t+\tau)^{-2}\,t^{(\gamma-r)q}\frac{e^{-q\frac{\tau}{2}}}{\tau^{\alpha q}}\,d\tau \le$$

$$\le c2^\mu\int_0^t (t+\tau)^{-2}\,\tau^{(\gamma'-r)q}\,d\tau + c\,2^\mu\, t^{(\gamma-r)q}\int_t^\infty (t+\tau)^{-2}\frac{d\tau}{\tau^{\alpha q}} \le c_1 t^{(\gamma'-r)q-1}.$$

Substituting this estimate in (25), we obtain

$$Q_{p,q}^{r,\gamma} = \frac{1}{A} Q_{p,q}^{r,\gamma'}$$

where the constant $\frac{1}{A}$ depends on $r, \gamma, \gamma'$ only and thus the assertion of the lemma is proved for $1 \leq q < \infty$.

For $q = \infty$ we argue analogously, using (11*b*) and (23*b*) instead of (11*a*) and (23*a*) respectively Q. E. D.

## 5. Definition of spaces $B_{p,q}^r$ for arbitrary real $r$ and theorem of isomorphism.

DEFINITION. *A $S'$-distribution $f$ belongs to the space $B_{p,q}^r(R_n)$, $1 < p < \infty$, $1 \leq q \leq \infty$; $-\infty < r < \infty$ if for a certain $\gamma > r$ the integral*

$$\int_0^\infty t^{\gamma - r\, q} \left\{ \int_{R_n} \left| \frac{\vartheta^\gamma}{\vartheta t^\gamma} F(x, t) \right|^p dx \right\}^{q/p} \frac{dt}{t} \Bigg\}^{1/q} \tag{26}$$

*where $F(x, t)$ is the methaharmonic continuation of $f$ in $R_{n+1}$, is finite.*

Provided with the norm

$$\|f\|_{B_{p,q}^r} = Q_{p,q}^{r,\, r+1}(f)$$

$B_{p,q}^r(R_n)$ becomes a Banach space.

REMARKS. 1) According to the main lemma, $Q_{p,q}^r(f)$ may be taken as a norm in $B_{p,q}^r(R_n)$ for every $\gamma > r$. Of course, such a norm is equivalent to (16). However the quantity $Q_{p,q}^{r,\, r+1}(f)$ changes smoothly with $r$ and besides has also another advantage (see below).

2) The definition makes sense also for $p = 1$, $p = \infty$. Moreover for these values of $p$ the theorem of isomorphism that will be proved later is valid.

3) In the particular case $p = q$, the space $B_{p,p}^0 = B_p^0$ may be characterized by the finiteness of the quantity

$$Q_{p,p}^{0,\gamma}(f) = \left\{ \int_0^\infty \int_{R_n} \left| \frac{\vartheta^\gamma}{\vartheta t^\gamma} F(x, t) \right|^p dx\, dt \right\}^{1/p}, \gamma = \frac{1}{p}.$$

But this condition means (as it may be proved), that the function $F(x, t)$ belongs to

$$L_p^\gamma (R_{n+1}^+) = \mathcal{H}_p^\gamma (R_{n+1}^+), \quad \gamma = \frac{1}{p}, \qquad \text{i. e.}$$

$$F \in L_p (R_{n+1}^+), \frac{\vartheta^\gamma}{\vartheta t^\gamma} F \in L_p (R_{n+1}^+)$$

and

$$\frac{\vartheta^\gamma}{\vartheta x_i^\gamma} F \in L_p (R_{n+1}^+), \qquad i = 1, \dots, n.$$

Thus the space $B_p^0 (R_n)$ consists exactly of the traces of the methaharmonic function $F \in L_p^{\frac{1}{p}} (R_{n+1})$. Comparing this assertion with theorem 7.1 of chapter II, one may conclude that the norms of the spaces $B_p^\gamma (R_{n+1}^+)$ and $L_p^\gamma (R_{n+1}^+)$ on the methaharmonic functions. We turn now to the proof of the theorem of isomorphism.

THEOREM 4. *The operator* $\mathcal{I}_\varrho$, $-\infty < \varrho < \infty$, *defines an isometric isomorphism between the spaces* $B_{p,q}^r$ *and* $B_{p,q}^{r+\varrho}$ *i. e. if* $f \in B_{p,q}^r$ *than* $\mathcal{I}_\varrho f \in B_{p,q}^{r+\varrho}$ (*and conversely*) *and*

$$\| \mathcal{I}_\varrho f \|_{B_{p,q}^{r+\varrho}} = \| f \|_{B_{p,q}^r} .$$

PROOF. Let $f \in B_p^r$, $\mathcal{I}_\varrho f = h$ and $F(x, t)$, $H(x, t)$ be the methaharmonic continuation of $f$ and $h$, respectively. In order to prove that $h \in B_{p,q}^{r+\varrho}$ it is sufficient (according to the previous definition) to evaluate the quantity $Q_{p,q}^{r+\varrho, r+\varrho+1} (h)$. Using the fact (see theorem 2) that

$$H(x, t) = F_{(\varrho)} (x, t) = T_{(\varrho)} F(x, t)$$

we have

$$\left\{ \int_0^\infty t^q \left( \int_{R_n} \left| \frac{\vartheta^{r+\varrho+1} H(x, t)}{\vartheta t^{r+\varrho+1}} \right|^p dx \right)^{q/p} \frac{dt}{t} \right\}^{1/q} =$$

$$= \left\{ \int_0^\infty t^q \left( \int_{R_n} \left| \frac{\vartheta^{r+1} (x, t)}{\vartheta t^{r+1}} \right|^p dx \right) \frac{dt}{t} \right\}^{1/p} .$$

It follows from this equality, that $h \in B_{p,q}^{r+}$ and $\| h \|_{B_{p,q}^{r+\varrho}} = \| f \|_{B_{p,q}^r}$. One must verify besides that the correspondence between $B_{p,q}$ and $B_{p,q}^{r+\varrho}$ is one-to-one. It is enough here to prove that every $h \in B_{p,q}^{r+\varrho}$ can be represented in the form

$$h = \mathcal{J}_{\varrho, f}, f \in B^r_{p,q}$$

and

$$\| h \|_{B^{r+\varrho}_{p,q}} = \| f \|_{B^r_{p,q}} .$$

Consider the function $T_{-\varrho} H(x, t) = v(x, t)$. It is a methaharmonic one and we have

$$(27) \qquad \left\{ \int_0^\infty t^q \left( \int_{R_n} \left| \frac{\vartheta^{r+1} v}{\vartheta t^{r+1}} \right|^p dx \right)^{q/p} \frac{dt}{t} \right\}^{1/q} =$$

$$= \left\{ \int_0^\infty t^q \left( \int_{R_n} \left| \frac{\vartheta^{r+\varrho+1} H}{\vartheta t^{r+\varrho+1}} \right|^p dx \right)^{q/p} \frac{dt}{t} \right\}^{1/q} .$$

Consequently by definition, its boundary values $\lim_{t \to 0} v(v, t)$ determines a function $f \in B^r_{p\,q}$. We recall now that the operation $T_{(r)}$ has the group property and can be substituted by $\mathcal{J}_{(r)}$.

Therefore

$$H(x, t) = T_{(\varrho)} v(x, t) = \mathcal{J}_\varrho (x, t).$$

Hence when $t \longrightarrow 0$, we obtain

$$h(x) = \mathcal{J}_\varrho f.$$

In addition the equality (27), means that

$$\| \mathcal{J}_\varrho f \|_{B^{r+\varrho}_{p,q}} = \| f \|_{B^r_{p,q}} .$$

This proves the theorem.

We note in conclusion that the isometric property of the operator $\mathcal{J}_{(\varrho)}$ fellows from our choice of the norm (see Remark 2).

## FOOTNOTES

(1) p. 25. $A_0 + A_1$ is a Banach space for the norm

$$\| a \|_{A_0+A_1} = \inf_{a=a_0+a_1} (\| a_0 \|_{A_0} + \| a_1 \|_{A_1}) .$$

(2) p. 25. $L_p(A)$ (resp. $L_p(0, \infty; A)$) denotes the space of strongly measurable functions $t \longrightarrow u(t)$ from $R$ (resp. $(0, \infty)$) to $A$, such that $\| u(t) \|_A$ belongs to $L_p(R)$ (resp. $L_p(0, \infty)$), $1 \leq p \leq \infty$. It is a Banach space when provided with the norm $\| \, \| u(t) \|_A \, \|_{L_p}$.

(3) p. 34. On the whole space $R^n$, the inclusion $[\mathcal{B}_{p_0}^{s_0}, \mathcal{B}_{p_1}^{s_1}]_\theta \subset \mathcal{B}_p^s$ is announced in [19]; the inverse inclusion follows by duality, using the Calderon's duality theorem : $[A_0, A_1]'_\theta = [A'_0, A'_1]_\theta$ (when, for instance, the spaces are reflexive); the similar results for spaces considered on $\Omega$ instead of $R^n$ follow easily.

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