ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

L. FUCHS

Riesz groups

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 19, nº 1 (1965), p. 1-34

<http://www.numdam.org/item?id=ASNSP_1965_3_19_1_1_0>

RIESZ GROUPS

by L. FUCHS (*)

Dedicated to my Father on his 80th birthday

Several authors have devoted their interest to investigating lattice-ordered groups, and recently the theory of lattice-ordered groups has made a great progress. There is a class of partially ordered groups which lies very closely to lattice-ordered groups and which however has not been dealt with systematically, though it deserves great interest because plenty of examples may be found for such groups in different fields of mathematics. This class consists of the directed groups G with the following interpolation property: if $a_1, a_2, b_1, b_2 \in G$ satisfy

$$a_1 \leqq b_1 , \ a_1 \leqq b_2 , \ a_2 \leqq b_1 , \ a_2 \leqq b_2 ,$$

then there exists some $c \in G$ such that

$$a_1 \leqq c \leqq b_1 , \quad a_2 \leqq c \leqq b_2 .$$

In his investigations on linear operators, F. RIESZ has called attention to such groups [13] (1), and this is the reason why we shall call them *Riesz groups*. He has introduced them by the refinement property: if $a_1, \ldots, a_m, b_1, \ldots, b_n$ are positive elements of G and

$$a_1 \ldots a_m = b_1 \ldots b_n ,$$

Pervenuto alla Redazione il 28 Agosto 1964.

(*) The author was supported by a grant from Consiglio Nazionale delle Ricerche at Centro Ricerche Fisica e Matematica in Pisa.

(1) Numbers in brackets refer to the bibliography given at the end of this paper.

then there exist positive elements c_{ij} $(i = 1, \dots, m; j = 1, \dots, n)$ such that

$$a_i = c_{i1} \dots c_{im} \quad \text{and} \quad b_j = c_{1j} \dots c_{mj}$$

for every i and j. Later BIRKHOFF [2] has established some properties of Riesz groups. For some recent applications we may refer to BAUER [1] and NAMIOKA [10].

The aim of the present paper is to lay down a systematic treatment of Riesz groups from the algebraic point of view. A large part of the discussion runs parallel to the theory of lattice-ordered groups. In order to ensure that certain theorems on Riesz groups contain important results on lattice-ordered groups as special cases, one has to consider Riesz groups not simply as partially ordered groups with some special type of order, but rather as partially ordered groups in which for certain pairs of elements « meet » or « union » operation is defined. Thus Riesz groups are to be regarded as algebraic systems with not everywhere defined operations « meet » and « union ». This fact causes some difficulties at several places. Another difficulty stems from the fact that while lattice-ordered groups form an equationally definable class of algebras, and so do those lattice ordered groups which are representable as subdirect products of fully ordered groups, the Riesz groups fail to have this property. Therefore, special care must be taken when subdirect representations are discussed.

First we lay down the most important terminologies and notations to be used throughout the paper (§ 1). Then we begin with different characterizations of Riesz groups (§ 2). It turns out that this class of partially ordered groups admits several equivalent definitions, showing that it is not only of importance from the point of view of applications, but it is at the same time a very natural generalization of the concept of lattice-ordered group. Some of the simplest examples of Riesz groups which are not lattice-ordered may be found in § 3. The next section (§ 4) is devoted to the notions of orthogonality and carrier; they are useful in Riesz groups as well. In § 5, the important concept of o-ideal is discussed. In Riesz groups the o-ideals play a similar role as the l-ideals do in lattice-ordered groups. The property of being a Riesz group is preserved on passing modulo o-ideals. The main result on o-ideals states that in Riesz groups they form a distributive sublattice of the lattice of all normal subgroups.

The next § 6 deals with extensions of commutative Riesz groups analogously to the Schreier extension theory of groups. Among the extensions of a Riesz group by another one, the Riesz groups can be characterized easily. The results of this section serve as tools for obtaining some theorems in the subsequent sections.

Of great importance are the Riesz groups in which two elements may have an intersection (or union) only if one is greater than or equal to the other. These Riesz groups, called *antilattices*, play the same role in the theory of Riesz groups as the fully ordered groups do in the lattice-ordered case. They are introduced in § 7, and in § 9 we get full descriptions of antilattices in the commutative case. First, it is shown that a commutative antilattice with isolated order is an extension of a trivially ordered group by a fully ordered group. The other structure theorem states that they can be obtained as subgroups of cartesian products of fully ordered groups where an element of this product is to be considered greater than e only if each of its components is greater than e. Exceptional elements, called *pseudo-identities* and *pseudo-positive* elements, are discussed in § 8.

In § 10 it is shown that a commutative Riesz group is subdirectly irreducible if and only if it is an antilattice. By making use of this result it is proved that to every commutative Riesz group there exists a meet and union preserving o-isomorphism with a subdirect product of antilattices. The next § 11 contains the discussion of the case when the subdirect product representations by means of antilattices are irredundant. Like in case of lattice-ordered groups, they are then unique up to o-isomorphisms.

The final § 12 deals with the analogue of the Conrad radical of lattice-ordered groups. Here the underlying group is supposed to be only directed and to have isolated order, and even in this rather general case the existence and some of the main properties of the Conrad radical can be established. (In general, we do not lay stress on formulating and proving the results in most general form.)

§ 1. Terminology and notation.

By a *partially ordered group* G we mean a group (whose operation will be written as multiplication) which is at the same time a partially ordered set under a relation $\leqq$, and the monotony law holds: $a \leqq b$ implies $ca \leqq cb$ and $ac \leqq bc$ for all $c \in G$. If G is a lattice under $\leqq$, then it is called a *lattice-ordered group*. The set of all $x \in G$ with $x \geqq e$, e the group identity, is the *positivity domain* $P = G^+$ of G. The symbol P^* will be used for P with e omitted. G^+ completely determines the partial order of G, since $a \leqq b$ if and only if $ba^{-1} \in G^+$. G is *trivially ordered* if $G^+ = e$. G^+ generates the group G if and only if G is *directed* in the sense that to $a, b \in G$ there is always a $c \in G$ satisfying $a \leqq c$, $b \leqq c$.

The partial order $\leqq$ is called *isolated* if $a^n \geqq e$ for some positive n implies $a \geqq e$. It is called *dense* if given $a < b$ there always exists some

$c \in G$ such that $a < c < b$. This amounts to requiring the same for $a = e$, and hence to $P^{*2} = P^*$. (Here and in the sequel multiplication of subsets in G means complex multiplication.)

For $a_1, \dots, a_n \in G$, $U(a_1, \dots, a_n)$ and $L(a_1, \dots, a_n)$ will denote the set of all upper and lower bounds of $a_1, \dots, a_n$ in G. The symbols $U^*(a_1, \dots, a_n)$ and $L^*(a_1, \dots, a_n)$ will be used to denote the sets of elements in G which are greater than and less than, respectively, each of $a_1, \dots, a_n$ (equality excluded). A subset S of G is an *upper* (*lower*) *class* if $a \in S$ implies $U(a) \subseteq S$ ($L(a) \subseteq S$). We say that S is *u-directed* (*l-directed*) if $a, b \in S$ implies the existence of an $x \in S$ such that $x \geqq a$, $x \geqq b$ ($x \leqq a$, $x \leqq b$). S is called *convex* if $a \leqq x \leqq b$ and $a, b \in S$, $x \in G$ imply $x \in S$.

Let G and G' be partially ordered groups and φ a mapping from G into G'. If φ is a group homomorphism which preserves order relation, then it is called an *o-homomorphism.* An *o*-homomorphism, which is surjective and under which the preimage of a positive element always contains a positive element is an *o-epimorphism.* If φ is a group isomorphism preserving order relation, we say φ is an *o-monomorphism.* Finally, if φ is a group isomorphism and if φ, φ^{-1} preserve order relation, then φ will be said to be an *o-isomorphism.*

If A is a convex normal subgroup of G, then the partial order $\leqq$ of G induces one in G/A : one puts $\bar{b} \leqq \bar{c}$ for the cosets $\bar{b}$, $\bar{c}$ mod A if and only if some $b \in \bar{b}$ and $c \in \bar{c}$ satisfy $b \leqq c$. Then the canonical map $b \longrightarrow bA$ is an *o*-epimorphism of G onto G/A. Conversely, if φ is an *o*-epimorphism of G onto some G' and if A is the kernel of φ, then A is a convex normal subgroup of G such that the *o*-isomorphism $G' \cong_o G/A$ holds.

Let G_λ be a family of partially ordered groups with λ ranging over some index set Λ. The cartesian product $C = \Pi^* G_\lambda$ of the G_λ is made into a partially ordered group by putting $g \leqq h$ between two elements of C if $g_\lambda \leqq h_\lambda$ for the components g_λ, h_λ of g, h in each G_λ. The direct product ΠG_λ is a partially ordered subgroup of C, and so is every subdirect product of the G_λ. If we define $g < h$ in the cartesian product C to mean that $g_\lambda < h_\lambda$ for *each* λ, then we call the arising partially ordered group the *mild cartesian product* of the G_λ. *Mild subdirect products* will mean subdirect products with this definition of order.

For the concepts not mentioned here we refer to [7].

§ 2. Characterizations of Riesz groups.

Now we turn to our main objective, i. e. to Riesz groups.

A partially ordered group G is called a *Riesz group* if it has the following two properties:

(i) it is directed;

(ii) it has the *interpolation property*: to all $a_1, a_2, b_1, b_2 \in G$ with $a_i \leqq b_j (i = 1, 2; j = 1, 2)$ there exists a $c \in G$ such that

$$a_i \leqq c \leqq b_j \qquad (i = 1, 2; j = 1, 2).$$

Property (ii) may be called the (2,2)-interpolation property, if in general we mean by the (m, n)-interpolation property that given $a_1, \dots, a_m$ and $b_1, \dots, b_n$ in G such that

$$a_i \leqq b_j \qquad \text{for} \quad i = 1, \dots, m; \; j = 1 \dots, n,$$

then there exists a $c \in G$ satisfying

$$a_i \leqq c \leqq b_j \qquad \text{for} \quad i = 1, \dots, m; \; j = 1, \dots, n.$$

Since property (i) may be viewed as the (2,0)-interpolation property, it follows at once by induction:

LEMMA 2.1. *A partially ordered group G is a Riesz group if and only if it enjoys the (m, n)-interpolation property for all integers $m, n \geqq 0$.*

Note that if, in addition to directedness, the $(2, \infty)$-interpolation property is also assumed (∞ means an arbitrary cardinality), then this is equivalent to the hypothesis of being lattice-ordered. It is clear that the (∞, ∞)-interpolation property amounts to conditional completeness. Thus, roughly speaking, Riesz groups are in the same ratio to lattice-ordered groups as these to complete lattice-ordered groups.

While lattice-ordered groups are necessarily Riesz groups, there are a lot of examples for Riesz groups which fail to be lattice-ordered. See § 3.

The main properties of Riesz groups are summarized in the following theorem.

THEOREM 2.2. *For a directed group G, the following conditions are equivalent* (2)*:*

(1) *G is a Riesz group;*

(2) *for all $a_1, \dots, a_m \in G$, the set $U(a_1, \dots, a_m)$ is l-directed;*

(3) *for all $a_1, \dots, a_m$ and $b_1, \dots, b_n \in G$ we have*

$$U(a_1, \dots, a_m) \; U(b_1, \dots, b_n) = U(a_1 b_1, \dots, a_1 b_n, \dots, a_m b_n);$$

(2) Of course, even the duals of (2)-(5) are equivalent with (1). Portions of this theorem have been published in [13], [2], [1]; cf. also [15].

(4) *the intervals* $[e, a]$ *are multiplicative :*

$$[e, a] \cdot [e, b] = [e, ab];$$

(5) *if* $a \in G$ *satisfies*

$$e \leqq a \leqq b_1 \dots b_n \qquad \text{with} \qquad b_i \geqq e,$$

then there exist elements $a_i \in G$ *such that*

$$a = a_1 \dots a_n \qquad \text{where} \qquad e \leqq a_i \leqq b_i .$$

(1) and (2) are equivalent. Assume (1) and let $b_1, b_2 \in U(a_1, \dots, a_m)$. Then $a_i \leqq b_j$ for $i = 1, \dots, m$, $j = 1, 2$ and by the $(m, 2)$-interpolation property some $c \in G$ satisfies $a_i \leqq c \leqq b_j$ for all i and j. Thus $c \in U(a_1, \dots, a_m)$, and $U(a_1, \dots, a_m)$ is l-directed. That (2) implies (1) follows on using the reverse argument.

(1) and (3) are equivalent. First assume (1), and note that in any G

$$U(a_1, \dots, a_m) \; U(b_1, \dots, b_n) \subseteq U(a_1 b_1, \dots, a_m b_n).$$

Thus it suffices to show that every $x \in U(a_1 b_1, \dots, a_m b_n)$ belongs to $U(a_1, \dots, a_m) \; U(b_1, \dots, b_n)$. Clearly, $a_i b_j \leqq x$, and so $x^{-1} a_i \leqq b_j^{-1}$ for all i and j. By the (m, n) interpolation property there is a $y \in G$ such that $x^{-1} a_i \leqq y \leqq b_j^{-1}$ for all i and j. Now $xy \in U(a_1, \dots, a_m)$ and $y^{-1} \in U(b_1, \dots, b_n)$, and thus $x \in U(a_1, \dots, a_m) \; U(b_1, \dots, b_n)$, indeed. Conversely, suppose (3) and let $a_i \leqq b_j$ for $i = 1, 2$; $j = 1, 2$. Then $e \in U(a_1 b_1^{-1}, a_1 b_2^{-1}, a_2 b_1^{-1}, a_2 b_2^{-1})$ implies $e = cc^{-1}$ with some $c \in U(a_1, a_2)$ and $c^{-1} \in U(b_1^{-1}, b_2^{-1})$. This c satisfies $a_i \leqq c \leqq b_j$ for all i and j.

(1) implies (4). It is enough to verify for a Riesz group G that if $e \leqq x \leqq ab$ for some $x \in G$ where $e \leqq a$, $e \leqq b$, then there exist elements $y \in [e, a]$, $z \in [e, b]$ such that $x = yz$. Now any one of xb^{-1}, e is less than or equal to any one of x, a, hence some $y \in G$ can be inserted between them. If we define $z = y^{-1} x$, then $e \leqq z \leqq b$ and (4) follows.

(4) implies (5). Property (4) gives by induction

$$[e, b_1 \dots b_n] = [e, b_1] \dots [e, b_n]$$

where $b_i \geqq e$. If a belongs to the left member, then it belongs to the right member. This is nothing else than (5).

Finally, (5) implies (1). Assume (5) and $e \leqq b_1$, $e \leqq b_2$, $a \leqq b_1$, $a \leqq b_2$. Then $e \leqq b_1 \leqq b_2 (a^{-1} b_1)$, and (5) implies the existence of a $c \geqq e$ such that $c^{-1} b_1 \geqq e$ and $c \leqq b_2$, $c^{-1} b_1 \leqq a^{-1} b_1$. This c lies between e, a and b_1, b_2.

This completes the proof.

In commutative groups we have a further equivalent property:

THEOREM 2.3. *A commutative directed group G is a Riesz group if and only if it satisfies*

(6) *if for positive $a_1, \dots, a_m$, $b_1, \dots, b_n$ in G*

$$a_1 \dots a_m = b_1 \dots b_n,$$

then there exist positive elements c_{ij} in G $(i = 1, \dots, m;\ j = 1, \dots, n)$ such that

$$a_i = c_{i1} \dots c_{in} \qquad (i = 1, \dots, m)$$

and

$$b_j = c_{1j} \dots c_{mj} \qquad (j = 1, \dots, n).$$

If G is a Riesz group and the positive elements a_i, b_j satisfy $a_1 \dots a_m = b_1 \dots b_n$, then we have

$$e \leqq a_1 \leqq b_1 \dots b_n.$$

(5) guarantees that there are elements $c_{1j} \in G$ such that $e \leqq c_{1j} \leqq b_j$ for every j and $a_1 = c_{11} \dots c_{1n}$. Now $c^*_{2j} = b_j c_{1j}^{-1}$ are certainly positive and satisfy

$$c^*_{21} \dots c^*_{2n} = b_1 \dots b_n a_1^{-1} = a_2 \dots a_m.$$

A simple induction on the number of the a_i establishes (6). Conversely, if a directed group G satisfies (6), then (5) follows at once.

Let us mention here:

PROPOSITION 2.4. *The direct (or the cartesian) product of partially ordered groups is a Riesz group if and only if each factor is a Riesz group. The mild cartesian product of dense Riesz groups is again a Riesz group.*

The proofs of the statements are straightforward and may be left to the reader.

It is known that every abelian group ([3]) can be embedded in a minimal divisible group and divisible groups are easy to handle. We now show that torsion free abelian Riesz groups can be embedded in divisible Riesz groups:

([3]) For the needed results on abelian groups we refer e. g. to [6].

PROPOSITION 2.5. *Let G be a torsion free abelian Riesz group and D its divisible hull. The order of G can be extended in a unique way to a minimal isolated order in D. Then D will again be a Riesz group.*

As usual, $a \in D$ is defined to be positive if for some natural integer n, $a^n \in G$ is positive. This makes D into a partially ordered group which is obviously again directed. If given $a_1, a_2, b_1, b_2 \in D$ such that $a_i \leqq b_j$ $(i = 1, 2; j = 1, 2)$, then choosing a positive integer n such that $a_i^n, b_j^n \in G$, we find a $c \in G$ satisfying $a_i^n \leqq c \leqq b_j^n$ for all i, j. The unique nth root of c lies between the a_i's and b_j's.

In particular, we see that the order of a torsion free abelian Riesz group can always be extended to a minimal isolated order under which it is again a Riesz group.

REMARK. If the definition of Riesz groups is formulated in a much more general way, a family of intermediate notions between Riesz groups and lattice-ordered groups arises. Let $\mathfrak{m}$ and $\mathfrak{n}$ be infinite cardinal numbers. By the $(\mathfrak{m}, \mathfrak{n})$-*interpolation property* we understand the following property of a partially ordered group G: if given two subsets A and B of G such that the cardinality of A is less than $\mathfrak{m}$, that of B is less than $\mathfrak{n}$ and

$$a \leqq b \quad \text{for all} \quad a \in A \quad \text{and} \quad b \in B,$$

then there exists a $c \in G$ satisfying

$$a \leqq c \leqq b \quad \text{for all} \quad a \in A \quad \text{and} \quad b \in B.$$

In this sense, Riesz groups are characterized by the $(\aleph_0, \aleph_0)$-interpolation property, and lattice-ordered groups of power $< \mathfrak{m}$ by the $(\aleph_0, \mathfrak{m})$ interpolation property. Plenty of our results can at once be extended mutatis mutandis to the general case.

§ 3. Examples.

Since lattice-ordered groups are necessarily Riesz groups and we are furnished with a lot of examples for lattice-ordered groups, in what follows we are going to exhibit only examples for Riesz groups which fail to have a lattice-order.

1. Let G be the additive group of complex numbers and let the positivity domain P consist of all $x + iy$ (x, y real) for which either $x = y = 0$ or $x > 0$, $y > 0$.

2. The same group G, but P now consists of all $x + iy$ for which either $x = y = 0$ or $x > 0$, $y \geqq 0$.

3. The same group G, now let positivity be defined such that P consists of 0 and of all $x + iy$ with $x > 0$ (y is arbitrary).

4. Let G be an arbitrary dense fully ordered group and Γ an arbitrary group with no order at all. If $\mathfrak{G}$ is a group which contains Γ as a normal subgroup such that $\mathfrak{G}/\Gamma \cong G$ (group-isomorphism), then $\mathfrak{G}$ may be ordered so that its positivity domain consists of the identity and of all the elements which belong to strictly positive cosets (ordering as in G [4]).

Thus a Riesz group may contain elements of finite order, and need not have isolated order.

5. Let G be the additive group of all polynomials (or rational functions) with real coefficients, and define $f \geqq 0$ if and only if $f(x) \geqq 0$ for each real number x in the closed interval $[0, 1]$.

6. The same group G, but let $f > 0$ mean that $f(x) > 0$ for every $x \in [0, 1]$.

7. Let G consist of the additive group of all real-valued functions which are defined and differentiable in the interval $[0, 1]$. For $f \in G$ set $f \geqq 0$ if $f(x) \geqq 0$ for each $x \in [0, 1]$.

8. The same group, but let $f > 0$ mean that $f(x) > 0$ everywhere in $[0, 1]$.

9. Harmonic functions in a region of the plane form an additive group in which we put $f \geqq 0$ if $f(x) \geqq 0$ for every x.

10. Let G be a group with a valuation w, i. e. w is a function defined on G with real values such that

(i) $w(ab) = w(a) + w(b)$ for all $a, b \in G$,

(ii) $w(e) = 0$.

In addition we assume

(iii) the set of values $w(a)$ is an infinite dense subset of the real numbers.

---

[4] Here G can be replaced by a dense antilattice.

Then putting $a \in P$ if and only if either $a = e$ or $w(a) > 0$. G is made into a Riesz group [5].

If, for instance, G is the free group with the free generators $a_1, \dots, a_n, \dots$ and if we define w as the valuation induced by

$$w(a_n) = \frac{1}{2^n} \qquad (n = 1, 2, \dots),$$

then we get a Riesz group on the countably generated free group. (The same can be done in the abelian case.)

§ 4. Orthogonality, carriers.

As usual in lattice-ordered groups, we call the elements a, b of any partially ordered group G *orthogonal* if

$$a \wedge b = e$$

which means nothing else than $L(a, b) = L(e)$. Orthogonality may be denoted as usual by the symbol $a \perp b$.

This definition of orthogonality is equivalent to the one introduced by KUTYEV [9]; he has defined orthogonality by the relation $Pa^{-1} \cap Pb^{-1} = P$.

Orthogonality in the general sense preserves several properties of orthogonality in lattice-ordered groups. Let us list some of them here.

(a) *If* $a \wedge b = e$ *and if* $c \geqq e$, *then*

$$L(a, bc) = L(a, c).$$

We have clearly

$$L(a, c) = L(a, (a \wedge b) c) = L(a, ac \wedge bc) = L(a, ac, bc) = L(a, bc).$$

(b) *If* $a \wedge b = e$ *and* $a \wedge c = e$, *then* $a \wedge bc = e$. This is a simple consequence of (a).

(c) *If* $a_1, \dots, a_n$ *are pairwise orthogonal elements, then* $a_1 \vee \dots \vee a_n$ *exists and*

$$a_1 \vee \dots \vee a_n = a_1 \dots a_n .$$

By (b), $a_1 \dots a_{n-1}$ is orthogonal to a_n. Hence from the identity $x(x \wedge y)^{-1} y = x \vee y$ we infer $a_1 \dots a_{n-1} \vee a_n = a_1 \dots a_{n-1} a_n$. A simple induction concludes the proof.

---

(5) Note that Example 10 is a special case of Example 4.

(d) *Orthogonal elements commute.* This follows from (c) in the special case $n = 2$.

(e) *If a Riesz group G is the direct product of its convex normal subgroups A and B, $G = A >< B$, then the positive elements of A are orthogonal to the positive elements of B.*

Let $a \in A$, $b \in B$ be positive elements. Then $e \in L(a, b)$. If $g \in G^+$ belongs to $L(a, b)$, then by convexity $g \in A$ and $g \in B$ whence $g = e$. If $g \in L(a, b)$ and if g were incomparable with e, then by the dual of (2) in Theorem 2.2, there would exist an $h \in L(a, b)$ such that $e < h$ and $g < h$ which has been shown to be impossible.

(f) *The set X^* of elements of G^+ orthogonal to every element of a subset X of G^+ is a convex subsemigroup, containing e, of G^+.*

Evidently, $e \in X^*$ and X^* is convex. (b) implies that it is a semigroup, in fact.

In trying to generalize the notion of orthogonality to non-positive elements, analoguously to the lattice-ordered case, a serious difficulty arises. This stems from the fact that in our present case the absolute of an element fails to exist in general. Though it can be replaced by a certain subset of G (see FUCHS [7], p. 77), which is adequate for certain purposes, it does not lead to a very natural concept of generalized orthogonality. Therefore we do not discuss it here.

On using orthogonality, the notion of *carrier* (*filet*) can be introduced in the same way as in lattice-ordered groups (cf. [8]).

The positive elements a, b of G are said to belong to the same *carrier* if $a \wedge x = e$ for some $x \in G$ implies $b \wedge x = e$ and viceversa. This subdivides G^+ into pairwise disjoint carriers; the one containing a is denoted by $a^\wedge$. It follows at once:

(A) *The carriers are convex subsemigroups of G^+.*

In fact, for positive a, b, $ab \wedge x = e$ if and only if $a \wedge x = e$ and $b \wedge x = e$.

Let $a^\wedge \leqq b^\wedge$ mean that $b \wedge x = e$ implies $a \wedge x = e$, for each $x \in G$. Then this definition is independent of the representatives a, b of $a^\wedge, b^\wedge$ and makes the set $\mathfrak{C}$ of carriers of G into a partially ordered set. The map $a \longrightarrow a^\wedge$ of G^+ onto $\mathfrak{C}$ is obviously isotone.

(B) *The union $a^\wedge \vee b^\wedge$ of two carriers $a^\wedge, b^\wedge$ always exists, and satisfies*

$$a^\wedge \vee b^\wedge = (ab)^\wedge \qquad (a \in a^\wedge,\ b \in b^\wedge).$$

The inequalities $a^\wedge \leqq (ab)^\wedge$, $b^\wedge \leqq (ab)^\wedge$ being obvious, let $c^\wedge$ satisfy $a^\wedge \leqq c^\wedge$, $b^\wedge \leqq c^\wedge$, and let $c \in c^\wedge$. Then $c \wedge x = e$ implies both $a \wedge x = e$ and $b \wedge x = e$. By (b) these imply $ab \wedge x = e$ whence $(ab)^\wedge \leqq c^\wedge$, as we wished to show.

(C) *If* $a^\wedge \leqq b^\wedge$ *and if* $a \in a^\wedge$, *then there is a* $b \in b^\wedge$ *such that* $a \leqq b$.

Taking some $b_1 \in b^\wedge$, we have, in view of (B), $b^\wedge = b_1^\wedge = b_1^\wedge \vee a^\wedge = (b_1 a)^\wedge$, thus $b = b_1 a$ is an element of the desired type.

(D) $\mathfrak{C}$ *is distributive in the sense that if* $a^\wedge \wedge b^\wedge$ *exists, then so does* $(a^\wedge \vee c^\wedge) \wedge (b^\wedge \vee c^\wedge)$ *for each* $c^\wedge \in \mathfrak{C}$ *and*

$$(a^\wedge \wedge b^\wedge) \vee c^\wedge = (a^\wedge \vee c^\wedge) \wedge (b^\wedge \vee c^\wedge).$$

Put $d^\wedge = a^\wedge \wedge b^\wedge$; then obviously $d^\wedge \vee c^\wedge \leqq a^\wedge \vee c^\wedge$ and $d^\wedge \vee c^\wedge \leqq b^\wedge \vee c^\wedge$. Assume that $x^\wedge$ exists with $x^\wedge \leqq a^\wedge \vee c^\wedge$ and $x^\wedge \leqq b^\wedge \vee c^\wedge$ which is not $\leqq d^\wedge \vee c^\wedge$. Then by (B) there is also one such that $d^\wedge \vee c^\wedge < x^\wedge$. Let $d \in d^\wedge$, $c \in c^\wedge$, and let $x \in x^\wedge$, $a \in a^\wedge$, $b \in b^\wedge$ satisfy $dc < x < ac$, $x < bc$, which can be achieved because of (C). Then $d^\wedge < (xc^{-1})^\wedge$, for equality would imply $(dc)^\wedge = d^\wedge \vee c^\wedge = (xc^{-1})^\wedge \vee c^\wedge = x^\wedge$, against hypothesis. But $xc^{-1} \in L(a, b)$ implies $(xc^{-1})^\wedge \leqq a^\wedge$, $(xc^{-1})^\wedge \leqq b^\wedge$, a contradiction to the choice of $d^\wedge$.

THEOREM 4.1. *If a partially ordered group G has a finite number of carriers, then the partially ordered set* $\mathfrak{C}$ *of its carriers is a Boolean algebra.*

By (B), $\mathfrak{C}$ is a union semi-lattice, therefore the existence of a minimal element $e^\wedge$ in $\mathfrak{C}$ and the assumed finiteness of $\mathfrak{C}$ imply that $\mathfrak{C}$ is a lattice. By (D) it is distributive. If $a_1^\wedge, \dots, a_n^\wedge$ $(a_i \in G^+)$ are the atoms of $\mathfrak{C}$, and if $b^\wedge \in \mathfrak{C}$ satisfies $a_1^\wedge, \dots, a_k^\wedge \leqq b^\wedge$, but $a_{k+1}^\wedge, \dots, a_n^\wedge \not\leqq b^\wedge$ then $c^\wedge = a_{k+1}^\wedge \vee \dots \vee a_n^\wedge$ will be the complement of $b^\wedge$ in $\mathfrak{C}$. For, $b^\wedge \wedge c^\wedge = (b^\wedge \wedge a_{k+1}^\wedge) \vee \dots \vee (b^\wedge \wedge a_n^\wedge) = e^\wedge$. Furthermore $u = bc$ $(b \in b^\wedge, c \in c^\wedge)$ satisfies $a_i^\wedge \leqq b^\wedge \vee c^\wedge = (bc)^\wedge = u^\wedge$ for every i whence $u \wedge x = e$ implies $a_i \wedge x = e$ for all i; thus $x^\wedge$ contains no atoms, $x = e$ and $u^\wedge$ is the maximal element of $\mathfrak{C}$.

§ 5. *o*-ideals.

The importance of the role played by *l*-ideals in lattice ordered groups, is well-known. In arbitrary partially ordered groups, in particular in Riesz groups, the *o*-ideals seem to have corresponding though not so important a role. We are going to mention the main properties of *o*-ideals.

Recall that a subset A of a partially ordered group G is called an *o ideal* if

(i) A is a normal subgroup of G;

(ii) A is a convex subset of G;

(iii) A is a directed set.

It is evident that an *o*-ideal of a lattice-ordered group is nothing else than an *l*-ideal. Note that

(A) *o-ideals contain unions and intersections of their elements whenever they exist in G.*

(B) Neither the union nor the intersection of two o-ideals need be an o-ideal.

(C) *The union of an ascending chain of o-ideals is again an o-ideal.* Therefore, if A is an o-ideal of G and $x \in G$ does not belong to A, then there exists an o-ideal B of G maximal with respect to the properties of containing A and excluding x.

(D) *o-ideals generated by sets of positive elements do have a meaning.* The convex hull of the normal subsemigroup with e generated by a given set of positive elements is obviously a convex normal subsemigroup S of P which must be contained in all o-ideals generated by the given set. The rest follows from

PROPOSITION 5.1. *There is a one-to-one correspondence between the o-ideals A of a partially ordered group G and all convex normal subsemigroups S of G^+ containing e. The correspondences are given by* (6)

$$\varphi: A \longrightarrow G^+ \cap A \quad \textit{and} \quad \psi: S \longrightarrow \{S\}$$

which are inverse to each other.

It is clear that if A is an o-ideal, then $G^+ \cap A$ is a convex normal subsemigroup with e. Also, $\{G^+ \cap A\} = A$, because A is directed; thus (7) $\varphi\psi$ is the identity. Now if S is as formulated, then $\{S\} = A$ is plainly a normal subgroup which is directed. To see convexity, let $x \in G$ satisfy $ab^{-1} \leqq x \leqq cd^{-1}$ $(a, b, c, d \in S)$. Then on right multiplication by bd we get $ad \leqq xbd \leqq cd^{-1}\,bd$ where $ad \in S$ and $c\,(d^{-1}\,bd) \in S$. So — in view of the convexity of S — we have $y = xbd \in S$. Thus $x = y\,(bd)^{-1} \in \{S\}$. Finally, $G^+ \cap \{S\} = S$, for if $a, b \in S$ satisfy $ab^{-1} \in G^+$, then $e \leqq ab^{-1} \leqq a$ implies $ab^{-1} \in S$. So $\psi\varphi$ is again the identity map.

Note that the o ideal corresponding to G^+ coincides with G if and only if G is directed. Also, the o-ideal generated by a family of o-ideals does have a meaning.

(E) *The canonical map of a partially ordered group G onto its factor group G/A with respect to an o-ideal A of G preserves unions and intersections.* If, for $x, y \in G$, $x \wedge y = z$ exists in G, then for the corresponding cosets mod A one has evidently $\bar{z} \leqq \bar{x}$ and $\bar{z} \leqq \bar{y}$. If the coset $\bar{u}$ satisfies $\bar{u} \leqq \bar{x}$

(6) $\{S\}$ denotes the subgroup generated by the subset S.

(7) In a product of mappings the left factor is followed by the right one.

and $\bar{u} \leqq \bar{y}$, then for some $u_1, u_2 \in \bar{u}$, $u_1 \leqq x$ and $u_2 \leqq y$. By the directedness of cosets, there is a $u \in \bar{u}$ such that $u \leqq u_1$, $u \leqq u_2$. Therefore $u \leqq x$ and $u \leqq y$, $u \leqq x \wedge y = z$, and thus $\bar{u} \leqq \bar{z}$. Consequently, $\bar{z} = \bar{x} \wedge \bar{y}$.

It should be noted that property (A) or (E) is not characteristic for o-ideals. In fact, there exists a larger class of convex subgroups which shares this property. Calling a subgroup C of G an *ω-ideal* if it is the intersection of a family of o-ideals A_α of G, it is obvious that C still contains unions and intersections of its elements if they exist. Furthermore, (E) also prevails, since G/C is canonically isomorphic to a subdirect product of the G/A_α, and since the natural homomorphisms $G \longrightarrow G/A_\alpha$ preserve unions and intersections, so does the map $G \longrightarrow G/C$ they induce. The ω-ideals have the advantage that the ω-ideal generated by an arbitrary subset of G has a well-defined meaning.

Next let G be an arbitrary partially ordered group, and consider the set $\mathfrak{O}$ of all o-ideals of G, partially ordered by inclusion. It is rather surprising that $\mathfrak{O}$ is a lattice (but it is only exceptionally a sublattice of all normal subgroups of G, cf. Theorem 5.6):

PROPOSITION 5.2. *If the set of all o-ideals of a partially ordered group G is ordered by inclusion, then it is a complete lattice.*

By what has been noted at the end of (D) it follows that the set $\mathfrak{O}$ of o-ideals is a complete semilattice under union. By a standard result in lattice theory, $\mathfrak{O}$ is then a complete lattice provided it has a minimum element which is now the case.

Up to now we have considered o-ideals in arbitrary partially ordered groups; in the rest of this section we shall be concerned with those in Riesz groups.

PROPOSITION 5.3. *The factor group G/A of a Riesz group G with respect to an o-ideal A is again a Riesz group.*

Let the cosets $\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2$ mod A satisfy $\bar{a}_i \leqq \bar{b}_j$ $(i = 1, 2, j = 1, 2)$. Then given $a_i \in \bar{a}_i$ arbitrarily, there exist elements $b_{ji} \in \bar{b}_j$ satisfying $a_i \leqq b_{ji}$ $(i = 1, 2; j = 1, 2)$. By the directedness of cosets, there are $b_j \in \bar{b}_j$ $(j = 1, 2)$ such that $b_{j1}, b_{j2} \leqq b_j$. We apply the interpolation property to the pairs a_1, a_2 and b_1, b_2 to conclude the existence of a $c \in G$ such that $a_i \leqq c \leqq b_j$ for all i, j. Then the coset $\bar{c}$ containing c satisfies $\bar{a}_i \leqq \bar{c} \leqq \bar{b}_j$ for all i, j.

PROPOSITION 5.4. *The intersection of a finite number of o-ideals is also one.*

We prove that $A \cap B$ is an o-ideal if so are A and B. We need only show that $A \cap B$ is directed. Let $x, y \in A \cap B$. For some $a \in A$ and $b \in B$ we have $x, y \leqq a$ and $x, y \leqq b$. Let c lie between the pairs x, y and a, b. By convexity, $c \in A$ and $c \in B$, and so c is an upper bound for x, y in $A \cap B$.

For infinite intersections the last proposition fails in general. For instance, let H_n $(n = 1, 2, \dots)$ be the fully ordered group of rationals and H_∞ the same group with the trivial order. Let G be the lexicographic product of the groups $H_1, \dots, H_n, \dots, H_\infty$ (in this order). It is easy to check that G is a Riesz group in which the lexicographic product A_n of $H_n, \dots, H_\infty$ is an o-ideal for $n = 1, 2, \dots$. But the intersection of all these A_n is equal to H_∞ which is not an o-ideal.

PROPOSITION 5.5. *The product of a finite number of o-ideals is again an o-ideal. The subgroup generated by an arbitrary set of o ideals is likewise an o-ideal.*

We show that AB is an o-ideal if so are A and B. Since the elements of AB are of the form ab $(a \in A, b \in B)$ and A, B are directed, it is evident that AB is directed. To see the convexity of AB, assume $x \in G$ satisfies $e \leqq x \leqq ab$ for some $a \in A$, $b \in B$; in view of directedness a and b may be assumed to be positive. Theorem 2.2, (5) ensures the existence of $a', b' \in G$ satisfying $e \leqq a' \leqq a$, $e \leqq b' \leqq b$ and $x = a'b'$. Here certainly $a' \in A$, $b' \in B$, and consequently, $x \in AB$. Since the subgroup generated by a family of o ideals is the set union of the subgroups generated by finite subfamilies, the second statement is an immediate consequence of the first one.

The next result is a generalization of the corresponding statement on l-ideals in lattice-ordered groups.

THEOREM 5.6. *The o-ideals of a Riesz group G form a distributive sublattice in the lattice of all normal subgroups of G.*

By virtue of Propositions 5.4 5.5, we need only verify the distributivity

$$A \cap \{B, C\} = \{A \cap B, A \cap C\}$$

for o-ideals A, B, C of G. It suffices to establish the inclusion $\subseteq$. Let $a = bc$ $(b \in B, c \in C)$ belong to the left member; without loss of generality $e < a$ may be assumed. By directedness, B and C contain positive elements b_1, c_1 such that $b \leqq b_1$, $c \leqq c_1$. Applying (5) of Theorem 2.2 to $e \leqq a \leqq b_1c_1$ we infer that $a = b_2c_2$ for some b_2, c_2 with $e \leqq b_2 \leqq b_1$, $e \leqq c_2 \leqq c_1$. Since $b_2 \in B$, $c_2 \in C$ and $b_2 \leqq a$, $c_2 \leqq a$, it follows that $b_2 \in A \cap B$, $c_2 \in A \cap C$, and thus $a \in (A \cap B)(A \cap C)$, as desired.

It is to be observed that even the infinite distributive law

$$A \cap \{\dots, B_\alpha, \dots\} = \{\dots, A \cap B_\nu, \dots\}$$

holds true when B_α runs over an arbitrary set of o-ideals of a Riesz group. This is an immediate consequence of the finite distributive law and the fact that a subgroup generated by infinitely many subsets is the set union of subgroups generated by a finite number of subsets.

In Riesz groups we also have:

PROPOSITION 5.7. *If G is a Riesz group and A is an o-ideal of G, and if $\bar{x}$, $\bar{y}$, $\bar{z}$ are cosets of A such that*

$$\bar{x} \leqq \bar{y} \leqq \bar{z},$$

then to every $x \in \bar{x}$ and to every $z \leqq \bar{z}$ satisfying $x \leqq z$ there exists a $y \in \bar{y}$ such that

$$x \leqq y \leqq z.$$

Under the given hypotheses, there exist $y_1, y_2 \in \bar{y}$ such that $x \leqq y_1$, and $y_2 \leqq z$. The coset $\bar{y}$ being directed, some $y_3 \in \bar{y}$ satisfies $y_1 \leqq y_3$ and $y_2 \leqq y_3$. By the interpolation property, some $y \in G$ lies between x, y_2 and y_3, z. But $y_2 \leqq y \leqq y_3$ implies $y \in \bar{y}$, so y is a desired element.

The following result may be of some interest.

PROPOSITION 5.8. *If G is a Riesz group and if $A_1, \ldots, A_n$ are o-ideals of G such that G as an abstract group is the direct product of the A_i, then G as a partially ordered group is also the direct product of the partially ordered groups A_i.*

We need to prove that $a_1 \ldots a_n \geqq e$ $(a_i \in A_i)$ implies $a_i \geqq e$ for each i. There exists to each a_j positive $b_j \in A_j$ such that $b_j \geqq a_j$, and we replace the $a_j\,(j \neq i)$ by these b_j to get $b_1 \ldots b_{i-1}\, a_i\, b_{i+1} \ldots b_n \geqq e$. We also write $a_i = b_i c_i^{-1}$ with positive b_i, c_i in A_i. Since c_i commutes with $b_j\,(j \neq i)$, we have

$$b_1 \ldots b_{i-1}\, b_i\, b_{i+1} \ldots b_n \geqq c_i\,.$$

Here the $b_j\,(j \neq i)$ must be orthogonal to c_i, in view of § 4 (*e*). On account of § 4 (*a*) we infer

$$L(c_i) = L(b_1 \ldots b_{i-1}\, b_i\, b_{i+1} \ldots b_n, c_i) = L(b_i, c_i)$$

which means $b_i \geqq c_i$, that is, $a_i \geqq e$.

§ 6. Extensions of commutative Riesz groups.

The Schreier extension problem for partially ordered groups has been considered by the author [5]. The lattice-ordered case has been investigated by R. TELLER [14]; here we need the extensions for commutative Riesz groups, discussed recently by TELLER [15]. For completeness' sake we prove here his main result (Theorem 6.1).

Let G and Γ be partially ordered groups and $\mathfrak{G}$ an extension of Γ by G. This means that $\mathfrak{G}$ can be regarded as the set of pairs (a, α) with $a \in G$, $\alpha \in \Gamma$ such that

$$(a, \alpha) = (b, \beta) \text{ is equivalent to } a = b, \ \alpha = \beta,$$

and

$$(a, \alpha) \cdot (b, \beta) = (ab, \alpha\beta f(a, b)) .$$

Here f denotes a factor set, i.e. a function from the product set $G \times G$ into Γ satisfying

(i) $f(a, e) = \varepsilon$ for all $a \in G$ (ε denotes the neutral element of Γ);

(ii) $f(a, b) = f(b, a)$ for all $a, b \in G$;

(iii) $f(a, b) f(ab, c) = f(a, bc) f(b, c)$ for all $a, b, c \in G$.

The order relation in $\mathfrak{G}$ can be given in terms of the sets:

$$P_a = \text{the set of all } \alpha \in \Gamma \text{ such that } (a, \alpha) \geqq (e, \varepsilon).$$

These P_a satisfy:

(iv) P_a is not void if and only if $a \geqq e$;

(v) $P_e = \Gamma^+$;

(vi) $P_a P_b f(a, b) \subseteq P_{ab}$.

The equivalence of extensions may be formulated in the obvious way; we shall not need this notion.

We are interested in Riesz groups. For them we have [15]:

THEOREM 6.1. *Let G and Γ be commutative Riesz groups, and $\mathfrak{G}$ a commutative extension of Γ by G, corresponding to $f(a, b), P_a$. Then $\mathfrak{G}$ is again a Riesz group if and only if*

(a) *P_a is l-directed for each $a \in G^+$;*

(b) *$P_a P_b f(a, b) = P_{ab}$ for all $a, b \in G^+$.*

First assume that $\mathfrak{G}$ is a Riesz group. Let $\alpha, \beta \in P_a$ and choose some $\gamma \in \Gamma$ with $\gamma \leqq \alpha$, $\gamma \leqq \beta$. Then each of (e, ε), (a, γ) is $\leqq$ each of (a, α), (a, β); consequently, some (d, δ) can be inserted between them. Evidently, $d = a$, and so $\delta \in P_a$ satisfies $\delta \leqq \alpha$, $\delta \leqq \beta$. Hence (a) is a necessary condition. To every $\gamma \in P_{ab}$ we can certainly find elements α, β such that $\alpha \in P_a$, $\beta \in P_b$ and $\gamma \leq \alpha\beta f(a, b)$, because the sets P are upper classes. Then

$$(e, \varepsilon) \leqq (ab, \gamma) \leqq (a, \alpha)(b, \beta)$$

implies the existence of $\alpha' \in P_a$ and $\beta' \in P_b$ such that $\gamma = \alpha' \beta' f(a, b)$. Thus $P_{ab} \subseteq P_a P_b f(a, b)$, and by (vi), we get (b).

Conversely, let G and Γ be Riesz groups, and let f and P satisfy (a) and (b). In order to prove that $\mathfrak{G}$ is again a Riesz group, let

$$(1) \qquad (e, \varepsilon) \leqq (c, \gamma) \leqq (a, \alpha)(b, \beta)$$

with $\alpha \in P_a, \beta \in P_b$. We show that $(c, \gamma) = (a', \alpha') \cdot (b', \beta')$ for some

$$\alpha' \in P_{a'}, \beta' \in P_{b'} \text{ with } (a', \alpha') \leqq (a, \alpha), (b', \beta') \leqq (b, \beta).$$

By the Riesz property of G, $c = a_0 b_0$ for some $a_0, b_0 \in G$ with $e \leqq a_0 \leqq a$, $e \leqq b_0 \leqq b$. We also have $(e, \varepsilon) \leqq (a_0, \alpha_0) \leqq (a, \alpha)$ for some $\alpha_0 \in P_{a_0}$, since

$$\alpha \in P_a = P_{aa_0^{-1}} P_{a_0} f(aa_0^{-1}, a_0) = P_{aa_0^{-1}} f(a_0^{-1}, a_0) f(a, a_0^{-1})^{-1} P_{a_0},$$

and so $\alpha = (\alpha\alpha_0^{-1}) \alpha_0$ with $\alpha\alpha_0^{-1} \in P_{aa_0^{-1}} f(a_0^{-1}, a_0) \; f(a, a_0^{-1})^{-1}$ and $\alpha_0 \in P_{a_0}$.

Similarly, $(e, \varepsilon) \leqq (b_0, \beta_0) \leqq (b, \beta)$ for some β_0. Since $\gamma \in P_c = P_{a_0 b_0} = P_{a_0} P_{b_0} f(a_0, b_0)$ and P_{a_0}, P_{b_0} are l-directed, α_0, β_0 may be assumed to be chosen so as to satisfy $\alpha_0 \beta_0 f(a_0, b_0) \leqq \gamma$. Then dividing by (a_0, α_0) and (b_0, β_0), we have reduced (1) to the case $c = e$.

In the case $c = e$ we have $\alpha\beta f(a, b) \gamma^{-1} \in P_{ab} = P_a P_b f(a, b)$ and so $\alpha\beta f(a, b) \gamma^{-1} = \alpha_1 \beta_1 f(a, b)$ for some $\alpha_1 \in P_a, \beta_1 \in P_b$. If $\alpha_2 \in P_a, \beta_2 \in P_b$ are chosen so as to satisfy $\alpha_2 \leqq \alpha, \alpha_2 \leqq \alpha_1, \beta_2 \leqq \beta, \beta_2 \leqq \beta_1$, then we have $\alpha_2 \beta_2 \leqq \alpha_1 \beta_1 \leqq \alpha\beta$. Thus the Riesz property of Γ implies that $\alpha_1 \beta_1 = \alpha_3 \beta_3$ with $\alpha_2 \leqq \alpha_3 \leqq \alpha, \beta_2 \leqq \beta_3 \leqq \beta$, and so $\alpha_3 \in P_a, \beta_3 \in P_b$. On putting $\alpha' = \alpha\alpha_3^{-1}$, $\beta' = \beta\beta_3^{-1}$, we have $(e, \alpha')(e, \beta') = (e, \gamma)$ where $(e, \varepsilon) \leqq (e, \alpha') \leqq (a, \alpha), (e, \varepsilon) \leqq (e, \beta') \leqq (b, \beta)$. This completes the proof.

Remarks. 1. If Γ is a fully ordered group, then condition (a) may be omitted.

2. If the group Γ is assumed only to have the interpolation property (and not to be directed), then $\mathfrak{G}$ will certainly be a Riesz group provided it is directed.

3. The Riesz group $\mathfrak{G}$ has the property that if $(a_i, \alpha_i) \leqq (b_j, \beta_j)$ for $i = 1, \dots, m; j = 1, \dots, n$, and if $c \in G$ satisfies $a_i \leqq c \leqq b_j$ for all i, j, then to this c we can find a $\gamma \in \Gamma$ satisfying

$$(a_i, \alpha_i) \leqq (c, \gamma) \leqq (b_j, \beta_j) \qquad \text{for all } i, j.$$

It suffices to show that we can find a $\gamma_0 \in \Gamma$ such that $(a_i, \alpha_i) \leqq (c, \gamma_0)$, for then setting $(b_{n+1}, \beta_{n+1}) = (c, \gamma_0)$ and doing the same with the a's, the Riesz property of $\mathfrak{G}$ settles the question. Since

$$\bigcap_{i=1}^{m} P_{ca_i^{-1}} \alpha_i f(a_i, a_i^{-1}) f(c, a_i^{-1})^{-1}$$

is not void in view of the upper class property of the P's, any γ_0 in the intersection will do.

4. By condition (b), it is sufficient to know the sets P_a for generators a of G^+. If G^+ is a free semigroup, and $a_i \in G^+$ are free generators, then prescribing P_{a_i} arbitrarily on the free generators so that they are upper classes and satisfy (a), P_c for every $c \in G^+$ can uniquely be determined on the basis of (b). They will obviously satisfy (b) and — as easily seen — also (a).

§ 7. Antilattices.

The concept of antilattices seems to be of fundamental importance in the theory of Riesz groups. They play essentially the same role as the fully ordered groups do in the theory of lattice-ordered groups. They are, roughly speaking, Riesz groups without proper union or intersection of elements.

A Riesz group H is said to be an *antilattice* if it satisfies the following condition:

(*) if $a \wedge b$ exists in H, then either $a \wedge b = a$ or $a \wedge b = b$.

This condition is equivalent to saying that the elements of an antilattice are meet-irreducible. Considering that $a \vee b = (a^{-1} \wedge b^{-1})^{-1}$, (*) implies that if $a \vee b$ exists in H, then either $a \vee b = a$ or $a \vee b = b$.

Every fully ordered group is clearly an antilattice. On the other hand, *if an antilattice is lattice-ordered, then it is a fully ordered group.* Examples for antilattices are the groups in Examples 1, 5, 6, 8 of § 3.

LEMMA 7.1. *For a Riesz group G, the following conditions are equivalent:*

(a) *G is an antilattice;*

(b) *if $a \wedge b = e$ for a, $b \in G$, then $a = e$ or $b = e$;*

(c) *P^* is l-directed.*

The proof is straightforward and may be left to the reader.

Note that if an antilattice H has an atom a ($> e$), then every positive element $> e$ must be $\geqq a$.

In general, the existence of the intersection $a \wedge b \wedge c$ does not imply that of $a \wedge b$. Therefore it is desirable to know whether or not a finite number of elements may have an intersection in antilattices.

PROPOSITION 7.2. *If in an antilattice $a_1 \wedge \ldots \wedge a_n = b$ exists, then $a_i = b$ for some i.*

This being true for $n = 1$, assume $n \geqq 2$. If $a_1 \wedge \ldots \wedge a_{n-1}$ exists, then on account of (b), either $a_1 \wedge \ldots \wedge a_{n-1} = b$ or $a_n = b$, and the assertion follows by induction on n. If $a_1 \wedge \dots \wedge a_{n-1}$ does not exist, then $L(a_1, \dots, a_{n-1})$ being u-directed, there exists a $c \in L(a_1, \dots, a_{n-1})$ such that $c > b$. We claim $c \wedge a_n = b$. Indeed, if $b' \leq c$ and $b' \leq a_n$, then $b' \leq a_i$ for every i whence $b' \leq b$. But then $c \wedge a_n = b$ implies $a_n = b$, as we wished to show.

We are going to introduce a topology in dense Riesz groups, the so-called open-interval topology. As a subbase of open sets in G we take the subsets

$$G, \ U^*(a), \ L^*(a) \quad \text{for each} \quad a \in G.$$

Then this is a HAUSDORFF topology if and only if there exists no $c \in G$ ($c \neq e$) such that $P^* c = P^*$ (i. e. no pseudo-identities exist in the sense of § 8), and in that case G is a topological group in the open-interval topology (8).

PROPOSITION 7.3. *A dense Riesz group without pseudo-identities is an antilattice if and only if it is not discrete in its open-interval topology.*

If the Riesz group H is not an antilattice and if $a \wedge b = e$ with $a \neq e \neq b$, then the open intervals $L^*(a)$, $L^*(b)$, $U^*(a^{-1})$, $U^*(b^{-1})$ have the intersection e, and so H is discrete in the open-interval topology. (The same conclusion holds when H is an antilattice, but is not dense, for then there is an atom in H.) Conversely, let H be a dense antilattice, without pseudo-identities. It is to be shown that for a_i, $b_j \in H$, the intersection

$$U^*(a_1, \dots, a_m) \cap L^*(b_1, \dots, b_n)$$

is either void or contains a closed interval $[c, d]$, $c < d$. Let x belong to the intersection. Since $U^*(a_1, \dots, a_m)$ is l-directed and contains no minimal element (by Proposition 7.2), there is a $c \in U^*(a_1, \dots, a_m)$ with $c < x$. Similarly we can find a $d \in L^*(b_1, \dots, b_n)$ with $d > x$. Then $[c, d]$ is an interval in the intersection. Q. e. d.

It should be observed that the proof shows that the open intervals (c, d) form a base for the open interval topology in the case covered by the last result.

Let H be a commutative torsion-free antilattice whose order is isolated. By Proposition 2.5, H can be embedded in a divisible commutative Riesz group D which has isolated order again. Here we point out that D will again be an antilattice, since the orthogonality of a and b would imply the orthogonality of a^n and b^m ($\in H$). Clearly, D is a dense antilattice.

The next result is worthwhile mentioning.

---

(8) Cf. [7], p. 32. For the results on topological groups see [11].

PROPOSITION 7.4. *Let* $\mathfrak{G}$ *be a commutative Riesz group that is an extension of a dense antilattice* Γ *by a dense antilattice* G, *and let* $\mathfrak{G}$ *be described in terms of* $f(a, b)$ *and* P_a *as in* § 6. $\mathfrak{G}$ *is again an antilattice exactly if the sets* $P_a\,(a > e)$ *are open in the open-interval topology of* Γ.

Since no intersections exist in Γ and in G, two elements of $\mathfrak{G}$ can be orthogonal only if they are of the form (a, α) with $a > e$ and (e, β). But $(a, \alpha) \wedge (e, \beta) = (e, \varepsilon)$ is equivalent to

$$\alpha^{-1} P_a \cap \beta^{-1} P_e = P_e$$

which can be written in the form

$$P_a \cap \gamma P_e = \alpha P_e \qquad \text{with} \qquad \gamma = \alpha\beta^{-1} .$$

This amounts to saying that $P_a \cap \gamma P_e$ has a minimal element α, since — as readily seen from the interpolation property — $P_a \cap \gamma P_e$ is again l-directed. We see that $\mathfrak{G}$ is an antilattice exactly if $P_a \cap \gamma P_e$ has no minimal element unless $\gamma \in P_a$. If $P_a \cap \gamma P_e$ has no minimal element, then for each $\alpha \in P_a \cap \gamma P_e$, some $\beta \in P_a \cap \gamma P_e$ exists with $\alpha \in U^*(\beta) \subset P_a'$, and P_a is open. Conversely, if P_a is open and $\alpha \in P_a \cap \gamma P_e$, then in case $\gamma \notin P_a$ we have $P_a \cap \gamma P_e = P_a \cap U^*(\gamma)$, thus α lies in the open set $P_a \cap U^*(\gamma)$ and hence it cannot be minimal there.

Finally, let us note the simple observation:

PROPOSITION 7.5. *If an antilattice is connected in the open-interval topology, then it has no o-ideals except the trivial ones.*

If the antilattice A satisfies the hypothesis, then every neighborhood of e generates A as a group. If given a $u > e$ in A, then the open interval (u^{-1}, u) generates A, i. e. to each $a \in A$ there exists an integer n such that $u^{-n} < a < u^n$. Hence the convex subgroup generated by u coincides with A.

§ 8. Pseudo-identities and pseudo-positive elements.

Let us turn our attention to elements which are exceptional in Riesz groups in the sense that such elements do not exist in the lattice-ordered case.

Let G be a Riesz group. An element $c \in G$, distinct from e, is called a *pseudo identity* if

$$cP^* = P^*$$

and is said to be *pseudo positive* if $c \notin P$ and

$$cP^* \subseteq P^*$$

where as always P^* denotes the set of all elements greater than e in G.

Since cP^* never contains c, every pseudo identity is pseudo-positive.

LEMMA 8.1. *If in G both c and c^{-1} are pseudo-positive, then c is a pseudo-identity.*

Hypothesis implies the inclusions

$$cP^* \subseteq P^* \text{ and } c^{-1}P^* \subseteq P^*.$$

The second one means $P^* \subseteq cP^*$ whence $cP^* = P^*$.

A lattice-ordered group G contains no pseudo-positive elements. For, if $c \in G$ were such, then $c \vee e$ would satisfy: $c < x$ implies $c \vee e \leqq x$. Hence c would be meet irreducible, and so G fully ordered. That in fully ordered groups no pseudo-positive elements may occur is quite obvious.

In Example 3 of § 8, the complex numbers $iy \neq 0$ are pseudo-identities. In Example 1 the numbers $x > 0$ and iy with $y > 0$ are pseudo-positive, and so are the functions f in Examples 6, 8 which are positive everywhere except for one place.

LEMMA 8.2. *In a Riesz group G the pseudo-identities form, together with e, a convex normal subgroup C. If $C \neq e$, then the factor group G/C is a dense antilattice without pseudo-identities.*

If C denotes the set consisting of e and the pseudo-identities of G, then $c, d \in C$ implies $(cd)\, P^* = cP^* = P^*$, i.e. $cd \in C$. Clearly, $c^{-1} \in C$ and $x^{-1}\, cx \in C$ for each $x \in G$. Thus C is a normal subgroup which is trivially ordered, and so convex. Since $a < b$ implies $ac < b$ and $a < bc$ for $c \in C$, we see that if $\bar{a} < \bar{b}$ for cosets $\bar{a}, \bar{b}$ mod C, then for all $a \in \bar{a}$, $b \in \bar{b}$ one has $a < b$. Hence the Riesz property of G/C is immediate.

Now if $e < a$ and if $c \in C$, $c \neq e$, then there exists some x between the pairs e, c and a, ac, and so $e < x < a$ shows that G is dense whence G/C is dense too. If for $\bar{a}, \bar{b} \in G/C$ the element $\bar{a} \wedge \bar{b}$ exists and differs from $\bar{a}, \bar{b}$, then for $a \in \bar{a}$, $b \in \bar{b}$, $d \in \bar{a} \wedge \bar{b}$ and $e \neq c \in C$ we have some $x \in G$ between d, dc and a, b. Now $\bar{d} < \bar{x} < \bar{a}$ and $\bar{x} < \bar{b}$ which is impossible. Hence G and G/C are dense antilattices. Since each element in a coset mod C which is a pseudo-identity in G/C is a pseudo-identity in G, the proof is completed.

By making use of this lemma, we can describe the Riesz groups with pseudo-identities.

THEOREM 8.3. *If the Riesz group G contains a pseudo-identity, then it is a dense antilattice. G contains a trivially ordered convex normal subgroup C such that the factor group G/C is a dense antilattice without pseudo-identities, and $g \in G$ is positive if and only if either $g = e$ or the coset $\bar{g}$ of g mod C is* *greater than $\bar{e}$ in G/C. Conversely, every group G which arises in this way from C and G/C is a dense antilattice.*

The statement concerning the positivity of g needs no verification. Assume that G is a partially ordered group that arises in the described way. Then strict inequalities between cosets mod C are equivalent to strict inequalities between arbitrary representatives of the cosets whence everything is clear.

Since C can be an arbitrary group, it follows that a Riesz group, moreover an antilattice need not be torsion-free.

We shall make use of the following characterization of pseudo-identities.

PROPOSITION 8.4. *Let G be a partially ordered group with isolated order. $c \in G$, $c \neq e$, is a pseudo-identity of G if and only if together with D, also $\{D, c\}$ is a trivially ordered subgroup of G.*

Since pseudo-identities c have the property that it is allowed to multiply by them one member of strict inequalities, it is clear that if D is trivially ordered, then so is $\{D, c\}$. Conversely, let $c \neq e$ have the indicated property, and let $p > e$. Then $d = cp$ cannot be incomparable with e, for otherwise $D = \{d\}$ would be a trivially ordered subgroup of G, but $\{D, c\}$ would not share this property, since $p \in \{D, c\}$. Hence either $cp < e$ or $cp > e$. The first case is by $c < cp < e$ excluded, thus $cp > e$ for all $p \in P^*$. We conclude that c is pseudo-positive. Similarly c^{-1} is pseudo-positive, and Lemma 8.1 implies that c is a pseudo-identity.

Concerning pseudo-positive elements we do not have much to say. The product of two pseudo-positive elements may be positive; but trivially, the set of all positive and pseudo-positive elements is a subsemigroup Q of G which is clearly normal and convex. If Q does not contain elements $\neq e$ along with their inverses, then Q defines a partial order of G in which G may again possess pseudo-positive elements. By means of Example 6 in § 3 it may be shown that Q need not define a Riesz order on G even if the order of G we started with has been one.

Finally, let us mention a method of constructing pseudo-positive elements. Let G be an arbitrary partially ordered group and N a non-trivial normal convex subgroup of G. If P is the positivity domain of G, then delete from P the elements of N with the exception of e. It is straightforward to check that with this smaller positivity domain P', G will in fact be a partially ordered group in which the elements of P not in P' are pseudo-positive. It is also easy to see that if G is a Riesz group, N an o-ideal of G such that G/N is a dense antilattice, then P' will make G into a Riesz group again.

§ 9. The structure of commutative antilattices.

In the special case of commutative antilattices it is possible to give more information about the structure, whenever the order is isolated. In view of Proposition 2.5 and a remark in § 7, we shall confine ourselves to divisible antilattices.

Let H be a divisible commutative Riesz group with isolated order; thus H is torsion-free. Among the subgroups of H which are trivially ordered there exist maximal ones; let C denote one of these. Then C is convex in H. Now H/C does not contain any trivially ordered subgroup $\neq e$, for the elements of H belonging to these cosets would form a trivially ordered subgroup of H which properly contains C. The order of $\overline{H/C}$ is again isolated, for if the coset $\bar{a}$ mod C satisfies $\bar{a}^n \geqq \bar{e}$ for some positive integer n, then for a representative $a \in \bar{a}$ we have $a^n \geqq c$ for some $c \in C$. Now C being obviously divisible, $c_0^n = c$ for some $c_0 \in C$. Hence $a^n \geqq c_0^n$, and thus $a \geqq c_0$, because H has isolated order. Therefore $\bar{a} \geqq \bar{e}$ and the order of H/C is in fact isolated. But then H/C must be fully ordered, since if $\bar{a} \in H/C$ were incomparable with $\bar{e}$ then the powers of $\bar{a}$ would form a trivially ordered subgroup of H/C. We conclude that H is an extension of a trivially ordered group C by a fully ordered group H/C.

Since, because of divisibility, H as an abstract group is isomorphic to the direct product of C and H/C, the representatives of the cosets can be chosen so that the factor set $f(a, b)$ collapses to the identity. By making use of Theorem 6.1 and Remark 2 in § 6, we see that if in the direct product of C and H/C the sets $P_a \subseteq C$ for positive $a \in H/C$ are subject to conditions of Theorem 6.1 and care is taken that a directed group will arise, then the arising group $H = C \times H/C$ will be a Riesz group, moreover an antilattice, if orthogonal elements $\neq e$ do not exist. Since C is trivially ordered and H/C need not be representable within H by a fully ordered subgroup, Theorem 6.1 is not applicable to the direct product. Hence we are led to:

THEOREM 9.1. *Let H be a divisible commutative Riesz group with isolated order. Then H as an abstract group is isomorphic to a direct product*:

$$H \cong G \times \Gamma$$

where G is fully ordered and Γ is trivially ordered. If H is considered as an extension of Γ by G with factor set $f(a, b) = \varepsilon$ for all $a\ b \in G$, then the partially ordered group $H^* \cong G \times \Gamma$ where P_a satisfy

(i) *P_a is not void if and only if $a \in G^+$,*

(ii) $P_e = \Gamma^+ = \varepsilon$,

(iii) $P_a = \Gamma$ *if* $a > e$,
has the property that the canonical map of H *into* H^* *is an o-monomorphism. Every group* H^* *which arises from a divisible fully ordered group* G *and from a trivially ordered group* Γ *in this way is a Riesz group.*

The case when H is o simple is worthwhile mentioning. Then the o-ideal generated by a positive element $a \neq e$ coincides with H. This means that to given $a, b > e$ in H, we can find a positive integer n such that $a^n > b$. Hence H/C is archimedean, and so it is o-isomorphic to a subgroup of the real numbers.

COROLLARY 9.2. *If the group* H *of the preceding theorem is o-simple, then* G *is o-isomorphic to a divisible subgroup of the real numbers. Conversely, if* G *is such a group, then* H^* *is o-simple.*

In Example 1 of § 3 we may take C to consists of all (a, b) with $b = 0$, and in Example 6 C can be chosen as the set of all polynomials vanishing at a fixed ξ in $[0,1]$.

There is another approach of getting information about commutative antilattices. This is a representation by means of fully ordered groups which will next be considered. Now the absence of pseudo identities must be assumed which is, by virtue of Theorem 8.3, not too restrictive a hypothesis.

Let H be a divisible commutative antilattice with isolated order having no pseudo-identities. We let C_ϱ run over all subgroups of H which are maximal with respect to the property of excluding some positive element $\neq e$ of H. We claim that the intersection of all these C_ϱ is just e. By way of contradiction, suppose that some $x \neq e$ belongs to each C_ϱ. This x possesses the property that if the subgroup A is trivially ordered, then so is $\{A, x\}$. Thus x would be a pseudo identity of H (cf. Proposition 8.4). Therefore $\cap C_\varrho = e$, indeed, and consequently, H is isomorphic to a subdirect product of the partially ordered groups H/C_ϱ. We have shown above that H/C_ϱ are fully ordered, so an o-monomorphism of H into a subdirect product of fully ordered groups H/C_ϱ arises.

This representation has the additional property that if $a < b$ in H, then we have $a_\varrho < b_\varrho$ for the components a_ϱ of a and b_ϱ of b in each H/C_ϱ. This follows from the fact that ba^{-1} never belongs to a C_ϱ. Thus we arrive at the following result:

THEOREM 9.3. *Let* H *be a divisible commutative antilattice whose order is isolated and which contains no pseudo-identities. Then* H *is o-isomorphic to a mild subdirect product of fully ordered groups.*

It is to be shown that the canonical map φ of H into the mild cartesian product of the H/C_ϱ carries only positive elements into positive elements

$\neq e$. If $\varphi(h) > e$ for some $h \in H$, then no C_ϱ may include h. Thus h is not incomparable with e, and since $h < e$ is absurd, it follows that $h > e$, indeed.

Note that a mild cartesian product of divisible fully ordered groups is necessarily an antilattice. Also, in Theorem 9.3 « antilattice » can be replaced by « Riesz group », but then the o-isomorphism does not preserve meets and unions.

In case the additional assumption is made that H is o-simple, we get from Corollary 9.2 and Theorem 9.3 :

COROLLARY 9.4. *Let H be as in Theorem 9.3 and assume H is o-simple. Then H is o-isomorphic to a subgroup of real-valued functions on some set Ξ where a function f is > 0 if and only if $f(\xi) > 0$ for all $\xi \in \Xi$.*

§ 10. Representation of commutative Riesz groups.

We wish to get a subdirect product representation of commutative Riesz groups such that it preserves not only group operations and order relations, but unions and intersections as well whenever these happen to exist. Since the class of Riesz groups is not equationally definable, there is nothing to guarantee the a priori existence of such a representation.

We begin with considering the subdirectly irreducible Riesz groups (9).

THEOREM 10.1. *A commutative Riesz group is subdirectly irreducible if and only if it is an antilattice.*

If the commutative Riesz group G is subdirectly reducible, then there exist non-trivial o-ideals A and B such that $A \cap B = e$. Then every positive element of A is orthogonal to every positive element in B whence G cannot be an antilattice. Thus an antilattice is subdirectly irreducible. Conversely, assume that G is not an antilattice. Then we can find elements a, b in G which are $> e$ and satisfy $a \wedge b = e$. The set of all positive $x \in G$ orthogonal to b generates an o-ideal A of G containing a, and the set of all positive $y \in G$ orthogonal to every x generates an o-ideal B containing b. We have $A \cap B = e$, since every positive element in the intersection is orthogonal to itself, and the intersection must again be an o-ideal on account of Proposition 5.4. We claim that the canonical map $\varphi:\ G \longrightarrow G/A \times G/B$ yields a subdirect product representation of G. Clearly, φ is bijective and order, union, intersection preserving. It remains to verify that $\varphi(g) \geqq e$

(9) By subdirect irreducibility we mean that the group is not properly representable as the subdirect product of *two* (or a finite number of) factor groups; for the sake of convenience we assume that the kernels are o-ideals. This is not essential for what follows. (If we omit the last hypothesis, then the « if » part of Theorem 10.1 should be cancelled.)

implies $g \geqq e$. Now $\varphi(g) \geqq e$ means that the coset of g both mod A and mod B contains positive elements, say $ga \geqq e$ and $gb \geqq e$ for $a \in A$, $b \in B$. By directedness, we may assume, without loss of generality, that $a \geqq e$ and $b \geqq e$. Since $a \wedge b = e$ exists, so does $ga \wedge gb = g(a \wedge b) = g$. Thus $ga, gb \geqq e$ implies $g \geqq e$, and this completes the proof.

Now we have come to the problem of getting an adequate subdirect product representation for commntative Riesz groups, namely one which gives the Lorenzen representation in the special case of lattice-ordered groups. In establishing the existence of such a representation, a slight modification of the proof, usually given for equationally definable class of algebras, is necessary.

The main result reads as follows.

THEOREM 10.2. *Let G be a commutative Riesz group. There exists a family H_λ $(\lambda \in \Lambda)$ of antilattices and an o-isomorphism φ of G onto a subdirect product of the H_λ such that φ preserves unions and intersections.*

Let g range over all elements of G which are not $\leqq e$. For each such g take an o-ideal $A(g)$ of G which is maximal with respect to the property of not intersecting $U(g)$: since $e \notin U(g)$, $A(g)$ does exist. We claim that $G/A(g)$ is an antilattice. It suffices to show, on account of Theorem 10.1, that if B, C are o ideals properly containing $A(g)$, then $B \cap C$ has the same property. Let $b \in B \cap U(g)$ and $c \in C \cap U(g)$; evidently, b and c may be chosen to be positive. By the interpolation property, we can intercalate between e, g and b, c some $a \in G$, and this a is clearly contained in each of B, C and $U(g)$, i.e. $B \cap C$ intersects $U(g)$. Now the intersection of all the $A(g)$ collapses to e, since if $h \neq e$ then either $A(h)$ or $A(h^{-1})$ exists and excludes h. Therefore, if we choose for the H_λ the antilattices $G/A(g)$, then the natural map φ of G into the cartesian product of the H_λ is an o-monomorphism preserving unions and intersections. What we still have to verify is that φ^{-1} is order-preserving too, or in other words, that $\varphi(g) \geqq e$ implies $g \geqq e$. But if $\varphi(g) \geqq e$ then g cannot be incomparable with e, for if it were so then in $G/A(g^{-1})$ the coset of g would contain, by the meaning of $\varphi(g) \geqq e$, a positive element, say $ga \geqq e$, $a \in A(g^{-1})$, in contradiction to the fact that $U(g^{-1})$ does not meet $A(g^{-1})$. This completes the proof of the theorem.

Observe that if G happens to be lattice-ordered, then Thorem 10.2 is equivalent to the Lorenzen representation theorem. For, in that case the groups H_λ — as union and intersection preserving images of G — must be lattice-ordered, and so they are fully ordered, indeed.

§ 11. Irredundant representations.

In Theorem 10.2 it has been shown that commutative Riesz groups can be represented as subdirect products of antilattices, preserving unions and intersections if exist. We are naturally interested in getting conditions under which the mentioned representations are shortest in the sense that they don't have any superfluous components and certain uniqueness statement can be established.

An o-isomorphism φ of a partially ordered group G into the cartesian product C of antilattices G_λ will be called a *representation* of G, if the kernels of the projections $G \longrightarrow G_\lambda$ are o-ideals of G. From Theorem 10.2 we know that a commutative Riesz group always has a representation.

Let ψ_λ denote the projection of C onto the cartesian product of the G_μ with $\mu \neq \lambda$. If for some λ, the composite map $\varphi\psi_\lambda$ is still an isomorphism of the abstract group G, then we call the component G_λ *superfluous*. If $\varphi\psi_\lambda$ is no longer an isomorphism of G, then G_λ is said to be an *essential* component. Obviously, G_λ is essential exactly if

$$(1) \qquad \bigcap_{\mu \neq \lambda} \operatorname{Ker} \varphi\varphi_\mu \neq e$$

holds true where φ_μ denotes the projection of C on G_μ. Clearly, there is nothing to prevent us from identifying G with a subgroup of C under φ.

If one tries to carry over the representation theory of commutative lattice-ordered groups, developed by JAFFARD, RIBENBOIM and CONRAD, to the case of Riesz groups, then an unsurmountable difficulty arises: the intersection of infinitely many o-ideals need not be an o-ideal again. In order to overcome this difficulty, one has the choice either to make restrictions on the representations to be considered or to assume that we are dealing with Riesz groups where any intersection of o-ideals is again an o-ideal. Since the second alternative seems to be the simpler and since this includes the most important examples, we are going to discuss Riesz groups with the mentioned property. For the sake of brevity, we shall call them *strong* Riesz groups; thus a Riesz group is *strong* if and only if its ω-ideals are o-ideals.

LEMMA 11.1. *If G_λ is an essential component of a strong Riesz group G, then for some $a \in G^+$ with $\varphi_\lambda(a) \neq e$, the carrier $a^\wedge$ is minimal in the partially ordered set $\mathfrak{C}$ of carriers of G.*

Now the left member of (1) is an o-ideal of G, since the kernels of φ_μ are such. If G_λ is an essential component in the representation of G, then

(1) holds, and hence the left member contains an $a \in G^+$, $a \neq e$. This a has only one component $\neq e$ in the representation, namely $\varphi_\lambda(a)$. The carrier $a^\wedge$ must be minimal in $\mathfrak{C}$, for if $b^\wedge \leqq a^\wedge$ and $b^\wedge \neq e^\wedge$, then we have also $\varphi_\lambda(b) \neq e$, for otherwise $a \wedge b = e$ would hold whence $b^\wedge = a^\wedge \wedge b^\wedge = e^\wedge$, a contradiction. If $b \wedge x = e$ for some $x \in G^+$, then necessarily $\varphi_\lambda(x) = e$, and so $a \wedge x = e$. This shows that $a^\wedge \leqq b^\wedge$, and $b^\wedge = a^\wedge$.

A representation in which every component is essential is called *irredundant.* We have as a main result:

THEOREM 11.2. *A commutative strong Riesz group G admits an irredundant representation if and only if it satisfies:*

(i) *the partially ordered set $\mathfrak{C}$ of its carriers is atomic,*

(ii) *if c_μ $(\mu \in M)$ is a set of positive elements in G such that to each $a \in P^*$ there is a $b \in P^*$ with $b \leqq a$ and $b \perp c_\mu$ for some μ, then $x \leqq c_\mu$ for all μ implies $x \leqq e$.*

Any two irredundant representations of G are o-isomorphic.

Assume that G has an irredundant representation with components G_λ $(\lambda \in \Lambda)$ which are antilattices, and let φ_λ denote the projection $G \longrightarrow G_\lambda$. Let $b^\wedge$ be an arbitrary carrier of G. not equal to $e^\wedge$, and let $b \in b^\wedge$. There is an index λ such that $\varphi_\lambda(b) > e$. The component G_λ being essential, some $a \in G^+$ satisfies $\varphi_\lambda(a) > e$ such that $a^\wedge$ is a minimal carrier (cf. Lemma 11.1). Since G_λ is an antilattice, $\varphi_\lambda(a)$ and $\varphi_\lambda(b)$ do not have an intersection, and so $a \wedge b$ fails to exist in G, But then some $c \in G^+$, $c \neq e$, satisfies $c \leqq a$ and $c \leqq b$. Therefore $c^\wedge \leqq a^\wedge$, $c^\wedge \leqq b^\wedge$, and hence the minimality of $a^\wedge$ implies $c^\wedge = a^\wedge$, i.e. $\mathfrak{C}$ is atomic.

If c_μ is a set as described in (ii) and if $x \leqq c_\mu$ for all μ, then let $a \in G^+$ such that its λ-th component is $> e$ and all its other components equal e. If c_μ is orthogonal to a, then the λ-th component of c_μ must be e. Thus $x \leqq c_\mu$ for every μ only if the components of x are $\leqq e$, and so $x \leqq e$.

Conversely, if $\mathfrak{C}$ is atomic, then let us consider the set of atoms $a_\lambda^\wedge$ (indexed by a certain set Λ) in $\mathfrak{C}$. The set of all elements of G^+ orthogonal to the elements of a fixed $a_\lambda^\wedge$ is the positivity domain of an o-ideal I_λ of G (Proposition 5.1 and § 4, (f)). This I_λ clearly contains all the elements of G contained in the carriers $a_\mu^\wedge$ with $\mu \neq \lambda$, but none in $a_\lambda^\wedge$. We claim that $G/I_\lambda = H_\lambda$ is an antilattice. If b, c are positive elements of G such that $bI_\lambda \wedge cI_\lambda = I_\lambda$ in H_λ and neither $bI_\lambda = I_\lambda$, nor $cI_\lambda = I_\lambda$, then neither b nor c is orthogonal to any $a_\lambda (\in a_\lambda^\wedge)$. Hence some $a'_\lambda \in a_\lambda^\wedge$ satisfies $a'_\lambda \leqq b$ and $a'_\lambda \leqq c$, and thus $a'_\lambda I_\lambda \leqq bI_\lambda$, $a'_\lambda I_\lambda \leqq cI_\lambda$, in contradiction to $bI_\lambda \wedge cI_\lambda = I_\lambda$. Now the intersection of all I_λ does not contain any positive element $\neq e$, for such an element must be orthogonal to each a_λ: consequently, $\bigcap I_\lambda = e$, since it is by hypothesis an o ideal. But the intersection of the I_μ with $\mu \neq \lambda$ is distinct from e for each fixed λ, because it contains the elements in $a_\lambda^\wedge$. The last two sentences show that G has an o-monomorphism φ into the cartesian product of the antilattices H_λ, φ is induced by the natural o-homomorphisms of G onto H_λ.

It remains to show that φ maps only positive elements upon positive elements. $x \in G$ is mapped by φ upon a positive element in the cartesian product of the H_λ if and only if the coset xI_λ contains a positive element, for each λ, that is, $xc_\lambda \geqq e$ for some $c_\lambda \in I_\lambda$ which may evidently be assumed to be positive. Now we have a set of positive elements $c_\lambda (\lambda \in \Lambda)$ which has the property described in (ii) because of the atomicity of $\mathfrak{C}$ and the definition of I_λ. Hence by (ii) $x^{-1} \leqq c_\lambda$ implies $x^{-1} \leqq e$, or $x \geqq e$.

Finally, to show uniqueness up to o-isomorphy, suppose that G has an irredundant representation by means of the antilattices G_μ, and let φ_μ denote the projection of G onto G_μ. By our hypothesis on G, the intersection $\bigcap_{\nu \neq \mu} \operatorname{Ker} \varphi_\nu$ is an o-ideal $\neq e$, hence it contains some positive b_μ; this b_μ has e for its νth component, $\nu \neq \mu$. Obviously, $b_\mu^\wedge$ is an atom in $\mathfrak{C}$, G_μ being an antilattice. Thus each component G_μ determines uniquely an atom in $\mathfrak{C}$. If $a^\wedge$ is an atom in $\mathfrak{C}$ and if $a \in a^\wedge$, then a cannot have two components $> e$, for if the μth and νth components of a were $> e$, then both $b_\mu^\wedge \leqq a^\wedge$ and $b_\nu^\wedge \leqq a^\wedge$, which would contradict the atomic character of $a^\wedge$. Hence each atom determines a component G_μ, and so atoms of $\mathfrak{C}$ and components G_μ of an irredundant representation are in a one-to-one correspondence. Moreover Ker φ_μ must be the subgroup generated by the positive elements of G orthogonal to b_μ, i.e. Ker $\varphi_\mu = I_\mu$ as defined above. Hence $G_\mu \cong_o G/I_\mu$ and G_μ is determined uniquely up to o-isomorphism.

The group in Example 7 of § 3 has a representation, but in no representation is it possible to find an essential component. Condition (ii) is always satisfied in the lattice-ordered case.

§ 12. The Conrad radical.

We can associate with each directed group G an ω-ideal of G which is reminiscent of the radical and has been discovered by P. CONRAD in the lattice ordered case. The discussion to be given here differs from Conrad's in that we introduce the radical as the union of certain elements of G, and then we show that it is the intersection of certain o-ideals.

Throughout this section let G denote a *directed group with isolated order*.

With a finite set $x_1, \dots, x_m \in G$ we associate the subset

$$(x_1, \dots, x_m)^\# = L(U(x_1, \dots, x_m)).$$

We have obviously the rules:

(a) $x_i \in (x_1, \dots, x_m)^{\#}$ for each i,

(b) if $x_i \in (y_1, \dots, y_m)^{\#}$ for $i = 1, \dots, n$, then $(x_1, \dots, x_n)^{\#} \subseteq (y_1, \dots, y_m)^{\#}$,

(c) $x_i \leqq y_j$ for $i = 1, \dots, m$; $j = 1, \dots, n$ implies $(x_1, \dots, x_m)^{\#} \subseteq (y_1, \dots, y_n)^{\#}$,

(d) $(ax_1 b, \dots, ax_m b)^{\#} = a(x_1, \dots, x_m)^{\#} b$,

(e) $e \in (a, a^{-1})^{\#}$ for each $a \in G$.

In order to verify (e), let $x \in U(a, a^{-1})$, that is, $x \geqq a$ and $x \geqq a^{-1}$. Then $x^2 \geqq e$, and $x \geqq e$ by isolatedness.

The following simple observation will be needed.

LEMMA 12.1. *If $A_1, \dots, A_m$ are o-ideals of a directed group G with isolated order and if $g > e$ belongs to the o-ideal generated by $A_1, \dots, A_m$, then there exist positive elements $a_i \in A_i$ such that*

$$g \in (a_1, \dots, a_m)^{\#}.$$

Since $(a_1, \dots, a_m)^{\#}$ is a lower class and since to each g in the o-ideal generated by $A_1, \dots, A_m$, there exist $g_i \in A_i^+$ such that $g < g_1 \dots g_m$, it suffices to establish the assertion for $g = g_1 \dots g_m$ with $g_i \in A_i^+$. If $m = 2$, then take $(g_1 g_2^{-1}, g_2 g_1^{-1})^{\#}$. By (e), this contains e, whence $g = g_1 g_2 \in (g_1^2, g_1 g_2 g_1^{-1} g_2)^{\#} = (a_1, a_2)^{\#}$ with $a_1 = g_1^2 \in A_1$ and $a_2 = g_1 g_2 g_1^{-1} g_2 \in A_2$. If $m > 2$, then $g = g_1 \dots g_m \in (a_1, a')^{\#}$ with $a_1 \in A_1$ and $a' \geqq e$ in the o-ideal generated by $A_2, \dots, A_m$. By induction, $a' \in (a_2, \dots, a_m)^{\#}$ for some $a_i \in A_i^+$. Since both a_1 and a' are contained in $(a_1, \dots, a_m)^{\#}$, property (b) shows that $(a_1, a')^{\#} \subseteq (a_1, \dots, a_m)^{\#}$. Hence the desired inclusion $g \in (a_1, \dots, a_m)^{\#}$ follows.

The following concept is fundamental for the radical. Call an element $a \in G$ *subordinate* to a positive element $g \in G$ if whenever

(1) $$g \in (g_1, \dots, g_n)^{\#}$$

holds for positive $g_i \in G$, then there is an index i such that

(2) $$a \in I(g_i)$$

where $I(g_i)$ denotes the o-ideal generated by g_i. The sign $a \blacktriangleleft g$ will be used to denote that a is subordinate to g. If g is not necessarily positive, then $a \blacktriangleleft g$ will mean that if $g = g_1 g_2^{-1}$ with positive g_1, g_2, then $a \blacktriangleleft g_1 g_2$ in the sense above. That this definition yields the same concept for positive g, will be clear from (v) below.

We have the obvious properties:

(i) $e \blacktriangleleft g$ for each $g \in G^+$;

(ii) $a \blacktriangleleft g$ implies $a \in I(g)$;

(iii) in fully ordered groups, $a \blacktriangleleft g$ if and only if $a \in I(g)$;

(iv) in lattice ordered groups, $a \blacktriangleleft g$ means that $|g| = g_1 \vee \ldots \vee g_n$ with $g_i \geqq e$ implies $a \in I(g_i)$ for some i;

(v) if $e \leqq a \blacktriangleleft g$ and if $e \leqq b \leqq a$, $e \leqq g \leqq h$, then $b \blacktriangleleft h$;

(vi) if $a \blacktriangleleft g$, then $x^{-1}ax \blacktriangleleft y^{-1}gy$ for arbitrary $x, y \in G$.

The elements of G to which a fixed $a \neq e$ is not subordinate form an o-ideal of G which can easily be characterized.

PROPOSITION 12.2. *In a directed group G with isolated order, $a \neq e$ is not subordinate to g exactly if $g \in \Omega(a)$ where $\Omega(a)$ denotes the o-ideal generated by all o-ideals of G that fail to contain a.*

Let a be not subordinate to g. Then there is a decomposition $g = x_1 x_2^{-1}$ with $x_1, x_2 \geqq e$ such that $x_1 x_2 \in (g_1, \ldots, g_n)^{\#}$ for some $g_i \in G^+$, but $a \notin I(g_i)$ for all i. Then $x_1 x_2 \in \Omega(a)$, since $g_1 \ldots g_n \in U(g_1, \ldots, g_n)$ implies $x_1 x_2 \leqq g_1 \ldots g_n$, and $x_1 x_2 \in I(g_1, \ldots, g_n)$. Thus $g \in \Omega(a)$. Conversely, if $g \in \Omega(a)$, then there exist a finite number of o-ideals $A_1, \ldots, A_n$ not containing a such that g belongs to the o-ideal they generate. If $g = x_1 x_2^{-1}$ with $x_1, x_2 \geqq e$ still in this o-ideal, then also $x_1 x_2$ belongs to the same o-ideal, and hence by Lemma 12.1 we have $x_1 x_2 \in (a_1, \ldots, a_n)^{\#}$ for suitable $a_i \in A_i^+$. But $a \notin I(a_i) \subseteq A_i$, so a is not subordinate to g.

The set of all g such that $a \blacktriangleleft g$ implies $a = e$ will be denoted by $R(G)$, and called the *Conrad radical* of G. We easily obtain the intersection property.

THEOREM 12.3. *The Conrad radical $R(G)$ of a directed group G with isolated order is the intersection of all $\Omega(a)$ with $a \neq e$ in G:*

$$R(G) = \bigcap_{a \neq e} \Omega(a).$$

Thus $R(G)$ is an ω-ideal of G.

If $a \blacktriangleleft g$ implies $a = e$, then by the preceding result $g \in \Omega(a)$ for every $a \in G$, $a \neq e$; and conversely.

In Example 1 of § 3 the Conrad radical is 0, since this group is o-simple. In Example 5 of § 3 the Conrad radical is the whole group, since a non zero polynomial cannot be subordinate to any polynomial.

The following terminology will be useful in our subsequent considerations. If $a \in G$ and if M is an o-ideal of G that is maximal with respect to the exclusion of a, then we call M a *regular o-ideal associated with a*. The intersection of all o-ideals of G properly containing M contains a, and thus it is the only ω-ideal M^* of G for which $M \subset M^*$ and there is no ω ideal between M and M^*. Obviously, $\Omega(a)$ is just the union of all regular o ideals of G associated with a.

Call an o-ideal A of G *essential* [4] if

(A) it is regular, and

(B) there is a $b \neq e$ in G such that $A \supseteq \Omega(b)$.

If to some $a \in G$ there is only one regular o-ideal M associated with a, then $M = \Omega(a)$, and hence M is essential.

LEMMA 12.4. *A regular o-ideal M is essential if and only if the intersection of all regular o-ideals of G not contained in M is different from e.*

Assume that the intersection of all regular o-ideals N not in M contains some element $a \neq e$. Then no regular o-ideal associated with a may occur among the N, whence M contains all regular o-ideals associated with a. Therefore $M \supseteq \Omega(a)$, and M is essential. Conversely, if $\Omega(a) \subseteq M$, then a must belong to the indicated intersection.

By making use of the concepts of regular and essential l-ideals, CONRAD [4] has shown that the radical is a lattice-invariant of a lattice-ordered group: it can be characterized lattice-theoretically in the lattice of all l-ideals of G (which is a complete sublattice of all normal subgroups of G). This result admits a generalization to our present case.

Let G be a directed group with isolated order, $\mathfrak{N}$ the lattice of its normal subgroups and $\mathfrak{O}$ the set of its o-ideals. Now the regular o-ideals of G can be characterized as elements of $\mathfrak{O}$ which cannot be represented as the intersection (taken in $\mathfrak{N}$) of any set of greater elements of $\mathfrak{O}$. By virtue of the last Lemma, the essential o-ideals can also be singled out by means of $\mathfrak{O}$ and $\mathfrak{N}$. The next result will show that the same holds for the CONRAD radical $R(G)$.

THEOREM 12.5. *The Conrad radical $R(G)$ of a directed group G with isolated order is the intersection of all essential o-ideals of G.*

From the definition it follows that the intersection of all essential o-ideals of G contains the intersection of all the $\Omega(a)$ $(a \neq e)$, and hence $R(G)$ owing to Theorem 12.3. To prove the converse, let $g \notin R(G)$, that is to say, $g \notin \Omega(a)$ for some $a \neq e$ in G. By Zorn's lemma, there is a regular o-ideal M, containing $\Omega(a)$, associated with g. This M is essential, and so the intersection of the essential o-ideals does not contain g either. This completes the proof.

Budapest and Pisa.

REFERENCES

[1] H. BAUER, *Geordnete Gruppen mit Zerlegungseigenschaft*, S.-B. Bayer. Akad. Wiss. Math.-Nat. Klasse, 1958, 25-36.

[2] G. BIRKHOFF, *Lattice-ordered groups*, Annals of Math., 43 (1942), 298-331.

[3] G. BIRKHOFF, *Lattice theory* (New York, 1948).

[4] P. CONRAD, *The relationship between the radical of a lattice-ordered group and complete distributivity* Pacific J. Math., 14 (1964), 493-499.

[5] L. FUCHS, *The extension of partially ordered groups*, Acta Math. Acad. Sci. Hung., 1 (1950), 118-124.

[6] L. FUCHS, *Abelian groups* (Budapest, 1958).

[7] L. FUCHS, *Partially ordered algebraic systems* (Oxford-London-New York-Paris, 1963).

[8] P. G. KONTOROVIČ-K. M. KUTYEV, *On the theory of lattice-ordered groups* (in Russian), Izv. Vys. Uč. Zav. Mat., 1959, 112-120.

[9] K. M. KUTYEV, *On regular lattice-ordered groups* (in Russian), Usp. Mat. Nauk, 11 : 1 (1956), 256.

[10] I. NAMIOKA, *Partially ordered linear topological spaces*, Memoirs Amer. Math. Soc., 24 (1957), 1-50.

[11] L. S. PONTRJAGIN, *Topologische Gruppen* (Leipzig, 1957/8).

[12] P. RIBENBOIM, *Théorie des groupes ordonnés* (Bahia Blanca, 1964).

[13] F. RIESZ, *Sur quelques notions fondamentales dans la théorie générale des opérations linéaires*, Annals of Math., 41 (1940), 174-206.

[14] J. R. TELLER, *On the extensions of lattice-ordered groups*, Pacific J. Math., 14 (1964), 709-718.

[15] J. R. TELLER, *On partially ordered groups satisfying the Riesz interpolation property* (to appear).