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GENERATING CURVES ON ABELIAN VARIETIES AND RIEMANN'S THETA-FUNCTION

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I. Introduction.

We shall show how to prove (and possibly illuminate) two well known theorems of Riemann and Poincaré about theta-functions, using Weil's ([9]) « intrinsic » approach to classical abelian varieties. Some variations of Matsusaka's criterion for jacobians ([2], [5]) are incidentally obtained.

II. Generating curves.

Let Z be a positive 1-cycle on an n-dimensional abelian variety A. Since we work modulo numerical equivalence, we may suppose that Z is a (possibly reducible) curve, all of whose components pass through the identity 0 of A. Let $\{Z\}$ be the smallest abelian subvariety containing Z, and call Z a generating curve if $\{Z\} = A$.

Let $Z^{(n)}$ denote the n-fold Pontrjagin product of Z with itself (see [10]). There is a non-negative integer $k(Z)$ such that, as a cycle, $Z^{(n)} = k(Z)\, n\,!\, A$.

PROPOSITION 1. Z is a generating curve $\Longleftrightarrow k(Z) \neq 0$.

PROOF. The implication $\Longleftarrow$ is obvious. For the converse, first consider the case of an irreducible Z, and let r be the smallest integer for which $Z^{(r+1)} = 0$. Then $|Z^{(r)}|$ (the support of $Z^{(r)}$) is invariant under translations by points of Z, and since it contains 0, must be a subgroup, so clearly must be equal to $\{Z\}$, and the proposition follows immediately in this case. Now for any Z, let its irreducible components be the Z_i, and $n_i = \dim \{Z_i\}$, so that by the preceding, there are positive integers k_i for which $Z_i^{(n_i)} = n_i\,!\, k_i \{Z_i\}$. Furthermore, one sees that the $\{Z_i\}$ generate $\{Z\}$.

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If the $\{Z_i\}$ are direct summands of $\{Z\}$ then $n = \Sigma_i n_i$, and $k(Z) = \Pi_i k_i \neq 0$. If $n = \Sigma_i n_i$, but the sum is not direct, then the direct sum of the $\{Z_i\}$ form a cover of degree $d > 1$ of $\{Z\}$, and $k(Z) = d\, \Pi_i k_i > 1$. If, finally, $n < \Sigma_i n_i$, for some $i \neq j$ $\dim \{Z_i\} \cap \{Z_j\} > 0$. Choose the smallest $r_i \leq n_j$ so that $Z_i^{(n_i)} \oplus Z_j^{(r_i+1)} = 0$ (where $\oplus$ means Pontrjagin product). Then by an argument similar to that above, we see that $| Z_i^{(n_i)} \oplus Z_j^{(r_j)} |$ is an abelian variety, in fact, it equals $\{Z_i + Z_j\}$. By symmetry, we can also find an $r_i \leq n_i$ so that $\{Z_i + Z_j\} = | Z_i^{(r_i)} \oplus Z_j^{(n_j)} |$.

Hence, in the formula

$$Z^{(n)} = \sum_{g_1 + \dots + g_q = n} \frac{n!}{g_1! \dots g_q!} Z_1^{(g_1)} \oplus \dots \oplus Z_q^{(g_q)}$$

the right-hand side has at least two nonvanishing terms, so in fact, $k(Z) \geq 2$.

PROPOSITION 2. *If $k(Z) = 1$, the Z_i are non-singular curves of genus n_i, and A is the direct sum of the $\{Z_i\}$, which are the jacobians of the Z_i.*

PROOF. The proof of Proposition 1 shows that A is the direct sum of the $\{Z_i\}$, and that $Z_i^{(n_i)} = n_i!\, \{Z_i\}$, so we are reduced to the case of Z irreducible. In that case, let J be the jacobian of its normalization $\bar{Z}$. Since Z is generating, the map $\bar{Z} \to Z$ gives rise to a surjection $J \to A$, so $\bar{Z}$ has genus $g \geq n$. But $k(Z) = 1$ means that the n-fold symmetric product of $\bar{Z}$ is birationally equivalent to A, so by a remark of Weil ([11] p. 37) $n = g$, and $J \to A$ is an isomorphism. In particular, $Z = \bar{Z}$.

III. The associated Kähler metric

Assume we are in the classical case, and define an hermitian form on $H^0(A, \Omega^1)$ (the holomorphic differentials) by

$$H(\alpha, \alpha') = \frac{\sqrt{-1}}{2} \int_Z \alpha \wedge \bar{\alpha}'$$

PROPOSITION 3. *Z is generating $\iff$ H is positive definite.*

PROOF. The implication $\Leftarrow$ is obvious. Conversely, let Z generate A, and J_j be the jacobians of the normalizations $\bar{Z}_j$ of the components Z_j of Z. The maps $\varphi_j : \bar{Z}_j \to Z_j$ induce a surjection $\varphi : \Pi_j J_j \to A$, whose dual map

$\widehat{\varphi}:\widehat{A}\longrightarrow \Pi_j \widehat{J}_j$ on the Picard varieties thus has finite kernel. The lifting of $\widehat{\varphi}$ to the universal covering spaces is the map $\Pi_j \varphi_j^*: H^0(A,\Omega^1)\longrightarrow \Pi_i H^0(J_j,\Omega^1)$ gotten by pulling back differentials, which, being linear and having a discrete kernel, is an injection. So if $\alpha \neq 0$ is a differential on A, some $\varphi_j^*(\alpha)\neq 0$, and

$$H(\alpha,\bar{\alpha}) \geq \frac{\sqrt{-1}}{2}\int_{\bar{Z}_j} \varphi_j^*(\alpha)\wedge\overline{\varphi_j^*(\alpha)} = \int_{\bar{Z}_j} \| \mathrm{Re}(\varphi_j^*(\alpha))\|^2 > 0$$

(where $\| \;\|$ is the Dirichlet norm) by a standard result about harmonic forms on Riemann surfaces.

Now let $\alpha_1,\dots,\alpha_n$ be an orthonormal basis for H. Then $(\Sigma_i\,\alpha_i)^2$ gives an hermitian form on the tangent space to A at 0, and hence an invariant Kähler metric. Associated with this is the closed 2-form $u=\frac{\sqrt{-1}}{2}\Sigma_i\,\alpha_i\wedge\bar{\alpha}_i$, and Hodge's adjoint operation $*$ on differentials forms of cohomology. An easy computation (see Weil [9] p. 20) shows that $*u=\left(\frac{\sqrt{-1}}{2}\right)^{n-1}\Sigma_i\,\Omega_i$, where $\Omega_i=\prod_{i\neq j}\alpha_j\wedge\bar{\alpha}_j$. Let $\gamma=\int_A u^n/n!$ be the volume of A for this metric.

Let $[X]$ denote the cohomology class of either a form or cycle, X, denote either intersection or cup products by juxtaposition, and preserve the notation $X\oplus Y$ and $X^{(r)}$ for Pontrjagin products of cohomology classes, as well as cycles. (It is not hard to see that the two are compatible). Then we have:

PROPOSITION 4. $*[u]=\gamma[Z]$

PROOF. Let η be a closed (1,1) form on A, hence cohomologous to $\frac{\sqrt{-1}}{2}\Sigma_{ij}\,a_{ij}\,\alpha_i\wedge\bar{\alpha}_j$, for constant a_{ij}. Now

$$\frac{\sqrt{-1}}{2}\,\alpha_i\wedge\bar{\alpha}_j\wedge\Omega_k=\begin{cases} u^n/n! & \text{if } i=j=k\\ 0 & \text{otherwise}\end{cases}$$

so $[\eta\wedge *u]=[\Sigma_j\,a_{jj}\,u^n/n!]$. But on the other hand,

$$\int_Z \eta=\Sigma_{ij}\frac{\sqrt{-1}}{2}\,a_{ij}\int_Z \alpha_i\wedge\bar{\alpha}_j=\Sigma_{ij}\,a_{ij}\,H(\alpha_i,\alpha_j)=\Sigma_j\,a_{jj}$$

and the proposition follows immediately.

PROPOSITION 5. $[Z^{(n-1)}] = \gamma^{1-n}(n-1)!\,[u]$.

PROOF. By a general result ([4]) one has $^*(XY) = (^*X) \oplus (^*Y)$ for X and Y any cohomology classes. Hence

$$[Z]^{(n-1)} = (^*[u]/\gamma)^{(n-1)} = \gamma^{1-n}\,{}^*([u]^{n-1}).$$

But by [9] p. 25 $[u]^{n-1} = (n-1)!\,{}^*[u]$, and our assertion follows.

COROLLARY 1. $k(Z) = \gamma^{-n}$, so if $\gamma \geq 1$ (in particular, if the metric is Hodge, $k(Z) = \gamma = 1$ and A is a product of jacobians).

PROOF. $[Z^{(n)}] = k(Z)\,n!\,[A]$, but also $[Z]^{(n)} = (^*[u]/\gamma)^{(n)} = \gamma^{-n}\,{}^*[u^n] = \gamma^{-n}!\,[A]$. The last assertion follows from Proposition 2.

Assume we are dealing with a product of jacobians, and let, as usual, $r!\,W_r = Z^{(r)}$ and $\Theta = W_{n-1}$.

COROLLARY 2. (Poincaré [6]. For numerical equivalence, in the abstract case, see [5]). $(n-r)!\,[W_r] = [\Theta^{n-r}]$.

PROOF. $(n-r)!\,[W_r] = \dfrac{(n-r)!}{r!} Z^{(r)} = \dfrac{(n-r)!}{r!}\,{}^*(^*[Z])^r = (n-r)!/r!\,{}^*[u^r]$.

But by an elementary algebraic identity ([1] p. 170) $^*[u^r] = r!/(n-r)!\,[u^{n-r}]$, from which the corollary follows.

COROLLARY 3. (Riemann [7]. See e. g. [3] for a rigorous presentation). Let ϑ be « Riemann's » theta-function. Then a translate of Θ is « cut out by » ϑ.

PROOF. Let ϑ cut out the positive divisor X. Examining its « factors of automorphy » we see that ϑ « belongs to » the hermitian form H, in the sense of Weil [9] ch. VI. Hence ([9] p. 112) $[X] = [u] = [\Theta]$, and so ([9] p. 115) X is linearly equivalent to a translate of Θ. But since $\gamma = 1$, $\lambda(X) = 1$ by the (Frobenius) Riemann-Roch theorem, and the corollary follows.

Note that ϑ is even, so $X = X^-$, the image of X under the endomorphism $x \longrightarrow -x$. On the other hand, $\Theta = \Theta_c$, where c is the « canonical point » (see Weil [8] p. 73). So if $\Theta = X_r$, then $2r = c$.

The coordinates of the point r are the traditional « Riemannian constants ».

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