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ON THE DOMINANCE OF PARTIAL DIFFERENTIAL OPERATORS II* [1]

MARTIN SCHECHTER

1. Introduction.

In [19] we began a study of certain types of inequalities for partial differential operators. In the present paper we continue along these lines but specialize to particular situations. This specialization allows us to remove some of the restrictions of [19] and is directed toward certain applications. Although our methods are related to those of [19], they are much simpler. Moreover, the present paper is self contained and is completely independent of [19].

The fact that solutions of elliptic boundary value problems are smooth up to the boundary is well known (cf., e. g., [2, 3, 11, 14, 17, 18]). Recently similar results were obtained for hypoelliptic operators (for definitions cf. Section 2). Hörmander [7] considered the case

$$P(D)\,v = 0 \text{ in } x_n > 0 \tag{1.1}$$

$$Q_j(D)\,v = 0 \text{ on } x_n = 0, \qquad 1 \leq j \leq r, \tag{1.2}$$

where the operators $P(D)$ and the $Q_j(D)$ have constant coefficients. (Here x_n is the last coordinate in Euclidean n-space.) Assuming that $P(D)$ is hypoelliptic, he gave a necessary and sufficient condition that every solution be infinitely differentiable in $x_n \geq 0$. His method was to study very carefully the solutions of the ordinary differential equation obtained by

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taking Fourier transforms with respect to the remaining coordinates $x_1, \dots, x_{n-1}$.

Peetre [13] considered the Dirichlet problem

$$P(x, D)\, v = f \text{ in } x_n > 0 \tag{1.3}$$

$$\frac{\partial^k v}{\partial x_n^k} = 0 \text{ on } x_n = 0, \qquad 0 \leq k < r, \tag{1.4}$$

with f infinitely differentiable on $x_n \geq 0$. He assumed that $P(x, D)$ is formally hypoelliptic, i. e., that it is a variable coefficient operator which for each x equals a constant coefficient hypoelliptic operator and is equally strong at each point (for a precise definition cfr., e. g., [8]). He obtains a sufficient condition that every solution v of (1.3), (1.4) should be infinitely differentiable in $x_n \geq 0$. This condition in also sufficient for the estimate

$$\| R(D)\, v \| \leq C (\| f \| + \| v \|) \tag{1.5}$$

to hold for all solutions, where $R(D)$ is any operator weaker than $P(x, D)$ and the norm is that of $L^2(x_n > 0)$ (cf. Section 2). In fact his proof of regularity was accomplished by means of an inequality similar to (1.5).

In this paper we are interested in extending the results of Peetre to the problem

$$P(D)\, v = f \quad \text{in} \quad x_n > 0 \tag{1.6}$$

$$Q_j(D)\, v = f_j \text{ on } x_n = 0, \qquad 1 \leq j \leq r, \tag{1.7}$$

with f and the f_j infinitely differentiable on $x_n \geq 0$ and $x_n = 0$, respectively. We do not assume that $P(D)$ is hypoelliptic but allow it to belong to a slightly larger class. We obtain sufficient conditions for the estimate

$$\| R(D)\, v \| \leq C \left(\| f \| + \sum_{j=1}^{r} \langle f_j \rangle + \| v \| \right) \tag{1.8}$$

to hold for all solutions v of (1.6), (1.7), where $\langle \cdot \rangle$ is an appropriate norm on $x_n = 0$. (Actually we prove a family of inequalities stronger than (1.8) (cf. Theorem 2.1).) This inequality enables us to prove that every solution of (1.6), (1.7) is infinitely differentiable in $x_n \geq 0$ (Theorem 5.1).

We mention two observations concerning inequality (1.8).

1. When $P(D)$ is elliptic, every operator of the same or lower order is weaker than $P(D)$. In this case (1.8) becomes a coerciveness inequality

(cf. Section 4 or [1, 2, 3, 14, 16]). In some instances the norm $\langle \cdot \rangle$ may be slightly stronger than the norm usually employed. However, and this is essential, inequality (1.8) may hold even though the $Q_j(D)$ do not cover $P(D)$ in the usual sense. The reason for this is that the lower order terms in the $Q_j(D)$ may play a role in determining the validity of (1.8). Examples where this is the case are given in Section 2. In brief, (1.8) is new even for elliptic operators.

2. Inequality (1.8) allows one to handle regularity for variable coefficient operators as well. If $P(x, D)$ is equally strong at all points, any error obtained by approximating it by a constant coefficient operator can be estimated by (1.8). This is essentially the method employed in treating formally hypoelliptic operators (cf. [8, 13]). We shall carry out the details elsewhere.

We also give another proof of the usual coerciveness inequality for (1.6), (1.7) when $P(D)$ is elliptic and all the operators are homogeneous. This proof is much simpler than any presently found in the literature.

In Section 2 we give pertinent definitions and state our main inequalities. Proofs are given in Section 3. In Section 5 we prove our regularity theorem by means of these results. Our short proof of the usual coerciveness inequality is given in Section 4.

2. The Main Results.

Let E^n denote n-dimensional Euclidean space with generic points $(x_1, \dots, x_n)$. We shall find it convenient to set $x = (x_1, \dots, x_{n-1})$, $y = x_n$ and denote points of E^n by (x, y). The half-spaces $y > 0$ and $y \geq 0$ are denoted by E^n_+ and $\bar{E}^n_+$, respectively.

Let $\mu = (\mu_1, \dots, \mu_n)$ be a multi-index of non-negative integers with length $|\mu| = \mu_1 + \dots + \mu_n$. Let D_j be the operator $\partial/i\partial x_j$, $1 \leq j \leq n$, and set

$$D_x = (D_1, \dots, D_{n-1}), \qquad D_y = D_n .$$

We consider a partial differential operator of the form

$$P(D) = P(D_x, D_y) = \sum_{|\mu| \leq q} a_\mu D_1^{\mu_1} \dots D_{n-1}^{\mu_{n-1}} D_y^{\mu_n},$$

where the coefficients a_μ are complex constants. The polynomial corresponding to $P(D_x, D_y)$ is

$$P(\xi, \eta) = \sum_{|\mu| \leq q} a_\mu \xi_1^{\mu_1} \dots \xi_{n-1}^{\mu_{n-1}} \eta^{\mu_n},$$

where $\xi = (\xi_1, \dots, \xi_{n-1})$. We shall also employ the notation

$$P^{(\mu)}(\xi, \eta) = \frac{\partial^{|\mu|} P(\xi, \eta)}{\partial \xi_1^{\mu_1} \dots \partial \xi_{n-1}^{\mu_{n-1}} \partial \eta^{\mu_n}},$$

$$P^{(0, \dots, 0)}(\xi, \eta) = P(\xi, \eta).$$

We make the following assumptions on $P(\xi, \eta)$.

Hypothesis 1. There are constants K_1 and K_2 such that

$$\sum_{\mu} | P^{(\mu)}(\xi, \eta) | \leq K_1 | P(\xi, \eta) |$$

whenever ξ and η are real and $|\xi| > K_2$, where $|\xi|^2 = \xi_1^2 + \dots + \xi_{n-1}^2$.

Hypothesis 2. Let $m \leq q$ be the highest power of η in $P(\xi, \eta)$ and let $a(\xi)$ denote its coefficient. Then there is a constant K_3 such that

$$K_3 | a(\xi) | \geq 1$$

for all real ξ.

We remark that Hypotheses 1 and 2 are surely satisfied if P is hypoelliptic. By definition P is hypoelliptic if

$$\frac{| P^{(\mu)}(\xi, \eta) |}{| P(\xi, \eta) |} \to 0$$

as $(\xi, \eta) \to \infty$ through real values whenever $|\mu| > 0$. This immediately implies Hypothesis 1. Hypothesis 2 follows from the well-known observation by Hörmander [6, p. 239] that in a hypoelliptic operator the coefficient of the highest power of η is independent of ξ.

That there are operators satisfying Hypotheses 1 and 2 which are not hypoelliptic is seen from simple examples. For instance the operator corresponding to the polynomial

$$P(\xi, \eta) = (1 + \xi_1^2 + \dots + \xi_{n-1}^2)(\eta - i)$$

is not hypoelliptic.

For each real vector ξ let $\tau_1(\xi), \dots, \tau_m(\xi)$ denote the roots of

$$P(\xi, z) = 0.$$

We claim that there is a positive constant c_m depending only on m such that

$$|\operatorname{Im} \tau_k(\xi)| \geq c_m K_1^{-1} \tag{2.1}$$

for $|\xi| > K_2$. In fact let η be any real number and consider the polynomial $\Phi(t) = P(\xi, \eta + tK_1^{-1})$ in t. Then

$$|\Phi'(t)| = |K_1^{-1} \partial P(\xi, \eta + tK_1^{-1}) / \partial\eta| \leq |\Phi(t)|$$

by Hypothesis 1. Let $t_1, \dots, t_m$ denote the roots of $\Phi(t)$. Then there is a positive constant c_m depending only on m such that

$$|t_k| \geq c_m, \qquad 1 \leq k \leq m,$$

(cf. Lemma 3.1 of [8]). But $t_k = K_1(\tau_k(\xi) - \eta)$. Setting $\eta = \operatorname{Re} \tau_k(\xi)$ we obtain (2.1).

Since the set $|\xi| > K_2$ is connected for $n > 2$, it follows that the number of roots $\tau_k(\xi)$ with positive imaginary parts is constant in this set for such n. An operator with this property is said to be *of determined type* (cf. Hörmander [7, p. 227]). A hypoelliptic operator of determined type is called *properly hypoelliptic* (cfr. Peetre [13, p. 337]). One sees from the reasoning above that in dimensions higher than the second every operator satisfying Hypothesis 1 is of determined type and hence every hypoelliptic operator is properly hypoelliptic (cf. also Hörmander [7, p. 227]). For $n = 2$ we make the additional

Hypothesis 3. $P(\xi, \eta)$ is determined type.

Let r be the number of roots $\tau_k(\xi)$ with positive imaginary parts (for $|\xi| > K_2$). By rearrangement if necessary we assume that

$$\operatorname{Im} \tau_k(\xi) > 0, \qquad 1 \leq k \leq r, \tag{2.2}$$

$$\operatorname{Im} \tau_k(\xi) < 0, \qquad r < k \leq m. \tag{2.3}$$

Set

$$P_+ = \prod_{k=1}^{r} (\eta - \tau_k(\xi)), \quad P_- = P / P_+ .$$

Let there be given r polynomials $Q_1(\xi, \eta), \dots, Q_r(\xi, \eta)$ of degree $< m$ in η. For each j we resolve $Q_j(\xi, \eta) / P(\xi, \eta)$ into partial fractions:

$$\frac{Q_j(\xi, \eta)}{P(\xi, \eta)} = \frac{Q_{j+}(\xi, \eta)}{P_+(\xi, \eta)} + \frac{Q_{j-}(\xi, \eta)}{P_-(\xi, \eta)}$$

and set

$$\alpha_{ij}(\xi) = \int_{-\infty}^{\infty} \frac{Q_{i+}\overline{Q_{j+}}}{|P_+|^2}\, d\eta, \qquad 1 \leq i,\ j \leq r$$

$$\beta_{ij}(\xi) = \int_{-\infty}^{\infty} \frac{Q_{i-}\overline{Q_{j-}}}{|P_-|^2}\, d\eta, \qquad 1 \leq i,\ j \leq r.$$

We consider the $r \times r$ Hermitian matrices

$$A = (\alpha_{ij}),\ B = (\beta_{ij})$$

and assume

Hypothesis 4. The set of operators $\{Q_j(D)\}_{j=1}^{r}$ covers $P(D)$. This means that for $|\xi| > K_2$, A is non singular a. e. and there is a constant K_4 such that

$$BA^{-1}B \leq K_4 B \qquad \text{a. e.} \tag{2.4}$$

REMARKS. This definition of covering is a generalization of that for the elliptic case. That A is non-singular is equivalent to saying that the $Q_{j+}(\xi, \eta)$ are linearly independent, i. e., that the $Q_j(\xi, \eta)$ are linearly independent *modulo* $P_+(\xi, \eta)$. The estimate (2.4) says, in a sense, that this independence is uniform in $|\xi| > K_2$. In the elliptic case (2.4) holds automatically when A^{-1} exists. A condition equivalent to (2.4) is given by Peetre [13]. He assumed that every linear combination Q of the Q_j should satisfy

$$\int_{-\infty}^{\infty} \left| \frac{Q_-}{P_-} \right|^2 d\eta \leq C \int_{-\infty}^{\infty} \left| \frac{Q_+}{P_+} \right|^2 d\eta$$

in $|\xi| > K_2$.

Let $C_0^\infty(\overline{E}_+^n)$ denote the set of complex valued functions which are infinitely differentiable in $\overline{E}_+^n$ and vanish for $|x|^2 + y^2$ sufficiently large. For s real we employ the family of norms

$$|v|_s = \left(\int_0^{\infty} \int_{|\xi| < \infty} (1 + |\xi|^2)^s\, |u(\xi, y)|^2\, d\xi\, dy \right)^{\frac{1}{2}},$$

where $u(\xi, y)$ is the Fourier transform of $v(x, y)$ with respect to the variables $x_1, \dots, x_{n-1}$:

$$u(\xi, y) = (2\pi)^{-\frac{n-1}{2}} \int e^{-i\xi x} v(x, y)\, dx$$

($\xi x = \xi_1 x_1 + \dots + \xi_{n-1} x_{n-1}$). For s a non-negative integer one sees easily that $|v|_s$ is equivalent to the sum of the $L^2(E_+^n)$ norms of all derivatives of $v(x, y)$ with respect to $x_1, \dots, x_{n-1}$ up to order s. In particular $|v|_0$ is equivalent to the $L^2(E_+^n)$ norm of v.

We shall also make use of the scalar products

$$\langle v_1, v_2 \rangle_s = \int (1 + |\xi|^2)^s u_1(\xi, 0) \overline{u_2(\xi, 0)}\, d\xi,$$

where u_i is the Fourier transform of v_i with respect to x, $i = 1, 2$. The corresponding norms are given by

$$\langle v \rangle_s^2 = \langle v, v \rangle_s .$$

A polynomial $R(\xi, \eta)$ is said to be weaker than $P(\xi, \eta)$ if

$$|R(\xi, \eta)| \leq \text{const.} \sum_\mu |P^\mu(\xi, \eta)|$$

for all real ξ, η. The corresponding operator $R(D)$ is said to be weaker than $P(D)$.

We can now state our main results.

THEOREM 2.1. *Under Hypotheses* 1 4, *for each operator $R(D)$ weaker than $P(D)$ and every positive number b there is a constant C such that*

$$\begin{aligned} |R(D)v|_s^2 \leq C\Big(&|P(D)v|_s^2 \\ &+ \sum_{i,j=1}^r \langle \alpha^{ij} Q_i(D)v, Q_j(D)v \rangle_s + |v|_{s-b}^2\Big) \end{aligned} \tag{2.5}$$

for all $v \in C_0^\infty(\bar{E}_+^n)$ and all real s, where the $\alpha^{ij}(\xi)$ are the elements of A^{-1}.

COROLLARY 2.1. *Under the same hypotheses there is a constant d depending only on P and the Q_j such that* (2.5) *may be replaced by*

$$|R(D)v|_s \leq C'\Big(|P(D)v|_s + \sum_{j=1}^r \langle Q_j(D)v \rangle_{s+d} + |v|_{s-b}\Big). \tag{2.6}$$

The proofs of these results are given in the next section. An application is given in Section 5. In Section 4 we discuss the elliptic case.

We now consider some illustrations. Assume that P is of second degree in η and that the coefficient of η^2 is one. Then

$$P(\xi, \eta) = (\eta - \tau_1(\xi))(\eta - \tau_2(\xi))$$

and we take the case

$$\operatorname{Im} \tau_1 > 0, \quad \operatorname{Im} \tau_2 < 0$$

for $|\xi| > K_2$. For the boundary condition we take

$$Q(\xi, \eta) = \eta + p(\xi),$$

where $p(\xi)$ is a polynomial in $\xi_1, \dots, \xi_{n-1}$. One easily checks that

$$Q_+ = (\tau_1 + p)/(\tau_1 - \tau_2), \quad Q_- = (\tau_2 + p)/(\tau_2 - \tau_1)$$

and hence

$$\int_{-\infty}^{\infty} \left| \frac{Q_+}{P_+} \right|^2 d\eta = \frac{\pi}{|\operatorname{Im} \tau_1|} \frac{|\tau_1 + p|^2}{|\tau_1 - \tau_2|^2},$$

$$\int_{-\infty}^{\infty} \left| \frac{Q_-}{P_-} \right|^2 d\eta = \frac{\pi}{|\operatorname{Im} \tau_2|} \frac{|\tau_2 + p|^2}{|\tau_2 - \tau_1|^2}.$$

Condition (2.4) now reduces to

$$\frac{|\tau_2 + p|^2}{|\operatorname{Im} \tau_2|} \leq K \frac{|\tau_1 + p|^2}{|\operatorname{Im} \tau_1|} \tag{2.7}$$

for $|\xi|$ sufficiently large. If p is real and the τ_i are pure imaginary, this reduces further to

$$|\tau_2| + \frac{p^2}{|\tau_2|} \leq K\left(|\tau_1| + \frac{p^2}{|\tau_1|}\right). \tag{2.8}$$

This in turn is valid if there is a constant K such that

$$|\tau_1| \leq K |\tau_2| \tag{2.9}$$

for $|\xi|$ sufficiently large and

$$p^2 \geq K^{-1} |\tau_1 \tau_2| - |\tau_1|^2 \tag{2.10}$$

for such ξ. Otherwise there must be a constant $K \geq \frac{1}{2}$ such that

$$K^{-1} |\tau_1 \tau_2| - |\tau_1|^2 \leq p^2 \leq K |\tau_1 \tau_2| \left(1 - K \left| \frac{\tau_2}{\tau_1} \right| \right)^{-1} \tag{2.11}$$

for $|\xi|$ large, where the right hand inequality in (2.11) need only be satisfied when the last expression is positive.

Peetre [13] observed that inequality (2.9) is necessary for an inequality slightly stronger than (1.5) to hold for Dirichlet boundary conditions. An example which violates it is

$$P(D) = (D_y - i\Delta'^2)(D_y + i\Delta'),$$

where

$$\Delta' = D_1^2 + \dots + D_{n-1}^2.$$

This operator is hypoelliptic and hence satisfies Hypotheses 1 and 2. If $p(\xi)$ is real and of degree ≤ 3, then condition (2.11) is satisfied and hence we have

$$|v|_{s+6} + |D_y v|_{s+4} + |D_y^2 v|_s \leq \text{const.} \, (|P(D) v|_s + \langle Q(D) v \rangle_{s+2} + |v|_{s-b}),$$

where the left hand side represents operators weaker than $P(D)$.

Another example shows how Theorem 2.1 gives new information for elliptic operators. Take

$$P(D) = \Delta \equiv D_1^2 + \dots + D_n^2.$$

Then

$$P(\xi, \eta) = (\eta - i|\xi|)(\eta + i|\xi|)$$

and (2.9) holds. If we take

$$Q(D) = D_y + cD_1^2 \qquad c \neq 0 \tag{2.12}$$

one verifies easily that $Q(D)$ does not cover Δ in the usual sense (cf. Section 4 or [1, 2, 3, 14, 16]). The reason is that the highest order term of $Q(D)$ is tangential and hence the characteristic polynomial of $Q(D)$ vanishes for some $\xi \neq 0$ (in fact whenever $\xi_1 = 0$). But nevertheless, Theorem 2.1 applies. Inequality (2.10) holds trivially and hence we have

$$|v|_{s+2} + |D_y v|_{s+1} + |D_y^2 v|_s \leq \text{const. } (|\Delta v|_s + \langle D_y v + cD_1^2 v\rangle_{s+\frac{1}{2}} + |v|_{s-b}).$$

One might explain this situation by saying that the boundary condition (2.12) is not elliptic with respect to the Laplacian but is hypoelliptic with respect to it (Hörmander [7, p. 245]).

3. Fourier Transform Methods.

We now give the proof of Theorem 2.1. Assume that $P(\xi, \eta)$ and the $Q_1(\xi, \eta), \dots, Q_r(\xi, \eta)$ satisfy Hypotheses 1-4 and let $R(\xi, \eta)$ be any polynomial weaker than $P(\xi, \eta)$. For $|\xi| > K_2$ it clearly satisfies

$$(3.1) \qquad |R(\xi, \eta)| \leq \text{const.} \Sigma \, |P^{(\mu)}(\xi, \eta)| \leq \text{const.} \, |P(\xi, \eta)| \, .$$

We consider functions $u(\xi, y)$ as functions of y with ξ as a (vector) parameter. We let $H^m(E^1)$ denote the completion of $C_0^\infty(E^1)$ with respect to the norm

$$\| u \|_m = \left(\sum_{k=0}^{m} \int_{-\infty}^{\infty} | D_y^k u(\xi, y)|^2 \, dy \right)^{\frac{1}{2}} .$$

For a particular $u(\xi, y) \in C_0^\infty(\overline{E}_+^1)$ set

$$U_j(\xi) = \sqrt{2\pi}\, Q_j(\xi, D_y)\, u(\xi, 0), \qquad 1 \leq j \leq r,$$

and let U be the column vector with components U_j. We are going to show that

$$(3.2) \qquad \int_0^\infty |R(\xi, D_y)\, u(\xi, y)|^2 \, dy \leq C \left\{ \int_0^\infty |P(\xi, D_y)\, u(\xi, y)|^2 \, dy + U^* A^{-1} U \right\}$$

for all $u(\xi, y) \in C_0^\infty(\overline{E}_+^1)$ and real ξ such that $|\xi| > K_2$.

For $|\xi| \leq K_2$ a simpler inequality holds. In fact we have

$$\int_0^\infty |D_y^m u(\xi, y)|^2 \, dy \leq K_3^2 \left\{ \int_0^\infty |P(\xi, D_y)\, u(\xi, y)|^2 \, dy + C^2 \sum_{k=0}^{m-1} \int_0^\infty |D_y^k u(\xi, y)|^2 \, dy \right\}$$

where C is an upper bound for the coefficients of $P(\xi, D_y)$ on the set $|\xi| \leq K_2$ (recall that K_3 is an upper bound for $|a(\xi)|^{-1}$, where $a(\xi)$ is the coefficient of η^m in $P(\xi, \eta)$). Thus

$$\sum_{k=0}^{m} \int_0^\infty |D_y^k u(\xi, y)|^2 dy \leq K \left\{ \int_0^\infty |P(\xi, D_y) u(\xi, y)|^2 dy + \sum_{k=0}^{m-1} \int_0^\infty |D_y^k u(\xi, y)|^2 dy \right\}.$$

Employing the well known inequality (cfr. [4, 11])

$$\sum_{k=1}^{m-1} \int_0^\infty |D_y^k u(\xi, y)|^2 dy \leq \varepsilon \int_0^\infty |D_y^m u(\xi, y)|^2 dy + K_\varepsilon \int_0^\infty |u(\xi, y)|^2 dy,$$

and taking $\varepsilon < \frac{1}{2} K^{-1}$ we have

$$\sum_{k=0}^{m} \int_0^\infty |D_y^k u(\xi, y)|^2 dy \leq K' \left\{ \int_0^\infty |P(\xi, D_y) u(\xi, y)|^2 dy + \int_0^\infty |u(\xi, y)|^2 dy \right\}.$$

Thus

$$\int_0^\infty |R(\xi, D_y) u(\xi, y)|^2 dy \tag{3.3}$$
$$\leq K'' \left\{ \int_0^\infty |P(\xi, D_y) u(\xi, y)|^2 dy + \int_0^\infty |u(\xi, y)|^2 dy \right\}$$

for all $u(\xi, y) \in C_0^\infty(\bar{E}_+^1)$ and $|\xi| \leq K_2$, where K'' depends only on bounds for $|a(\xi)|^{-1}$ and the coefficients of $R(\xi, D_y)$ and $P(\xi, D_y)$ on $|\xi| \leq K_2$.

Once (3.2) is proved we see, in view of (3.3), that

$$\int_{|\xi|<\infty} (1 + |\xi|^2)^s \int_0^\infty |R(\xi, D_y) u(\xi, y)|^2 dy\, d\xi \tag{3.4}$$
$$\leq C \left\{ \int_{|\xi|<\infty} (1 + |\xi|^2)^s \int_0^\infty |P(\xi, D_y) u(\xi, y)|^2 dy\, d\xi \right.$$

$$+\int_{|\xi|>K_2}(1+|\xi|^2)^s\,U^*A^{-1}U\,d\xi$$
$$+(1+K_2^2)^b\int_{|\xi|<K_2}(1+|\xi|^2)^{s-b}\int_0^\infty |u(\xi,y)|^2\,dy\,d\xi\Big\}.$$

This immediately implies Theorem 2.1. For let $v(x,y)$ be any function in $C_0^\infty(\overline{E}_+^n)$ and let $u(\xi,y)$ be its Fourier transform with respect to $x_1,\ldots,x_{n-1}$. Substituting into (3.4) we obtain the desired result.

The proof of Theorem 2.1 can therefore be made to depend on (3.2). We now prove (3.2) by a method due to Peetre [13].

Let $u(\xi,y)$ be any function in $C_0^\infty(\overline{E}_+^1)$. The trick is to find an extension $u_1(\xi,y)$ of u to $H^m(E^1)$ such that

$$\int_{-\infty}^{\infty}|P(\xi,D_y)\,u_1(\xi,y)|^2\,dy\le C\Big\{\int_0^\infty|P(\xi,D_y)\,u(\xi,y)|^2\,dy+U^*A^{-1}U\Big\}\tag{3.5}$$

for $|\xi|>K_2$, where C is independent of u and ξ. For then by (3.1) and Parseval's identity

$$\int_0^\infty|R(\xi,D_y)\,u(\xi,y)|^2\,dy\le\int_{-\infty}^{\infty}|R(\xi,D_y)\,u_1(\xi,y)|^2\,dy$$
$$=\int_{-\infty}^{\infty}|R(\xi,\eta)\,\tilde{u}_1(\xi,\eta)|^2\,d\eta\le\text{const.}\int_{-\infty}^{\infty}|P(\xi,\eta)\,\tilde{u}_1(\xi,\eta)|^2\,d\eta$$
$$=\text{const.}\int_{-\infty}^{\infty}|P(\xi,D_y)\,u_1(\xi,y)|^2\,dy,$$

where

$$\tilde{h}(\xi,\eta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\eta y}\,h(\xi,y)\,dy$$

denotes the Fourier transform with respect to y. This gives (3.2) when combined with (3.5).

Let $u_1(\xi,y)$ be an extension of $u(\xi,y)$ to $H^m(E^1)$, i. e. a function in $H^m(E^1)$ which equals u on $\overline{E}_+^1$. Define

$$f(\xi,y)=\begin{cases}P(\xi,D_y)\,u(\xi,y) & \text{for } y\ge 0\\ 0 & \text{for } y<0\end{cases}$$

and set

$$g(\xi, y) = P(\xi, D_y)\, u_1(\xi, y) - f(\xi, y).$$

We have

$$P(\xi, \eta)\, \tilde{u}_1(\xi, \eta) = [P(\xi, D_y)\, u_1(\xi, y)]^{\sim} = \tilde{f} + \tilde{g}$$

and hence by Parseval's relation

$$\int_{-\infty}^{\infty} |P(\xi, D_y)\, u_1(\xi, y)|^2\, dy = \int_{-\infty}^{\infty} |\tilde{f}|^2\, d\eta + \int_{-\infty}^{\infty} |\tilde{g}|^2\, d\eta.$$

The problem is now reduced to proving

$$\int_{-\infty}^{\infty} |\tilde{g}|^2\, d\eta \leq C \left\{ \int_{-\infty}^{\infty} |\tilde{f}|^2\, d\eta + U^* A^{-1} U \right\}, \tag{3.6}$$

where the constant C is not permitted to depend upon u, u_1 or ξ.

In proving (3.6) we shall employ the two identities

$$\int_{-\infty}^{\infty} \frac{Q}{P}\, \tilde{f}\, d\eta = \sqrt{2\pi}\, Q^-(\xi, D_y)\, u(\xi, 0) \tag{3.7}$$

$$\int_{-\infty}^{\infty} \frac{Q}{P}\, \tilde{g}\, d\eta = \sqrt{2\pi}\, Q^+(\xi, D_y)\, u(\xi, 0) \tag{3.8}$$

holding for any polynomial $Q(\xi, \eta)$ of degree $< m$ in η, where

$$\frac{Q}{P} = \frac{Q_+}{P_+} + \frac{Q_-}{P_-}, \quad Q^+ = Q_+ P_-, \quad Q^- = Q_- P_+$$

We note that the second relation (3.8) follows from the first and the inversion formula for Fourier transforms:

$$h(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iy\eta}\, \tilde{h}(\xi, \eta)\, d\eta. \tag{3.9}$$

In fact

$$\int_{-\infty}^{\infty} \frac{Q}{P} \tilde{g}\, d\eta = \int_{-\infty}^{\infty} \frac{Q}{P} \{[P(\xi, D_y) u_1]^{\sim} - \tilde{f}\}\, dy$$

$$= \int_{-\infty}^{\infty} [Q(\xi, D_y) u_1]^{\sim} d\eta - \int_{-\infty}^{\infty} \frac{Q}{P} \tilde{f}\, d\eta = \sqrt{2\pi}\, Q(\xi, D_y) u_1(\xi, 0)$$

$$- \sqrt{2\pi}\, Q^-(\xi, D_y) u(\xi, 0) = \sqrt{2\pi}\, Q^+(\xi, D_y) u(\xi, 0),$$

since $Q = Q^+ + Q^-$.

In order to prove (3.7) we define $u(\xi, y)$ to be zero for $y < 0$ and observe that

$$[D_y u]^{\sim} = \eta \tilde{u} + \frac{i}{\sqrt{2\pi}} u(\xi, 0), \tag{3.10}$$

which is easily obtained by integration by parts. Secondly, we note that when $h(\xi, y)$ is discontinuous, the left hand side of (3.9) should be replaced by $\frac{1}{2}[h(\xi, y+) + h(\xi, y-)]$. By taking $h = u$ and $y = 0$ we have

$$u(\xi, 0) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \tilde{u}(\xi, \eta)\, d\eta, \tag{3.11}$$

where the integral is taken in the Cauchy principal value sense.

Set

$$P_k(\xi, \eta) = \frac{P(\xi, \eta)}{\eta - \tau_k(\xi)}, \qquad 1 \leq k \leq m.$$

Then by (3.10)

$$\tilde{f} = (\eta - \tau_k)[P_k(\xi, D_y) u]^{\sim} + \frac{i}{\sqrt{2\pi}} P_k(\xi, D_y) u(\xi, 0).$$

Expanding into partial fractions we have [2]

$$\frac{Q}{P} = \sum_{k=1}^{m} \frac{q_k(\xi)}{\eta - \tau_k(\xi)}$$

[2] For simplicity we assume that the roots $\tau_k(\xi)$ are simple. The reader will have no difficulty filling in the details for the general case.

and note that

$$\frac{Q^-}{P} = \frac{Q_-}{P_-} = \sum_{k=r+1}^{m} \frac{q_k}{\eta - \tau_k}.$$

Hence

$$Q^- = \sum_{k=r+1}^{m} q_k P_k.$$

Substituting the above expressions in the left hand side of (3.7) we obtain

$$\int_{-\infty}^{\infty} \frac{Q}{P} \tilde{f}\, d\eta = \sum_{k=1}^{m} q_k \int_{-\infty}^{\infty} \frac{\tilde{f}}{\eta - \tau_k}\, d\eta = \sum_{k=1}^{m} q_k \int_{-\infty}^{\infty} [P_k(\xi, D_y)\, u]^{\sim}\, d\eta$$

$$+ \frac{i}{\sqrt{2\pi}} \sum_{k=1}^{m} q_k P_k(\xi, D_y)\, u(\xi, 0) \int_{-\infty}^{\infty} \frac{d\eta}{\eta - \tau_k}$$

$$= \sqrt{\frac{\pi}{2}} \sum_{k=1}^{m} q_k P_k(\xi, D_y)\, u(\xi, 0) + \frac{i}{\sqrt{2\pi}} \sum_{k=1}^{m} q_k P_k(\xi, D_y)\, u(\xi, 0)\, (\pi i \operatorname{sgn} I_m \tau_k)$$

$$= \sqrt{2\pi} \sum_{k=r+1}^{m} q_k P_k(\xi, D_y)\, u(\xi, 0) = \sqrt{2\pi}\, Q^-(\xi, D_y)\, u(\xi, 0),$$

where we have employed (3.11). This gives (3.7).

Once (3.7) and (3.8) are known, we proceed as follows. Write $\tilde{g}$ in the form

$$\tilde{g} = \sum_{j=1}^{r} \lambda_j\, \overline{Q_{j+}} / \overline{P_+} + \tilde{\psi} \tag{3.12}$$

where the coefficients λ_j depend on ξ while ψ satisfies

$$\int_{-\infty}^{\infty} \frac{Q_{j+}}{P_+} \tilde{\psi}\, d\eta = 0, \qquad 1 \leq j \leq r. \tag{3.13}$$

We are going to show that we may always choose u_1 so that $\psi \equiv 0$. Assuming this for the moment, we proceed. By (3.8)

$$\int_{-\infty}^{\infty} \frac{Q_{j+}}{P_+} \tilde{g}\, d\eta = \sqrt{2\pi}\, Q_j^+(\xi, D_y)\, u(\xi, 0) \equiv U_j^+.$$

But by (3.12)

$$\int_{-\infty}^{\infty} \frac{Q_{j+}}{P_+} \tilde{g}\, d\eta = \sum_{j=1}^{r} \alpha_{jk} \lambda_k$$

and hence

$$\lambda_i = \sum_{j=1}^{r} \alpha^{ij}\, U_j^+,$$

where (α^{ij}) is the inverse of A. This gives

$$\begin{aligned} \int_{-\infty}^{\infty} |\tilde{g}|^2\, d\eta &= \sum_{i=1}^{r} \overline{\lambda}_i \int \frac{Q_{i+}}{P_+} \tilde{g}\, d\eta \\ &= \sum_{i,j=1}^{r} \overline{\alpha}^{ij}\, U_i^+\, \overline{U}_j^+ = U^{+*} A^{-1} U^+, \end{aligned} \tag{3.14}$$

where U^+ is the column vector with components U_j^+. (We have employed the fact that A is Hermitian). This is almost what we want. In fact it gives (3.2) with U replaced by U^+. Of course the Q_j^+ automatically satisfy Hypothesis 4 if A^{-1} exists.

A simple argument now gives us the form we desire. Set $U^- = U - U^+$. Then

$$U^{+*}A^{-1}U^+ = (U - U^-)^* A^{-1} (U - U^-) \leq K(U^* A^{-1} U + U^{-*} A^{-1} U^-).$$

The idea is to estimate the unwanted expression $U^{-*} A^{-1} U^-$ in terms of $\tilde{f}$. This may be done as follows. Write $\tilde{f}$ in the form

$$\tilde{f} = \sum_{j=1}^{r} \gamma_j \overline{Q_{j-}} / \overline{P_-} + \tilde{\Phi},$$

where

$$\int_{-\infty}^{\infty} \frac{Q_{j-}}{P_-} \tilde{\Phi}\, d\eta = 0, \qquad 1 \leq j \leq r,$$

and set $f_1 = f - \Phi$. Then

$$\int_{-\infty}^{\infty} |\tilde{f}_1|^2\, d\eta = \sum_{j=1}^{r} \overline{\gamma}_j \int_{-\infty}^{\infty} \frac{Q_{j-}}{P_-} \tilde{f}\, d\eta = \sum_{i,j=1}^{r} \beta_{ij} \gamma_i \overline{\gamma}_j = \gamma^* B \gamma,$$

where γ is the column vector with components γ_j. Now in view of Hypothesis 4

$$\gamma^* BA^{-1} B\gamma \leq K_4 \int_{-\infty}^{\infty} |\tilde{f}_1|^2 \, d\eta \leq K_4 \int_{-\infty}^{\infty} |\tilde{f}|^2 \, d\eta .$$

We now merely observe that $B\gamma = U^-$ since

$$\int_{-\infty}^{\infty} \frac{Q_{j-}}{P_-} \tilde{f} \, d\eta = U_j^- , \qquad 1 \leq j \leq r,$$

by (3.7). Hence

$$U^{-*} A^{-1} U^- \leq K_4 \int_{-\infty}^{\infty} |f|^2 \, dy. \tag{3.15}$$

This, combined with (3.14) gives the final result.

It therefore remains only to show that we may choose u_1 in such a way that $\psi = 0$. We begin by taking u_1 in $C_0^\infty(E^1)$. In this case g is infinitely differentiable in $y \leq 0$ and vanishes for $-y$ large. One then verifies easily that $\tilde{g}$ is bounded in $\operatorname{Im} \eta \geq 0$ and

$$\tilde{g} \to 0 \quad \text{as} \quad \operatorname{Im} \eta \to \infty.$$

The same is therefore true for the corresponding $\tilde{\psi}$. A simple contour integration shows that

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\tilde{\psi} \, d\eta}{\eta - \tau_k} = \tilde{\psi}(\xi, \tau_k), \qquad 1 \leq k \leq r.$$

But every polynomial Q of degree $< r$ in η can be expressed in the form

$$Q = \sum_{j=1}^{r} c_j Q_{j+} ,$$

where the c_j depend only on ξ. In particular this is true for $Q = P_k / P_-$, $1 \leq k \leq r$. Hence by (3.13)

$$\int_{-\infty}^{\infty} \frac{\tilde{\psi} \, d\eta}{\eta - \tau_k} = 0, \qquad 1 \leq k \leq r.$$

Thus $\widetilde{\psi}(\xi, \tau_k) = 0$ for each of the roots with positive imaginary parts. This shows that

$$\widetilde{w} = \frac{\widetilde{\psi}}{P}$$

is bounded in $\operatorname{Im} \eta \geq 0$. Moreover, w is in $H^m(E^1)$ since

$$(1 + \eta^2)^m \mid \widetilde{w} \mid^2 = \frac{(1 + \eta^2)^m}{\mid P \mid^2} \mid \widetilde{\psi} \mid^2$$

and $(1 + \eta^2)^m \mid P \mid^{-2}$ is bounded (cf. [20]). Hence by the Paley-Wiener theorem (cf. [12, p. 11]) w vanishes for $y \geq 0$. Since

$$[P(\xi, D_y)\, w]^{\sim} = P(\xi, \eta)\, \widetilde{w} = \widetilde{\psi},$$

$P(\xi, D_y)\, w$ equals ψ. Setting $u_1' = u_1 - w$, we see that u_1' is an extension of u to $H^m(E^1)$. In addition $g' = P(\xi, D_y)\, u_1' - f = g - \psi$ and we see that u_1' has the desired properties. This completes the proof of Theorem 2.1.

PROOF OF COROLLARY 2.1. We merely let d be any exponent such that

$$A^{-1} \leq \text{const.}\, (1 + \mid \xi \mid^2)^d\, I,$$

where I is the identity matrix. This immediately gives (2.6).

4. The Elliptic Case.

Coerciveness inequalities for elliptic operators have proved to be very useful tools in the study of boundary value problems and other investigations (cf., e. g., [2, 3, 9, 14, 17, 18]). Such inequalities for various situations have been proved by several authors (cf., e. g., [1, 2, 3, 14, 16]). In the case of L^2 estimates for one operator, the usual method is to reduce the problem to showing that

$$\mid R(D)\, v \mid_0 \leq C \left(\mid P(D)\, v \mid_0 + \sum_{j=1}^{r} \langle Q_j(D)\, v \rangle_{m - m_j - \frac{1}{2}} \right) \tag{4.1}$$

holds for all $v \in C_0^\infty(\bar{E}_+^n)$ under the following assumptions:

a) $P(\xi, \eta)$ and the $Q_j(\xi, \eta)$ are homogeneous polynomials of degree m and $m_j < m$, respectively.

b) $P(\xi, \eta)$ is elliptic, i. e., there are no real ξ, η satisfying $|\xi|^2 + \eta^2 \neq 0$ and $P(\xi, \eta) = 0$.

c) $P(\xi, \eta)$ is properly elliptic, i. e., is of determined type. Thus m is even and $r = m/2$.

d) The $Q_j(D)$ cover $P(D)$, i. e., they are linearly independent modulo P_+.

e) $R(\xi, \eta)$ is any homogeneous polynomial of degree m.

Actually, assumption *c*) is needed only for $n = 2$. A simple argument shows that every elliptic operator is properly elliptic in dimensions higher than the second (cf. [10, 2]).

The reduction of the problem to (4.1) is standard and easily carried out; the major difficulty is in proving (4.1).

Of course the inequality (4.1) is merely a special case of (2.6) and follows immediately from Theorem 2.1. However, when one is concerned only with proving (4.1), the argument can be made even simpler than that of the last section. The proof given below is much simpler than any presently found in the literature. The only result of Section 3 which we shall employ is formula (3.7) (actually even this is not needed).

We first note that the function $R(\xi, \eta)/P(\xi, \eta)$ is continuous on the surface $|\xi|^2 + \eta^2 = 1$ in E^n. Hence there is a constant K_5 such that

$$|R(\xi, \eta)| \leq K_5 |P(\xi, \eta)| \tag{4.2}$$

for all such ξ, η. By homogeneity this extends to all ξ, η such that $|\xi|^2 + \eta^2 \neq 0$.

Secondly we observe that by multiplying each $Q_j(\xi, \eta)$ by an appropriate power of $|\xi|$, we may assume that each $Q_j(\xi, \eta)$ is homogeneous (3) of degree $m - 1$. Resolving into partial fractions we have (4)

$$\frac{R}{P} = \sum_{k=1}^{m} \frac{e_k}{\eta - \tau_k}, \quad \frac{Q_j}{P} = \sum_{k=1}^{m} \frac{q_{jk}}{\eta - \tau_k},$$

where

$$e_k(\xi) = R(\xi, \tau_k) \Big/ \frac{\partial P(\xi, \tau_k)}{\partial \eta}, \quad q_{jk}(\xi) = Q_j(\xi, \tau_k) \Big/ \frac{\partial P(\xi, \tau_k)}{\partial \eta},$$

$$1 \leq j \leq r, \quad 1 \leq k \leq m.$$

(3) The Q_j may no longer be polynomials in the ξ_i, but this does not affect the argument.

(4) Cf. footnote 2.

One easily verifies from the homogeneity of $P(\xi, \eta)$ that each root $\tau_k(\xi)$ is homogeneous in ξ of degree one. It follows, therefore, that the same is true of each $e_k(\xi)$. Likewise each $q_{ik}(\xi)$ is homogeneous in ξ of degree zero. In particular, it follows that there are constants K_6 and K_7 such that

$$|e_k(\xi)| \leq K_6 |\xi|, \qquad 1 \leq k \leq m \tag{4.3}$$

$$K_7^{-1} |\xi| \leq |\operatorname{Im} \tau_k(\xi)| \leq K_7 |\xi|, \qquad 1 \leq k \leq m. \tag{4.4}$$

Let $u(\xi, y)$ be the Fourier transform of $v(x, y)$ with respect to the variables $x_1, \dots, x_{n-1}$ and define it to be zero for $y < 0$. As before we consider $u(\xi, y)$ as a function of y with ξ a parameter. We define f as in Section 3. A simple integration by parts gives

$$\tilde{f} = (\eta - \tau_k)[P_k(\xi, D_y) u]^{\sim} + W_k, \qquad 1 \leq k \leq m, \tag{4.5}$$

where

$$W_k = \frac{i}{\sqrt{2\pi}} P_k(\xi, D_y) u(\xi, 0), \qquad 1 \leq k \leq m.$$

(Recall that $\tilde{f}$ is the Fourier transform of f with respect to y and $P_k = P/(\eta - \tau_k)$). Since

$$R = \sum_{k=1}^{m} e_k P_k.$$

we have by (4.5)

$$\begin{aligned}[R(\xi, D_y) u]^{\sim} &= \sum_{k=1}^{m} e_k [P_k(\xi, D_y) u]^{\sim} \\ &= \sum_{k=1}^{m} e_k \frac{\tilde{f} - W_k}{\eta - \tau_k} = \frac{R}{P} \tilde{f} - \sum_{k=1}^{m} \frac{e_k W_k}{\eta - \tau_k}\end{aligned}$$

and hence

$$|[R(\xi, D_y) u]^{\sim}| \leq K_5 |\tilde{f}| + K_6 |\xi| \sum_{k=1}^{m} \frac{|W_k|}{|\eta - \tau_k|}.$$

Thus by Parseval's formula and (4.4)

$$\begin{aligned}\int_0^\infty |R(\xi, D_y) u|^2 dy &\leq C\left(\int_{-\infty}^{\infty} |\tilde{f}|^2 d\eta + |\xi|^2 \sum_{k=1}^{m} |W_k|^2 \int_{-\infty}^{\infty} \frac{d\eta}{|\eta - \tau_k|^2}\right) \\ &\leq C'\left(\int_{-\infty}^{\infty} |\tilde{f}|^2 d\eta + |\xi| \sum_{k=1}^{m} |W_k|^2\right).\end{aligned} \tag{4.6}$$

Since $\operatorname{Im} \tau_k < 0$ for $r < k \leq m$ we have by (3.7)

$$\int_{-\infty}^{\infty} \frac{\tilde{f}}{\eta - \tau_k}\, d\eta = \int_{-\infty}^{\infty} \frac{P_k}{P} \tilde{f}\, d\eta = -\, 2\pi i\, W_k, \qquad r < k \leq m.$$

Hence by (4.4) and Schwarz's inequality

$$|\, W_k\, |^2 \leq \text{const.}\ |\, \xi\, | \int_{-\infty}^{\infty} |\tilde{f}\,|^2\, d\eta, \qquad r < k \leq m.$$

Therefore (4.6) becomes

$$\int_0^{\infty} |\, R\,(\xi,\, D_y)\, u\, |^2\, dy \leq C \left(\int_{-\infty}^{\infty} |\tilde{f}\,|^2\, d\eta + |\, \xi\, | \sum_{k=1}^{r} |\, W_k\, |^2 \right). \tag{4.7}$$

Next we observe that there is no complex vector $\omega = (\omega_1, \dots, \omega_r) \neq 0$ such that

$$\sum_{k=1}^{r} q_{jk}\, \omega_k = 0, \qquad 1 \leq j \leq r. \tag{4.8}$$

For otherwise there would be a complex vector $\lambda = (\lambda_1, \dots, \lambda_r) \neq 0$ such that

$$\sum_{j=1}^{r} \lambda_j\, q_{jk} = 0, \qquad 1 \leq k \leq r,$$

and hence

$$P^{-1} \sum_{j=1}^{r} \lambda_j\, Q_j = \sum_{j=1}^{r} \sum_{k=1}^{m} \frac{\lambda_j\, q_{jk}}{\eta - \tau_k} = \sum_{k=r+1}^{m} \sum_{j=1}^{r} \frac{\lambda_j\, q_{jk}}{\eta - \tau_k}$$

showing that $\Sigma\, \lambda_j\, Q_j$ is a multiple of P_+, contradicting assumption *d*). Thus the expression

$$\sum_{j=1}^{r} \left| \sum_{k=1}^{r} q_{jk}\, \omega_k \right|^2$$

is positive on the compact set $\sum_1^r |\, \omega_k\, |^2 = 1,\ |\, \xi\, | = 1$. Hence by the homogeneity properties of the q_{jk} we have

$$\sum_{k=1}^{r} |\, \omega_k\, |^2 \leq K_8 \sum_{j=1}^{r} \left| \sum_{k=1}^{r} q_{jk}\, \omega_k \right|^2 \tag{4.9}$$

for all ω and ξ. Since

$$U_j^+ = \sqrt{2\pi}\, Q_j^+ (\xi, D_y)\, u(\xi, 0) = -\, 2\pi i \sum_{k=1}^{r} q_{jk}\, W_k\,, \qquad 1 \leq j \leq r$$

we have by (4.7) and (4.9)

$$\int_0^\infty | R(\xi, D_y)\, u |^2\, dy \leq C \left(\int_{-\infty}^{\infty} | \tilde{f} |^2\, d\eta + | \xi | \sum_{j=1}^{r} | U_j^+ |^2 \right). \tag{4.10}$$

Now by another application of (3.7)

$$U_j^- = \int_{-\infty}^{\infty} \frac{Q_j}{P} \tilde{f}\, d\eta, \qquad 1 \leq j \leq r,$$

and hence

$$| U_j^- |^2 \leq \int_{-\infty}^{\infty} \left| \frac{Q_j}{P} \right|^2 d\eta \int_{-\infty}^{\infty} | \tilde{f} |^2\, d\eta, \qquad 1 \leq j \leq r.$$

One easily checks that the expression

$$| \xi | \int_{-\infty}^{\infty} \left| \frac{Q_j}{P} \right|^2 d\eta$$

is homogeneous in ξ of degree zero and hence is bounded by a constant for all real ξ. Thus

$$| \xi | \sum_{j=1}^{r} | U_j^- |^2 \leq K_9 \int_{-\infty}^{\infty} | \tilde{f} |^2\, d\eta. \tag{4.11}$$

Combining (4.10) and (4.11) and employing the triangle inequality we finally obtain

$$\int_0^\infty | R(\xi, D_y)\, u |^2\, dy \leq C \left(\int_{-\infty}^{\infty} | \tilde{f} |^2\, d\eta + | \xi | \sum_{j=1}^{r} | U_j |^2 \right). \tag{4.12}$$

If we now integrate with respect to ξ we obtain (4.1). This completes the proof.

5. Regularity.

The power of Theorem 2.1 is in its ability to prove regularity up to the boundary of solutions of equations with variable coefficients. We shall save a detailed discussion of this for a future publication. At the moment we shall content ourselves with proving regularity of solutions of constant coefficient equations. This will illustrate the ideas without getting involved in technical difficulties.

Let $P(D)$ and $Q_j(D)$, $1 \leq j \leq r$, be given constant coefficient operators satisfying Hypotheses 1-4 of Section 2. For the sake of simplicity we shall even assume that Hypothesis 2 is replaced by

Hypothesis 2′. The coefficient of the highest power of η in $P(\xi, \eta)$ is a constant.

We shall consider solutions $v(x, y)$ of

$$P(D)\, v = f \qquad \text{in } E^n_+ \tag{5.1}$$

$$Q_j(D) v = f_j \qquad \text{on } E^{n-1}, \qquad 1 \leq j \leq r, \tag{5.2}$$

where E^{n-1} is considered as the hyperplane $y = 0$ in E^n. For each integer $k \geq 0$ let $C^k(\overline{E}^n_+)$ denote the set of functions having derivatives up to order k continuous in E^n_+. For each real s we let $T^s(E^n_+)$ denote the completion $C_0^\infty(\overline{E}^n_+)$ with respect to the norm $|\cdot|_s$. Clearly each $T^s(E^n_+)$ is a Hilbert space.

Let q be the highest order of the operators $P(D)$ and the $Q_j(D)$. We have

THEOREM 5.1. *Assume that $f \in C_0^\infty(\overline{E}^n_+)$ and that each $f_j \in C_0^\infty(E^{n-1})$. If $v \in C^q(\overline{E}^n_+) \cap T^0(E^n_+)$ is a solution of* (5.1), (5.2), *then v is infinitely differentiable in $\overline{E}^n_+$.*

In proving Theorem 5.1 we shall find it convenient to employ Friedrichs' mollifiers [5] with respect to the variables $x_1, \dots, x_{n-1}$. Let $j(x)$ be any non-negative function in $C_0^\infty(E^{n-1})$ which vanishes for $|x| > 1$ and such that

$$\int j(x)\, dx = 1.$$

Set

$$j_\varepsilon(x) = \varepsilon^{1-n} j(x/\varepsilon), \qquad \varepsilon > 0,$$

and

$$J_\varepsilon w(x) = \int j_\varepsilon (x - z)\, w(z)\, dz$$

for any function $w \in L^2(E^{n-1})$. Since

$$| J_\varepsilon w(x) - w(x) | = \left| \int j_\varepsilon (x - z) [w(z) - w(x)]\, dz \right|$$

$$\leq \max_{|z - x| < \varepsilon} | w(z) - w(x) |,$$

we see that $J_\varepsilon w$ approaches w as $\varepsilon \to 0$ uniformly on any compact set where w is continuous.

We next note that for $v \in T^s(E^n_+)$

(5.3) $$| J_\varepsilon v |_s \leq | v |_s, \qquad \langle J_\varepsilon v \rangle_s \leq \langle v \rangle_s,$$

This follows from the fact that the Fourier transform of $J_\varepsilon v$ is

$$(2\pi)^{\frac{1-n}{2}} \int e^{-i\xi x \varepsilon} j(x)\, dx$$

times the Fourier transform of v and

$$\left| \int e^{-i\xi x \varepsilon} j(x)\, dx \right| \leq \int j(x)\, dx = 1.$$

The same reasoning shows that for $v \in T^0(E^n_+)$ and each fixed $\varepsilon > 0$ the function $J_\varepsilon v$ is in $T^s(E^n_+)$ for every s.

Our main step in proving Theorem 5.1 is to establish

LEMMA 5.1. *Under the hypotheses of Theorem* 5.1 *for each s there is a constant K such that*

(5.4) $$\sum_{k=0}^{m} | D_y^k J_\varepsilon v |_s \leq K$$

for all $\varepsilon > 0$, where m is the order of $P(D)$ with respect to y.

We assume Lemma 5.1 for the moment and show how it implies Theorem 5.1. Note that the constant K depends on v and s but not on ε.

We first make use of the Sobolev type inequality

$$\max_{\bar{E}^n_+} | v(x, y) | \leq K(| v |_t + | D_y v |_t) \tag{5.5}$$

holding for all $v \in C^1(\bar{E}^n_+) \cap T^t(E^n_+)$ whenever $t > (n-1)/2$. This follows from the fact that

$$v(x, y) = (2\pi)^{\frac{1-n}{2}} \int e^{i\xi x} u(x, \xi)\, d\xi,$$

where $u(\xi, y)$ is the Fourier transform of $v(x, y)$ with respect to x. Thus

$$| v(x, y) | \leq \text{const.} \int (1 + | \xi |^2)^{-t/2} (1 + | \xi |^2)^{t/2} | u(\xi, y) | \, d\xi$$

and

$$| v(x, y) |^2 \leq \text{const.} \int \frac{d\xi}{(1 + | \xi |^2)^t} \int (1 + | \xi |^2)^t | u(\xi, y) |^2 \, d\xi.$$

Since

$$\int \frac{d\xi}{(1 + | \xi |^2)^t} < \infty$$

and

$$| u(\xi, y) |^2 \leq \text{const.} \left(\int_0^\infty (| u(\xi, y) |^2 + | D_y u(\xi, y) |^2)\, dy \right),$$

(5.5) follows immediately.

Applying (5.5) to (5.4) for large s we have

$$\max_{\bar{E}^n_+} | D^\mu_x J_s v | \leq K$$

for $| \mu | + (n-1)/2 < s$, where D^μ_x denotes differentiation with respect to the variables x only. We shall refer to it as an x-derivative. We see that the family of functions $\{ J_s v \}$ has bounded x-derivatives of orders $< s - (n-1)/2$ in $\bar{E}^n_+$. Hence for each fixed y and each compact subset Φ of E^{n-1}, the x-derivatives of orders $< s - (n-1)/2 - 1$ are equicontinuous. Thus there is a subfamily such that all x-derivatives converge uniformly on Φ. Since $J_s v$ converges uniformly to v on Φ, it follows that v has x-derivatives of orders $< s - (n-1)/2 - 1$ on Φ. Since s and Φ were arbitrary, we see that v is infinitely differentiable with respect to x in $\bar{E}^n_+$.

It remains to consider derivatives with respect to y. By applying the same reasoning to derivatives $D_y^k v$ with $k < m$, we see that they too are infinitely differentiable with respect to x. Moreover, (5.1) can be written in the form

$$D_y^m v = c^{-1} f - \text{terms of derivatives of } v \text{ of orders} < m \text{ with respect to } y, \tag{5.6}$$

where c is the constant coefficient mentioned in Hypothesis 2′. From (5.6) we see that $D_y^m v$ is infinitely differentiable with respect to the x variables. Differentiating (5.6) with respect to y, we see that the same is true of $D_y^{m+1} v$. Continuing in this way we see that each derivative $D_y^k v$ exists and is infinitely differentiable with respect to x. This completes the proof of Theorem 5.1.

It now remains only to prove Lemma 5.1. By (2.6) and (5.3) we have

$$\begin{aligned} | R(D) J_\varepsilon v |_s &\leq C\left(| J_\varepsilon f |_s + \sum_{j=1}^{r} \langle J_\varepsilon f_j \rangle_{s+d} + | J_\varepsilon v |_{s-q} \right) \\ &\leq C' \left(| f |_s + \sum_{j=1}^{r} \langle f_j \rangle_{s+d} + | J_\varepsilon v |_{s-q} \right). \end{aligned} \tag{5.7}$$

(Here we have taken $l = q$ and made use of the fact that J_ε commutes with differentiation). We note that $\partial^m P(\xi, \eta)/\partial \eta^m$ is weaker than $P(\xi, \eta)$ and hence we may take $R(\xi, \eta) = 1$ in (5.7). This gives

$$| J_\varepsilon v |_s \leq C\left(| f |_s + \sum_{j=1}^{r} \langle f_j \rangle_{s+d} + | J_\varepsilon v |_{s-q} \right). \tag{5.8}$$

Since $v \in T^0(E_+^n)$, we have

$$| J_\varepsilon v |_q \leq \text{const.}$$

If we now reapply (5.8) for the values $s = 2q, 3q, \dots$, we obtain

$$| J_\varepsilon v |_s \leq \text{const.} \tag{5.9}$$

for each real s. We note next that $\partial^{m-1} P(\xi, \eta)/\partial \eta^{m-1}$ is weaker than $P(\xi, \eta)$ and is of the form

$$m!\, c\eta + p_1(\xi),$$

where $p_1(\xi)$ is a polynomial in ξ only. By (5.7) and (5.9) we see that

$$| m!\, cD_y J_\varepsilon v + p_1(D) J_\varepsilon v |_s \leq \text{const.}$$

for each real s. But

$$| p_1(D) J_\varepsilon v |_s \leq \text{const.}\ | J_\varepsilon v |_{s+q}$$

and hence another application of (5.9) gives

$$| D_y J_\varepsilon v |_s \leq \text{const.}$$

for each real s. Repeating the process m times we eventually come to (5.4). This completes the proof.

Added in proof. The methods of this paper can be applied to systems of equations as well. Moreover, Hypothesis 1 can be relaxed considerably. Details will be given in a future publication.

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