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ON $\mathfrak{p}$ -EQUATIONS AND NORMAL EXTENSIONS OF FINITE $\mathfrak{p}$ -TYPE (II) THE ANALOGY OF THE RIEMANN'S PROBLEM

HISASI MORIKAWA (*)

§ 1. Introduction.

1.1 Let $\mathcal{M}$ be a closed Riemann surface and Σ be the direct system of all the finite subsets in $\mathcal{M}$, where the order in Σ is defined by the set theoretical inclusion. If $S \subset S'$ ($S, S' \in \Sigma$), there exists the canonical homomorphism $\varphi_{S, S'}$ of the fundamental group $\pi_1(\mathcal{M} - S')$ onto the fundamental group $\pi_1(\mathcal{M} - S)$. We denote by $G(\mathcal{M})$ the inverse limit of $\{\pi_1(\mathcal{M} - S) \mid S \in \Sigma\}$ with respect to the homomorphisms $\{\varphi_{S, S'} \mid S \subset S'\}$ and by φ_S the canonical homomorphism of $G(\mathcal{M})$ onto $\pi_1(\mathcal{M} - S)$. We denote by $K(\mathcal{M})$ the field of meromorphic functions on $\mathcal{M}$ and by $\mathcal{D}(\mathcal{M})$ the set of all the linear homogeneous ordinary differential equations with coefficients in $K(\mathcal{M})$. We denote by $\Omega(\mathcal{M})$ the set of all the solutions of certain non-zero elements in $\mathcal{D}(\mathcal{M})$. Then it easily checked that $\Omega(\mathcal{M})$ is a commutative $K(\mathcal{M})$ -algebra by the usual sum, the product and the multiplication of the elements of $K(\mathcal{M})$. The topological group $G(\mathcal{M})$ operates continuously on the discrete ring $\Omega(\mathcal{M})$ as follows: Let f be any element in $\Omega(\mathcal{M})$ and σ be any element in $G(\mathcal{M})$. Let S be the set of all the singularities of f on $\mathcal{M}$ and $\gamma(\sigma)$ be the closed path on $\mathcal{M} - S$ of which homotopy class in $\pi_1(\mathcal{M} - S)$ is the image $\varphi_S(\sigma)$ of σ by the canonical homomorphism φ_S . Then the image f^σ of f by σ is defined by the analytic continuation of f along the closed path $\gamma(\sigma)$. Therefore we can regard $\Omega(\mathcal{M})$ as a $C[G(\mathcal{M})]$ -module, where $\mathbb{C}$ is the field

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of complex numbers and we mean by a $G(\mathcal{M})$ -module a discrete module on which $G(\mathcal{M})$ operates continuously.

In these notations and terminologies the classical Riemann problem is formulated as follows ⁽¹⁾ :

Does there exist a $\mathbb{C}[G(\mathcal{M})]$ -submodule in $\Omega(\mathcal{M})$ which is isomorphic to a given $\mathbb{C}[G(\mathcal{M})]$ -module of finite dimension over $\mathbb{C}$?

1.2 We shall explain the analogy of the Riemann's problem in the ring of Witt vectors. Let p be a prime number and A be field of characteristic p . Let $\bar{A}'$ be the separable algebraic closure of A and $G(A)$ be the Galois group of $\bar{A}'/A$, where $G(A)$ is considered as a discrete group. We mean by a Witt vector with coefficients in $\bar{A}'$ an infinite ordered set $(\alpha_0, \alpha_1, \alpha_2, \dots)$ of elements α_i ($i = 0, 1, 2, \dots$) in $\bar{A}'$. Putting $\mathbf{0} = (0, 0, 0, \dots)$, $\mathbf{1} = (1, 0, 0, \dots)$, $\mathbf{p} = (0, 1, 0, \dots)$, $\mathbf{p}^n = (0, \overset{n}{\dots}, 0, 1, 0, \dots)$, we write $\sum_{i=0}^{\infty} \alpha_i p^{-i} \mathbf{p}^i$ instead of $(\alpha_0, \alpha_1, \alpha_2, \dots)$. E. Witt introduced the sum, the difference and the product of two Witt vectors by means of systems of infinite polynomials with coefficients in the prime field $GF(p)$

$$\{\Phi_{+,i}(x_0, \dots, x_{i-1}; y_0, \dots, y_{i-1})\}, \{\Phi_{-,i}(x_0, \dots, x_{i-1}; y_0, \dots, y_{i-1})\},$$

$\{\Phi_{0,i}(x_0, \dots, x_{i-1}, y_0, \dots, y_{i-1})\}$ as follows ⁽²⁾ :

$$\left(\sum_{i=0}^{\infty} \alpha_i p^{-i} \mathbf{p}^i\right) + \left(\sum_{i=0}^{\infty} \beta_i p^{-i} \mathbf{p}^i\right) = \sum_{i=0}^{\infty} \gamma_{+,i} p^{-i} \mathbf{p}^i,$$

$$\left(\sum_{i=0}^{\infty} \alpha_i p^{-i} \mathbf{p}^i\right) - \left(\sum_{i=0}^{\infty} \beta_i p^{-i} \mathbf{p}^i\right) = \sum_{i=0}^{\infty} \gamma_{-,i} p^{-i} \mathbf{p}^i,$$

$$\left(\sum_{i=0}^{\infty} \alpha_i p^{-i} \mathbf{p}^i\right) \cdot \left(\sum_{i=0}^{\infty} \beta_i p^{-i} \mathbf{p}^i\right) = \sum_{i=0}^{\infty} \gamma_{\cdot,i} p^{-i} \mathbf{p}^i,$$

$$(1) \quad \gamma_{+,i} = \alpha_i + \beta_i + \Phi_{+,i}(\alpha_0, \dots, \alpha_{i-1}; \beta_0, \dots, \beta_{i-1}),$$

$$(2) \quad \gamma_{-,i} = \alpha_i - \beta_i + \Phi_{-,i}(\alpha_0, \dots, \alpha_{i-1}; \beta_0, \dots, \beta_{i-1}),$$

$$(3) \quad \gamma_{\cdot,i} = \alpha_i p^i \beta_i + \alpha_i \beta_0 p^i + \Phi_{\cdot,i}(\alpha_0, \dots, \alpha_{i-1}; \beta_0, \dots, \beta_{i-1}).$$

⁽¹⁾ The Riemann's problem formulated in the classical terminology can be seen [2], II, 365 p. p. 383-384.

⁽²⁾ See [3].

By means of these operations all the Witt vectors with coefficients in $\bar{A}'$ form a commutative integral domain $W(\bar{A}')$. We call $W(\bar{A}')$ the ring of Witt vectors with coefficients in $\bar{A}'$. For any subring A in $\bar{A}'$ the ring $W(A)$ of Witt vectors with coefficients in A is naturally considered as a subring of $W(\bar{A}')$. Since the ring Z_p of p-adic integers can be regarded as the ring of Witt vectors with coefficients in the prime field $GF(p)$, Z_p is considered as a subring of $W(\bar{A}')$. We denote by $K(\bar{A}')$ (resp. $K(A)$) the quotient field of $W(\bar{A}')$ (resp. $W(A)$). More generally for any subfield A in $\bar{A}'$ we denote by $K(A)$ the quotient field of the ring $W(A)$ of Witt vectors with coefficients in A . The discrete group $G(A)$ operates continuously on $W(\bar{A}')$ as follows:

$$(4) \quad \left(\sum_{i=0}^{\infty} \alpha_i p^{-i} p^i \right)^{\sigma} = \sum_{i=0}^{\infty} (\alpha_i p^{-i})^{\sigma} p^i, \quad (\sigma \in G(A)).$$

Hence $W(\bar{A}')$ is regarded as a $Z_p[G(A)]$ -module.

In these notations and terminologies the analogy of Riemann's problem is formulated as follows:

Does there exist a $Z_p[G(A)]$ -submodule in $W(\bar{A}')$ which is isomorphic to a given $Z_p[G(A)]$ -module of finite rank over Z_p ?

In the present paper we shall solve this problem. Our main theorem is as follows:

MAIN THEOREM. If $K(A)$ is transcendental over $\mathbb{Q}_p$, there exists a $Z_p[G(A)]$ -module in $W(\bar{A}')$ which is isomorphic to a given $Z_p[G(A)]$ -module of finite rank over Z_p .

§ 2. Proof of the main theorem.

2.1. We shall begin by the theorem on normal base⁽³⁾:

Let L/K be a finite separable normal extension of a field K . Then there exists a base (called normal base) of L/K which consists of all the conjugates of an element in L over K .

For a finite separable extension of L/K we denote by $G(L/K)$ the Galois group of L/K and by $Tr_{L/K}$ the trace map of L into K , i. e. $Tr_{L/K}(\alpha) = \sum_{\sigma \in G(L/K)} \alpha^{\sigma}, (\alpha \in L)$.

⁽³⁾ See some standard textbooks on algebra.

LEMMA 1. Let L/K be a finite separable normal extension. Then the trace map $\text{Tr}_{L/K}$ is surjective.

PROOF. Let $\{\omega^\sigma \mid \sigma \in G(L/K)\}$ be a normal base of L/K and a be any element in K . Then there exists a unique system $\{c_\sigma \mid \sigma \in G(L/K)\}$ of elements in K such that $a = \sum_{\sigma} c_\sigma \omega^\sigma$. Since $a^\tau = a$ for τ in $G(L/K)$, we have $c_\sigma = c_{\sigma\tau}$ for σ, τ in $G(L/K)$. This shows that $c_\sigma = c$ for every σ in $G(L/K)$ with an element c in K . Namely

$$a = \sum_{\sigma} (c\omega)^\sigma = \text{Tr}_{L/K}(c\omega).$$

LEMMA 2. Let L/K be a finite separable normal extension. Let L_1 and L_2 be normal subfields of L over K such that $L_1 \cap L_2 = K$. If elements α in L_1 and β in L_2 satisfy $\text{Tr}_{L_1/K}(\alpha) = \text{Tr}_{L_2/K}(\beta)$, then there exists an element γ in L such that $\text{Tr}_{L_1/L_2}(\gamma) = \alpha$ and $\text{Tr}_{L_2/L_1}(\gamma) = \beta$.

PROOF. In view of Lemma 1 it is enough to prove Lemma 2 for the case $L = L_1 L_2$. By the condition in Lemma 2 the Galois group $G(L_1 L_2/K)$ is the direct product $G(L_1/K) \times G(L_2/K)$. We choose normal basis $\{\omega^\sigma \mid \sigma \in G(L_1/K)\}$ and $\{\lambda^\tau \mid \tau \in G(L_2/K)\}$ of L_1/K and L_2/K , respectively. Then $\{\omega^\sigma \lambda^\tau \mid \sigma \in G(L_1/K), \tau \in G(L_2/K)\}$ form a normal base of $L_1 L_2/K$. Put $\alpha = \sum_{\sigma} a_\sigma \omega^\sigma$ and $\beta = \sum_{\tau} b_\tau \lambda^\tau$ with coefficients in K . Let us consider the following system of linear equations in $\{X_{\sigma, \tau}\}$:

$$\text{Tr}_{L_1 L_2/L_1}(\sum_{\sigma, \tau} X_{\sigma, \tau} \omega^\sigma \lambda^\tau) = \sum_{\sigma} a_\sigma \omega^\sigma,$$

$$\text{Tr}_{L_1 L_2/L_2}(\sum_{\sigma, \tau} X_{\sigma, \tau} \omega^\sigma \lambda^\tau) = \sum_{\tau} b_\tau \lambda^\tau,$$

where $G(L_1 L_2/K)$ operates on $\{X_{\sigma, \tau}\}$ trivially. This system is equivalent to

$$\sum_{\tau} X_{\sigma, \tau} = (\text{Tr}_{L_2/K}(\lambda))^{-1} a_\sigma, \quad (\sigma \in G(L_1/K)),$$

$$\sum_{\sigma} X_{\sigma, \tau} = (\text{Tr}_{L_1/K}(\omega))^{-1} b_\tau, \quad (\tau \in G(L_2/K)).$$

Since $\text{Tr}_{L_1/K}(\alpha) = \text{Tr}_{L_2/K}(\beta)$, we have

$$(\sum_{\sigma} a_\sigma) \text{Tr}_{L_1/K}(\omega) = (\sum_{\tau} b_\tau) \text{Tr}_{L_2/K}(\lambda)$$

and

$$\text{Tr}_{L_2/K}(\lambda)^{-1} a_\varepsilon - \text{Tr}_{L_1/K}(\omega)^{-1} (\sum_{\tau \neq \varepsilon} b_\tau) = (\text{Tr}_{L_1/K}(\omega))^{-1} b_\varepsilon - (\text{Tr}_{L_2/K}(\lambda))^{-1} \sum_{\sigma \neq \varepsilon} a_\sigma,$$

where ε is the unit element in $G(L_1 L_2/K)$.

Putting

$$c_{\sigma, \tau} = \begin{cases} (Tr_{L_1/K}(\lambda))^{-1} a_{\sigma}, & \text{for } \sigma \neq \varepsilon \text{ and } \tau = \varepsilon \\ (Tr_{L_1/K}(\omega))^{-1} b_{\tau}, & \text{for } \tau \neq \varepsilon \text{ and } \sigma = \varepsilon \\ 0, & \text{for } \sigma \neq \varepsilon \text{ and } \tau \neq \varepsilon \\ (Tr_{L_1/K}(\lambda))^{-1} a_{\varepsilon} - (Tr_{L_1/K}(\omega))^{-1} \sum_{\varrho \neq \varepsilon} b_{\varrho}, & \text{for } \sigma = \varepsilon \text{ and } \tau = \varepsilon, \end{cases}$$

we have a solution $(c_{\sigma, \tau})$ of the above equations in K . Hence the element $\gamma = \sum_{\sigma, \tau} c_{\sigma, \tau} \omega^{\sigma} \lambda^{\tau}$ satisfies the condition in Lemma 2.

We shall formulate Lemma 2 to the problem in the rings of Witt vectors:

LEMMA 3. Let A be a finite separable normal extension of $\mathbb{A}$. Let A_1 and A_2 be normal subfields of A over $\mathbb{A}$ such that $A_1 \cap A_2 = \mathbb{A}$. If elements α in $W(A_1)$ and β in $W(A_2)$ satisfy $Tr_{K(A_1)/K(A)}(\alpha) = Tr_{K(A_2)/K(A)}(\beta)$, then there exists an element γ in $W(A)$ such that $Tr_{K(A)/K(A_1)}(\gamma) = \alpha$ and $Tr_{K(A)/K(A_2)}(\gamma) = \beta$.

PROOF. It is sufficient to show that the coefficients $\gamma_0, \gamma_1, \dots$ in the expansion $\sum_{i=0}^{\infty} \gamma_i^{p^{-i}} p^i$ of γ in Lemma 3 are successively constructed. Put $\alpha = \sum_{i=0}^{\infty} \alpha_i^{p^{-i}} p^i$ and $\beta = \sum_{i=0}^{\infty} \beta_i^{p^{-i}} p^i$. Then, since $Tr_{K(A_1)/K(A)}(\alpha) = Tr_{K(A_2)/K(A)}(\beta)$, we have $Tr_{K(A_1)/K(A)}(\alpha_0 \mathbf{1}) \equiv Tr_{K(A_2)/K(A)}(\beta_0 \mathbf{1}) \pmod{\mathfrak{p}}$ and $Tr_{A_1/A}(\alpha_0) = Tr_{A_2/A}(\beta_0)$. Hence, by virtue of Lemma 2, we have γ_0 in A such that $Tr_{A/A_1}(\gamma_0) = \alpha_0$ and $Tr_{A/A_2}(\gamma_0) = \beta_0$, namely

$$\begin{aligned} Tr_{K(A)/K(A_1)}(\gamma_0 \mathbf{1}) &\equiv \alpha_0 \mathbf{1} \equiv \alpha \\ &\pmod{\mathfrak{p}}. \\ Tr_{K(A)/K(A_2)}(\gamma_0 \mathbf{1}) &\equiv \beta_0 \mathbf{1} \equiv \beta \end{aligned}$$

Assume we have already $\gamma_0, \dots, \gamma_{n-1}$ in A such that

$$\begin{aligned} Tr_{K(A)/K(A_1)}\left(\sum_{i=0}^{n-1} \gamma_i^{p^{-i}} p^i\right) &\equiv \alpha \\ &\pmod{\mathfrak{p}^n}. \\ Tr_{K(A)/K(A_2)}\left(\sum_{i=0}^{n-1} \gamma_i^{p^{-i}} p^i\right) &\equiv \beta \end{aligned}$$

Put

$$\alpha - \text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A_1)} \left(\sum_{i=0}^{n-1} \gamma_i^{p^{-i}} \mathbf{p}^i \right) = \sum_{i=0}^{\infty} \alpha_i' p^{-n-i} \mathbf{p}^{n+i}$$

and

$$\beta - \text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A_2)} \left(\sum_{i=0}^{n-1} \gamma_i^{p^{-i}} \mathbf{p}^i \right) = \sum_{i=0}^{\infty} \beta_i' p^{-n-i} \mathbf{p}^{n+i}.$$

Then, since $\text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A)} (\alpha) = \text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A)} (\beta)$, we have

$$\text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A)} \left(\sum_{i=0}^{\infty} \alpha_i' p^{-n-i} \mathbf{p}^{n+i} \right) = \text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A)} \left(\sum_{i=0}^{\infty} \beta_i' p^{-n-i} \mathbf{p}^{n+i} \right),$$

and thus $\text{Tr}_{A_1/A} (\alpha_0') = \text{Tr}_{A_2/A} (\beta_0')$. Therefore by virtue of Lemma 2 we have γ_n in A such that

$$\alpha - \text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A_1)} \left(\sum_{i=0}^{\infty} \gamma_i^{p^{-i}} \mathbf{p}^i \right) \equiv \text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A_1)} (\gamma_n^{p^{-n}} \mathbf{p}^n) \pmod{\mathbf{p}^{n+1}},$$

$$\beta - \text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A_2)} \left(\sum_{i=0}^{n-1} \gamma_i^{p^{-i}} \mathbf{p}^i \right) \equiv \text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A_2)} (\gamma_n^{p^{-n}} \mathbf{p}^n)$$

namely

$$\alpha \equiv \text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A_1)} \left(\sum_{i=0}^n \gamma_i^{p^{-i}} \mathbf{p}^i \right) \pmod{\mathbf{p}^{n+1}}.$$

$$\beta \equiv \text{Tr}_{\mathbf{K}(A)/\mathbf{K}(A_2)} \left(\sum_{i=0}^n \gamma_i^{p^{-i}} \mathbf{p}^i \right)$$

2.2. We shall first prove the following Lemma and apply it together with Lemma 2 to construct of the matrix solution of $\mathbf{A}^\sigma = \mathbf{A}\mathbf{M}(\sigma)$, ($\sigma \in G(A)$) in $\mathbf{W}(\bar{A})$.

LEMMA 4. Let L/K be a finite separable normal extension and $\{N(\sigma) \mid \sigma \in G(L/K)\}$ be a representation of the Galois group by non-singular matrices with coefficients in K . Let ω be an element in L such that all the conjugates of ω over K form a normal base. Then the matrix $\sum_{\sigma \in G(L/K)} N(\sigma^{-1}) \omega^\sigma$ is non-singular.

PROOF. It is sufficient to prove Lemma 4 for every irreducible representation in K . Since every irreducible representation appears in the regular representation $\{R(\sigma) \mid \sigma \in G(L/K)\}$ as an irreducible component, it is sufficient to prove $\det \left(\sum_{\sigma} R(\sigma^{-1}) \omega^\sigma \right) \neq 0$. Giving an order $\sigma_1 > \dots > \sigma_n$ in

$G(L/K)$, we shall calculate $(i \succ j)$ -element of $\sum_{\sigma} R(\sigma^{-1}) \omega^{\sigma}$:

$$\sum_{\sigma} \delta(\sigma_i \sigma \sigma_j^{-1}) \omega^{\sigma} = \omega^{\sigma_i^{-1} \sigma_j},$$

where $\delta(\varepsilon) = 1$ for the unit element ε and $\delta(\tau) = 0$ for $\tau \neq \varepsilon$. Since $\{\omega^{\sigma} \mid \sigma \in G(L/K)\}$ form a normal base of L/K , the matrix of which $(i \succ j)$ element is $\omega^{\sigma_i^{-1} \sigma_j}$ is not singular. This proves Lemma 4.

Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ be a representation of $G(\Delta)$ by non singular $r \times r$ -matrices with coefficients in $\mathbb{Z}_p$. We denote by $\Gamma(m)$ the subgroup $\{\sigma \in G(\Delta) \mid \mathbf{M}(\sigma) \equiv \text{identity} \pmod{\mathfrak{p}^m}\}$ in $G(\Delta)$ and $\Delta(m)$ the subfield of $\bar{\Delta}'$ consisting of all the elements fixed by every element in $\Gamma(m)$, $(m = 1, 2, \dots)$. Then $\Gamma(m)$ are normal subgroups of finite index and $\Delta(m)/\Delta$ are finite separable normal extensions of Δ .

For each m we choose a system of representatives of $G(\Delta)/\Gamma(m)$ in $G(\Delta)$ and we understand by $\sum_{\sigma \bmod \Gamma(m)}$ that the sum is taken over all σ running through the representatives of $G(\Delta)/\Gamma(m)$.

THEOREM 1. Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ be a representation of $G(\Delta)$ by non-singular $r \times r$ -matrices with coefficients in $\mathbb{Z}_p$. Then there exists a non-singular matrix $\mathbf{A}$ with coefficients in $\mathbf{W}(\bar{\Delta}')$ such that $\mathbf{A}^{\sigma} = \mathbf{M}(\sigma) \mathbf{A}$, $(\sigma \in G(\Delta))$.

PROOF. We use the notations $\Gamma(m)$, $\Delta(m)$, $\sum_{\sigma \bmod \Gamma(m)}$ in the above. In view of Lemma 1 and the theorem of normal base, there exists a system $(\omega_1, \omega_2, \dots)$ of elements in $\mathbf{W}(\bar{\Delta}')$ such that

- 1) $\omega_m \in \mathbf{W}(\Delta(m))$, $(m = 1, 2, \dots)$, $\omega_1 = \omega, \mathbf{1}$.
- 2) all the conjugates of ω_1 over Δ form a normal base of $\Delta(1)/\Delta$,
- 3) $\text{Tr}_{\mathbf{K}(\Delta(m+1))/\mathbf{K}(\Delta(m))}(\omega_{m+1}) = \omega_m$, $(m = 1, 2, \dots)$.

Put

$$\mathbf{A}_m = \sum_{\sigma \bmod \Gamma(m)} \mathbf{M}(\sigma^{-1}) \omega_m^{\sigma}, \quad (m = 1, 2, \dots),$$

where $\mathbf{1}$ is the identity in $\mathbf{W}(\bar{\Delta}')$. Since $\omega_m \in \mathbf{K}(\Delta(m))$ and $\mathbf{M}(\varrho) \equiv \text{identity} \pmod{\mathfrak{p}^m}$ for ϱ in $\Gamma(m)$, the class of $\mathbf{A}_m \pmod{\mathfrak{p}^m}$ is independent of the choice of the representatives of $G(\Delta)/\Gamma(m)$. Moreover we have the following set important relations:

$$\mathbf{A}_{m+1} \equiv \mathbf{A}_m \pmod{\mathfrak{p}^m}, \quad (m = 1, 2, \dots),$$

because

$$\begin{aligned}
 A_{m+1} &\equiv \sum_{\sigma \bmod \Gamma(m+1)} M(\sigma^{-1}) \omega_m^\sigma \\
 &\equiv \sum_{\tau \bmod \Gamma(m)} M(\tau^{-1}) \sum_{\substack{\varrho \bmod \Gamma(m+1) \\ \varrho \in \Gamma(m)}} M(\varrho^{-1}) \omega_{m+1}^{\varrho\tau} \\
 &\equiv \sum_{\tau \bmod \Gamma(m)} M(\tau^{-1}) \sum_{\substack{\varrho \bmod \Gamma(m+1) \\ \varrho \in \Gamma(m)}} \omega_{m+1}^{\varrho\tau} \bmod p^m \\
 &\equiv \sum_{\tau \bmod \Gamma(m)} M(\tau^{-1}) \text{Tr}_{K(D(m+1))/K(D(m))} (\omega_{m+1})^\tau \\
 &\equiv \sum_{\tau \bmod \Gamma(m)} M(\tau^{-1}) \omega_m \equiv A_m \bmod p^m.
 \end{aligned}$$

Therefore there exists the limit $A = \lim_{m \rightarrow \infty} A_m$ such that $A \equiv A_m \bmod p^m$.

($m = 1, 2, \dots$). On the other hand $A_m^\sigma = \sum_{\tau \bmod \Gamma(m)} M(\tau^{-1}) \omega_m^{\tau\sigma} \equiv$
 $\equiv M(\sigma) \sum_{\tau \bmod \Gamma(m)} M(\tau^{-1}) \omega_m^\tau \equiv M(\sigma) A_m \bmod p^m$, ($\sigma \in G(D)$); $m = 1, 2, \dots$),

hence we have $A^\sigma = M(\sigma) A$, ($\sigma \in G(D)$). By virtue of Lemma 4 A_1 is non-singular, so A is also non-singular. This completes the proof of Theorem 1.

By an argument based on the same ideas as in the proof of Theorem 1 we have the following theorem:

THEOREM 2. Let $\{M(\sigma) \mid \sigma \in G(D)\}$ be a representation of $G(D)$ by non-singular $r \times r$ -matrices with coefficients in $\mathbb{Z}_p$ and B_1 be a $r \times r$ -matrix with coefficients in $D(1)$ such that $B_1^\sigma = M(\sigma) B_1$, ($\sigma \in G(D)$), where $\overline{M}(\sigma)$ is the reduction of $M(\sigma)$ modulo p and $D(1)$ the subfield of $\overline{D'}$ consisting of all the elements in $\overline{D'}$ fixed by the element σ such that $\overline{M}(\sigma) \equiv \text{identity} \bmod p$. Then there exists a matrix B with coefficients in $W(\overline{D'})$ such that

- 1) B_1 is the reduction of B modulo p ,
- 2) $B^\sigma = \overline{M}(\sigma) B$, ($\sigma \in G(D)$).

PROOF. On this proof $\omega_1 = \omega_1 \mathbf{1}, \omega_2 \dots$ denote the same elements of $\overline{D'}$ as in the proof of Theorem 1. Since $\sum_{\tau \bmod \Gamma(1)} \overline{M}(\tau^{-1}) \omega_1^\tau$ is non-singular (Lemma 4), we can put

$$O = \left(\sum_{\tau \bmod \Gamma(1)} \overline{M}(\tau^{-1}) \omega_1^\tau \right)^{-1} B_1,$$

$$B_m = \left(\sum_{\sigma \bmod \Gamma(m)} M(\sigma^{-1}) \omega_m^\sigma \right) (O \mathbf{1}), \quad (m = 1, 2, \dots).$$

Then C is a matrix with coefficients in Δ and $C \cdot 1$ is a matrix with coefficients in $W(\Delta)$. Moreover $B_m \equiv A_m(C \cdot 1) \pmod{\mathfrak{p}^m}$ and B_1 is the reduction of B_m modulo $\mathfrak{p}$, where A_m is the same as in the proof of Theorem 1. Hence putting $B = \lim_{m \rightarrow \infty} B_m$, we have a B satisfying the conditions in Theorem 2.

2.3 In order to solve the problem affirmatively, we need to prove the existence of a vector $\mathfrak{a} = (\alpha_1, \dots, \alpha_r)$ with coefficients in $W(\bar{\Delta}')$ such that 1) $\mathfrak{a}^\sigma = M(\sigma) \mathfrak{a}$, ($\sigma \in G_\Delta$) and 2) $\alpha_1, \dots, \alpha_r$ are linearly independent over $\mathbb{Q}_p$. The existence of non-zero vector satisfying 1) is guaranteed by Theorem 1. We shall first notice the existence of vectors satisfying 1) and 2) for the irreducible representations $\{M(\sigma) \mid \sigma \in G(\Delta)\}$ of $G(\Delta)$ and, then under the assumption that $K(\Delta)/\mathbb{Q}_p$ is transcendental, we shall prove the existence of vectors satisfying 1) and 2) by the induction on the number of irreducible components in the representations.

THEOREM 3. Let $\{M(\sigma) \mid \sigma \in G(\Delta)\}$ be an irreducible representation of $G(\Delta)$ by $r \times r$ -matrices with coefficients in $\mathbb{Z}_p$. Then there exists a system of elements $\xi_1, \dots, \xi_r$ in $W(\bar{\Delta}')$ such that 1)

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix}^\sigma = M(\sigma) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix}, \quad (\sigma \in G(\Delta)),$$

2) $\xi_1, \dots, \xi_r$ are linearly independent over $\mathbb{Q}_p$.

PROOF. In view of Theorem 1 there exists a non-zero vector $\mathfrak{a} = (\alpha_1, \dots, \alpha_r)$ with coefficients in $W(\bar{\Delta}')$ such that

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}^\sigma = M(\sigma) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}, \quad (\sigma \in G(\Delta)).$$

Let V be $\mathbb{Z}_p[G(\Delta)]$ -module spanned by $\alpha_1, \dots, \alpha_r$ over $\mathbb{Z}_p$ in $W(\bar{\Delta}')$. If the rank of V is less than r , the representation $\{M(\sigma) \mid \sigma \in G(\Delta)\}$ is not irreducible. This shows the linearly independentness of $\alpha_1, \dots, \alpha_r$ over $\mathbb{Q}_p$.

In the induction process we shall need the following lemma:

LEMMA 5. Let ξ be a transcendental element in a field L over a subfield K and $\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_{r-s}, \gamma_1, \dots, \gamma_s$ be elements in L such that $\{\alpha_1, \dots, \alpha_s\}$ and $\{\beta_1, \dots, \beta_{r-s}\}$ are sets of linearly independent elements over

K . Then there exists a positive integer n_0 such that for every $n \geq n_0$ the elements $\alpha_1 \xi^n + \gamma_1, \dots, \alpha_s \xi^n + \gamma_s, \beta_1, \dots, \beta_{r-s}$ are linearly independent over K .

PROOF. We shall give a proof from algebraic geometry. Put $M = \overline{K}(\xi, \alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_{r-s}, \gamma_1, \dots, \gamma_s)$, where $\overline{K}$ denotes the algebraic closure of K . We choose a normal projective variety V defined over $\overline{K}$ as a model of the function field $M/\overline{K}$. We denote by Σ the $\overline{K}$ -module spanned by $\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_{r-s}, \gamma_1, \dots, \gamma_s$ in M over $\overline{K}$ and denote by d the maximal of the degrees of the polar divisors of elements in Σ . For a prime divisor Y on V we mean by v_Y the valuation defined as follows: If the multiplicity of Y in the principal divisor (f) is n_Y , the value $v_Y(f)$ is given by $n_Y \deg Y$. We choose a prime divisor X on V such that $v_X(\xi) \geq 1$. Since the degree of polar divisors of elements in Σ is at most d , we have $-d \leq v_X(g) \leq d$ ($g \neq 0$ in Σ). We put $n_0 = 2d + 1$. Let n be a positive integer not less than n_0 and assume that there exist $a_1, \dots, a_s, b_1, \dots, b_{r-s}$ in K such that $\sum_i a_i (\alpha_i \xi^n + \gamma_i) + \sum_j b_j \beta_j = 0$. Then, since $\sum_i a_i \gamma_i + \sum_j b_j \beta_j \in \Sigma$, we have the two cases:

1) $\sum_i a_i \gamma_i + \sum_j b_j \beta_j = 0$, 2) $-d \leq v_X(\sum_i a_i \gamma_i + \sum_j b_j \beta_j) \leq d$. If $\sum_i a_i \gamma_i + \sum_j b_j \beta_j = 0$, it follows that $(\sum_i a_i \alpha_i) \xi^n = 0$ and $\sum_i a_i \alpha_i = 0$. Since $\alpha_1, \dots, \alpha_s$ are linearly independent over K , we have $a_1 = \dots = a_s = 0$. By the linear independence of $\beta_1, \dots, \beta_{r-s}$ over K we have $b_1 = \dots = b_{r-s} = 0$, because $\sum_j b_j \beta_j = 0$. Let us assume $-d \leq v_X(\sum_i a_i \gamma_i + \sum_j b_j \beta_j) \leq d$. Then $v_X((\sum_i a_i \alpha_i) \xi^n) = v_X(\sum_i a_i \gamma_i + \sum_j b_j \beta_j) \leq d$. On the other hand, since $\sum_i a_i \alpha_i \in \Sigma$ and $\sum_i a_i \alpha_i \neq 0$, we have $v_X((\sum_i a_i \alpha_i) \xi^n) = nv_X(\xi) + v_X(\sum_i a_i \alpha_i) \geq n - d \geq 2d + 1 - d = d + 1$. This is a contradiction. Therefore $\alpha_1 \xi^n + \gamma_1, \dots, \alpha_s \xi^n + \gamma_s, \beta_1, \dots, \beta_{r-s}$ are linearly independent over K .

The next lemma is the reduction of the problem.

LEMMA 6. Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\mathcal{A})\}$ be a representation of $G(\mathcal{A})$ by non-singular $r \times r$ -matrices with coefficients in $\mathbf{Z}_p$ such that

$$\mathbf{M}(\sigma) \begin{pmatrix} \overbrace{\mathbf{N}_1(\sigma)}^s & \overbrace{\mathbf{A}(\sigma)}^{r-s} \\ \mathbf{0} & \mathbf{N}_2(\sigma) \end{pmatrix} \begin{matrix} s \\ r-s \end{matrix}, \quad (\sigma \in G(\mathcal{A})).$$

Let $\mathcal{A}(m)$ be the subfield of $\overline{\mathcal{A}}$ consisting of all the elements fixed by the subgroup $\Gamma(m) = \{\sigma \in G(\mathcal{A}) \mid \mathbf{M}(\sigma) \equiv \text{identity mod } \mathfrak{p}^m\}$ ($m = 1, 2, \dots$). Assume

two systems $(\mathfrak{a}_1, \mathfrak{a}_2, \dots)$ and $(\mathfrak{b}_1, \mathfrak{b}_2, \dots)$ of vectors with coefficients in $W(\bar{\Delta}')$ satisfy

- 1) the coefficients of $\mathfrak{a}_m$ and $\mathfrak{b}_m$ belong to $W(\Delta(m))$,
- 2) $\text{Tr}_{K(\Delta(m+1))/K(\Delta(m))}(\mathfrak{a}_{m+1}) = \mathfrak{a}_m$, $\text{Tr}_{K(\Delta(m+1))/K(\Delta(m))}(\mathfrak{b}_{m+1}) = \mathfrak{b}_m$,
- 3) putting ⁽⁴⁾

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_s \end{pmatrix} = \lim_{m \rightarrow \infty} \sum_{\sigma \bmod \Gamma(m)} N_1(\sigma^{-1}) \mathfrak{a}_m^\sigma, \quad \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{r-s} \end{pmatrix} = \lim_{m \rightarrow \infty} \sum_{\sigma \bmod \Gamma(m)} N_2(\sigma^{-1}) \mathfrak{b}_m^\sigma,$$

the sets $\{\alpha_1, \dots, \alpha_s\}$ and $\{\beta_1, \dots, \beta_{r-s}\}$ are sets of linearly independent elements over $\mathbb{Q}_p$. Then, if $K(\Delta)$ is transcendental over $\mathbb{Q}_p$, there exists a system of vectors $(\mathfrak{x}_1, \mathfrak{x}_2, \dots)$ such that 1) the coefficients of $\mathfrak{x}_m$ belong to $W(\Delta(m))$, 2) $\text{Tr}_{K(\Delta(m+1))/K(\Delta(m))}(\mathfrak{x}_{m+1}) = \mathfrak{x}_m$, 3) putting

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix} = \lim_{m \rightarrow \infty} \sum_{\sigma \bmod \Gamma(m)} \mathbf{M}(\sigma^{-1}) \mathfrak{x}_m^\sigma,$$

the elements $\xi_1, \dots, \xi_r$ are linearly independent over $\mathbb{Q}_p$.

PROOF. Put

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_s \end{pmatrix} = \lim_{m \rightarrow \infty} \sum_{\sigma \bmod \Gamma(m)} \mathbf{A}(\sigma^{-1}) \mathfrak{b}_m^\sigma$$

and apply Lemma 5 to $\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_{r-s}, \gamma_1, \dots, \gamma_s$, $L = K(\bar{\Delta}')$ and $K = \mathbb{Q}_p$. Then there exists a non-zero element η in $W(\Delta)$ such that $\alpha_1 \eta + \dots + \alpha_s \eta + \gamma_s, \beta_1, \dots, \beta_{r-s}$ are linearly independent over $\mathbb{Q}_p$. Therefore, putting

$$\mathfrak{x}_m = \begin{pmatrix} \mathfrak{a}_m \eta \\ \mathfrak{b}_m \end{pmatrix}, \quad (m = 1, 2, \dots),$$

we get a system of vectors $(\mathfrak{x}_1, \mathfrak{x}_2, \dots)$ satisfying the conditions in Lemma 6.

Applying Lemma 6 successively we have our main theorem:

THEOREM 4 (the main theorem). If $K(\Delta)$ is transcendental over $\mathbb{Q}_p$, there exists a $\mathbb{Z}_p[G(\Delta)]$ -submodule in $W(\Delta')$ which is isomorphic to a given $\mathbb{Z}_p[G(\Delta)]$ -submodule of finite rank over $\mathbb{Z}_p$.

PROOF. We shall prove the next assertion:

(A) Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ be any representation of $G(\Delta)$ by non-singular $r \times r$ -matrices with coefficients in $\mathbb{Z}_p$. Let $\Gamma(m)$ be the subgroup

(4) The conditions 1) and 2) imply the existence of limits in 3).

$\{\sigma \in G(\Delta) \mid \mathbf{M}(\sigma) \equiv \text{identity} \pmod{\mathfrak{p}^m}\}$ and $\Delta(m)$ the subfield in Δ' consisting of all the elements fixed by every element in $\Gamma(m)$. Then, if $\mathbf{K}(\Delta)/\mathbf{Q}_p$ is transcendental, there exists a system of vectors $(\mathbf{x}_1, \mathbf{x}_2, \dots)$ such that

- 1) the coefficients of $\mathbf{x}_m$ belong to $\mathbf{W}(\Delta(m))$,
- 2) $\text{Tr}_{\mathbf{K}(\Delta(m+1))/\mathbf{K}(\Delta(m))}(\mathbf{x}_{m+1}) = \mathbf{x}_m$,
- 3) putting ⁽⁵⁾

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix} = \lim_{m \rightarrow \infty} \sum_{\sigma \in \Gamma(m)} \mathbf{M}(\sigma^{-1}) \mathbf{x}_m^\sigma,$$

the elements $\xi_1, \dots, \xi_r$ are linearly independent over $\mathbf{Q}_p$.

The vector $(\xi_1, \dots, \xi_r)$ in (Δ) satisfies the condition in Theorem 4, hence it is enough to prove (A) by the induction on the number of irreducible components. First we assume $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ is irreducible. Then, if we denote by $\mathbf{A}_1, \mathbf{A}_2, \dots$ the same matrices as in the proof of Theorem 1 and denote by $\mathbf{a}_1, \mathbf{a}_2, \dots$ the first column vectors of $\mathbf{A}_1, \mathbf{A}_2, \dots$, respectively, then by the argument in the proof of Theorem 3 the system $(\mathbf{a}_1, \mathbf{a}_2, \dots)$ satisfies the condition in (A). We assume (A) for the case in which the number of irreducible components is less than n .

Let $\{\mathbf{M}(\sigma) \mid \sigma \in G(\Delta)\}$ be a representation of $G(\Delta)$ by non-singular $r \times r$ -matrices with coefficients in $\mathbf{Z}_p$ such that i)

$$\mathbf{M}(\sigma) = \begin{pmatrix} \overbrace{\mathbf{N}_1(\sigma)}^s & \overbrace{\mathbf{A}(\sigma)}^{r-s} \\ 0 & \mathbf{N}_2(\sigma) \end{pmatrix} \begin{matrix} s \\ r-s \end{matrix}, \quad (\sigma \in G(\Delta))$$

ii) $\{\mathbf{N}_1(\sigma) \mid \sigma \in G(\Delta)\}$ is irreducible, iii) the number of irreducible components in $\{\mathbf{N}_2(\sigma) \mid \sigma \in G(\Delta)\}$ is $n-1$. We denote by $\Delta(i, m)$ the subfield of $\overline{\Delta'}$ consisting of all the elements fixed by every element in $\Gamma(i, m) = \{\sigma \in G(\Delta) \mid \mathbf{N}_i(\sigma) \equiv \text{identity} \pmod{\mathfrak{p}^m}\}$, ($i=1, 2; m=1, 2, \dots$). By the induction assumption there exist systems of vectors $(\mathbf{a}_1, \mathbf{a}_2, \dots)$ and $(\mathbf{b}_1, \mathbf{b}_2, \dots)$ such that

- 1) the coefficients of $\mathbf{a}_m$ (resp. $\mathbf{b}_m$) belong to $\mathbf{W}(\Delta(1, m))$. (resp. $\mathbf{W}(\Delta(2, m))$),
- 2) $\text{Tr}_{\mathbf{K}(\Delta(1, m+1))/\mathbf{K}(\Delta(1, m))}(\mathbf{a}_{m+1}) = \mathbf{a}_m$,
 $\text{Tr}_{\mathbf{K}(\Delta(2, m+1))/\mathbf{K}(\Delta(2, m))}(\mathbf{b}_{m+1}) = \mathbf{b}_m$,
- 3) putting ⁽⁶⁾

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_s \end{pmatrix} = \lim_{m \rightarrow \infty} \sum_{\sigma \in \Gamma(1, m)} \mathbf{N}_1(\sigma^{-1}) \mathbf{a}_m^\sigma$$

⁽⁵⁾, ⁽⁶⁾ see ⁽⁴⁾.

and

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{r-s} \end{pmatrix} = \lim_{m \rightarrow \infty} \sum_{\sigma \bmod \Gamma(2, m)} N_2(\sigma^{-1}) \mathbf{b}_m^\sigma,$$

the sets $\{\alpha_1, \dots, \alpha_s\}$ and $\{\beta_1, \dots, \beta_{r-s}\}$ are sets of linearly independent elements over $\mathbb{Q}_p$.

Put $\Delta^{(1)} = \bigcup_m \Delta(1, m)$ and $\Delta^{(2)} = \bigcup_m \Delta(2, m)$. Then there exists a non-decreasing arithmetic function Φ such that $\Delta(1, \Phi(m)) \supset \Delta^{(1)} \cap \Delta(m)$ and $\Delta(2, \Phi(m)) \supset \Delta^{(2)} \cap \Delta(m)$, ($m = 1, 2, \dots$), because $\Delta(m)$ ($m = 1, 2, \dots$) are finite extensions over Δ . We shall inductively show that we can choose vectors $\mathbf{c}_m$ and $\mathbf{d}_m$ with coefficients in $\mathbf{W}(\Delta(m))$ such that

$$(B_m) \quad \begin{cases} \text{Tr}_{\mathbb{K}(\Delta(1, \Phi(m))/\mathbb{K}(\Delta^{(1)} \cap \Delta(m)))}(\mathbf{a}_{\Phi(m)}) = \text{Tr}_{\mathbb{K}(\Delta(m))/\mathbb{K}(\Delta^{(1)} \cap \Delta(m))}(\mathbf{c}_m), \\ \text{Tr}_{\mathbb{K}(\Delta(2, \Phi(m))/\mathbb{K}(\Delta^{(2)} \cap \Delta(m)))}(\mathbf{b}_{\Phi(m)}) = \text{Tr}_{\mathbb{K}(\Delta(m))/\mathbb{K}(\Delta^{(1)} \cap \Delta(m))}(\mathbf{d}_m), \\ \text{Tr}_{\mathbb{K}(\Delta(m+1))/\mathbb{K}(\Delta(m))}(\mathbf{c}_{m+1}) = \mathbf{c}_m, \quad \text{Tr}_{\mathbb{K}(\Delta(m+1))/\mathbb{K}(\Delta(m))}(\mathbf{d}_{m+1}) = \mathbf{d}_m. \end{cases}$$

Since $\text{Tr}_{\mathbb{K}(\Delta(1, \Phi(1))/\mathbb{K}(\Delta^{(1)} \cap \Delta(1)))}(\mathbf{a}_{\Phi(1)})$ is a known element in $\mathbb{K}(\Delta^{(1)} \cap \Delta(1))$ applying Lemma 3 to $\Delta_1 = \Delta_2 = \Delta^{(1)} \cap \Delta(1)$, $\Delta = \Delta_{(1)}$ and $\alpha = \beta = \text{Tr}_{\mathbb{K}(\Delta(1, \Phi(1))/\mathbb{K}(\Delta^{(1)} \cap \Delta(1)))}(\mathbf{a}_{\Phi(1)})$, we have an element $\mathbf{c}_1$ with coefficients in $\mathbf{W}(\Delta(1))$ satisfying (B_1) . Assume we have already $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_{m-1}$ such that the coefficients of $\mathbf{c}_l$ belong to $\mathbf{W}(\Delta(l))$ and $\mathbf{c}_l$ satisfy (B_l) , ($l = 1, 2, \dots, m-1$). Then, since $(\Delta^{(1)} \cap \Delta(m)) \cap \Delta(m-1) = \Delta^{(1)} \cap \Delta(m-1)$ and

$$\begin{aligned} & \text{Tr}_{\mathbb{K}(\Delta^{(1)} \cap \Delta(m))/\mathbb{K}(\Delta^{(1)} \cap \Delta(m-1))}(\text{Tr}_{\mathbb{K}(\Delta(1, \Phi(m))/\mathbb{K}(\Delta^{(1)} \cap \Delta(m)))}(\mathbf{a}_{\Phi(m)})) \\ &= \text{Tr}_{\mathbb{K}(\Delta(m-1))/\mathbb{K}(\Delta^{(1)} \cap \Delta(m+1))}(\mathbf{c}_{m-1}), \end{aligned}$$

we can use Lemma 3 and get a vector $\mathbf{c}_m$ with coefficients in $\mathbf{W}(\Delta(m))$ such that

$$\text{Tr}_{\mathbb{K}(\Delta(m))/\mathbb{K}(\Delta^{(1)} \cap \Delta(m))}(\mathbf{c}_m) = \text{Tr}_{\mathbb{K}(\Delta(1, \Phi(m))/\mathbb{K}(\Delta^{(1)} \cap \Delta(m)))}(\mathbf{a}_{\Phi(m)})$$

and

$$\text{Tr}_{\mathbb{K}(\Delta(m+1))/\mathbb{K}(\Delta(m))}(\mathbf{c}_{m+1}) = \mathbf{c}_m.$$

By the same method we have $\mathbf{d}_1, \mathbf{d}_2, \dots$ satisfying the conditions. By virtue of the last condition in (B_m) the limits

$$\lim_{m \rightarrow \infty} \sum_{\sigma \bmod \Gamma(m)} N_1(\sigma^{-1}) \mathbf{c}_m^\sigma \quad \text{and} \quad \lim_{m \rightarrow \infty} \sum_{\sigma \bmod \Gamma(m)} N_2(\sigma^{-1}) \mathbf{d}_m^\sigma$$

exist and by the first two conditions in (B_m) we have

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_s \end{pmatrix} = \lim_{m \rightarrow \infty} \sum_{\sigma \bmod \Gamma(m)} N_1(\sigma^{-1}) \mathbf{c}_m^\sigma, \quad \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{r-s} \end{pmatrix} = \lim_{m \rightarrow \infty} \sum_{\sigma \bmod \Gamma(m)} N_2(\sigma^{-1}) \mathbf{d}_m^\sigma.$$

Therefore, applying Lemma 6 to $(\mathbf{c}_1, \mathbf{c}_2, \dots)$ and $(\mathbf{d}_1, \mathbf{d}_2, \dots)$, we have a system of vectors $(\mathbf{x}_1, \mathbf{x}_2, \dots)$ satisfying the conditions in (A). This completes the proof of the main theorem.

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