

ANNALI DELLA SCUOLA NORMALE SUPERIORE DI PISA *Classe di Scienze*

HISASI MORIKAWA

On a definition of abelian variety

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^e série, tome 18, n° 1 (1964), p. 161-163

http://www.numdam.org/item?id=ASNSP_1964_3_18_1_161_0

© Scuola Normale Superiore, Pisa, 1964, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (<http://www.sns.it/it/edizioni/riviste/annaliscienze/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

ON A DEFINITION OF ABELIAN VARIETY

HISASI MORIKAWA(*)

1. In the present note we shall give the following criterion of abelian variety which does not contain the associative law:

THEOREM. Let V be an irreducible projective variety defined over a field k . Let f be an everywhere defined regular map of $V \times V$ onto V , 1 be an k -rational point on V such that $f(a, 1) = f(1, a) = a$, ($a \in V$), and g be an everywhere defined regular map of V into V such that $f(a, g(a)) = 1$, ($a \in V$), where f and g are also defined over k . Then if for every a in V the regular map $T_a: x \rightarrow f(x, a)$ of V into V is a biregular map of V onto V , it follows that V is an abelian variety with the law of composition $(f, 1, g)$.

This theorem suggests that it is seldom possible to give a nice law of composition on a given irreducible projective variety.

2. **PROOF OF THEOREM.** Let $(f, 1, g)$ be a law of a composition on an irreducible projective variety V satisfying the condition in Theorem and k be the field of definition of V and the law of composition $(f, 1, g)$. For the sake of simplicity we put $a \circ b = f(a, b)$ and $a_r^{-1} = g(a)$, ($a, b \in V$). By virtue of the condition in Theorem there exists a map $T: a \rightarrow T_a$ of V into the group G of automorphism of the variety V . Since $(T_b T_a^{-1})(a) = b$ for every a and b in V , the group G operates on V transitively. Hence V is a non-singular irreducible projective variety, and thus by virtue of Matsusaka's result⁽¹⁾ G contains the largest irreducible algebraic group G_0 defined over k . We denote by K and H the subgroups $\{\sigma \in G \mid \sigma(1) = 1\}$ and $\{\sigma \in G_0 \mid \sigma(1) = 1\}$, respectively. Put $\xi(\sigma) = \sigma(1)$ and $\eta(\sigma) = T_{\sigma(1)}^{-1} \sigma$, ($\sigma \in G_0$). Then ξ and η are regular maps of G_0 into V and H , respectively, because G_0 is also transi-

Pervenuto alla Redazione il 22 Novembre 1963.

(*) The author was sponsored by Consiglio Nazionale delle Ricerche at Centro Ricerche Fisica e Matematica in Pisa.

⁽¹⁾ See [1] p. 45-48.

tive on V and for a generic point σ in G_0 over k the unit element e in G_0 is the unique specialization of $\eta(\sigma)$ over the specialization $\sigma \rightarrow e$. The maps ξ and η are also defined over k . Since $\sigma = T_{\xi(\sigma)} \eta(\sigma)$, ($\sigma \in G_0$) and $T_a(1) = a$, the maps $\sigma \rightarrow \xi(\sigma) \times \eta(\sigma)$, $\sigma \rightarrow T_{\xi(\sigma)} \times \eta(\sigma)$ are biregular maps of G_0 onto $V \times H$ and $T(V) \times H$, respectively, where $T(V)$ means the image of V in G_0 by T . By virtue of the structure theorem of algebraic group⁽²⁾ there exists an irreducible linear group L in G_0 such that L is defined over k and the quotient $A = G_0/L$ is an abelian variety. We shall next prove that $T(V) \cap L$ is zero-dimensional. Assume for a moment that $T(V) \cap L$ contains an irreducible subvariety W of dimension at least one and w be a generic point of W over a common field k' of definition of G_0 , $T(V)$, W and L . Then, since the linear group L is an affine variety and $T(V)$ is a complete variety, the variety W is not complete and there exists a specialization t of w over k' such that $t \in T(V)$ and $t \notin L$. This is a contradiction, because $t \in G_0$ and L is closed in G_0 . This shows that $T(V) \cap L = \{T_{c_1}, T_{c_2}, \dots, T_{c_N}\}$ with points $c_1, c_2, \dots, c_N$ in V . Let φ be the natural homomorphism of G_0 onto A and put $\lambda(a) = \varphi(T_a)$, ($a \in V$), then λ is a regular map of V into A such that $\lambda(1) = 0$, where 0 is the origin of A . We denote by μ the regular map of $V \times V$ into A defined by $\mu(a \times b) = \lambda(a \circ b)$, ($a, b \in V$), then by the property of a map of a product variety into an abelian variety⁽³⁾ there exist two regular maps ϱ_1 and ϱ_2 of V into A such that $\varrho_1(1) = 0$ and $\mu(a \times b) = \varrho_1(a) + \varrho_2(b)$, ($a, b \in V$). Since $\varrho_1(1) + \varrho_2(1) = \mu(1 \times 1) = \lambda(1 \circ 1) = \lambda(1) = 0$, $\varrho_2(1)$ is also the origin of A . We shall show $\varrho_1(a) = \varrho_2(a) = \lambda(a)$, $\lambda(a \circ b) = \lambda(a) + \lambda(b)$, $\lambda(a_r^{-1}) = -\lambda(a)$ as follows:

$$\begin{aligned} \varrho_1(a) &= \varrho_1(a) + \varrho_2(1) = \mu(a \times 1) = \lambda(a \circ 1) = \lambda(a), \varrho_2(a) = \varrho_1(1) + \varrho_2(a) \\ &= \mu(1 \times a) = \lambda(1 \circ a) = \lambda(a), \lambda(a \circ b) = \varrho_1(a) + \varrho_2(b) = \lambda(a) + \lambda(b), \\ \lambda(a) + \lambda(a_r^{-1}) &= \varrho_1(a) + \varrho_2(a_r^{-1}) = \mu(a \times a_r^{-1}) = \lambda(a \circ a_r^{-1}) = \lambda(1) = 0. \end{aligned}$$

This shows that $H \supseteq L$ implies Theorem. Next we shall show that λ is a finite regular map of V onto the image $\lambda(V)$ of V in A by λ . Since $\lambda(a) = \varphi(T_a)$, the relation $\lambda(a) = \lambda(b)$ implies $0 = \lambda(a) - \lambda(b) = \lambda(a) + \lambda(b_r^{-1}) = \lambda(a \circ b_r^{-1}) = \varphi(T_{a \circ b_r^{-1}})$ and $T_{a \circ b_r^{-1}} \in T(V) \cap L$. Hence $a \circ b_r^{-1} = c_i$ with a c_i in $\{c_1, \dots, c_N\}$ and $a = T_{b_r^{-1}}^{-1} c_i$. Namely $\lambda(a) = \lambda(b)$ if and only if $a = T_{b_r^{-1}}^{-1} c_i$ with a c_i in $\{c_1, \dots, c_N\}$. This pro-

(2) See [2] p. 425.

(3) See [3] II.

ves the finiteness of λ . Finally we shall prove $H \supseteq L$. Assume for a moment $H \not\supseteq L$. Then, since L and H are irreducible, the image $\bar{L}$ of L in V by the natural map: $G_0 \rightarrow V \times H \rightarrow V$ is at least one-dimensional. Moreover, since λ is a finite regular map of V onto $\lambda(V)$, the image $\lambda(\bar{L})$ is also at least one dimensional. This is a contradiction, because the image of linear group in an abelian variety is always zero-dimensional. This completes the proof of Theorem.

3. COROLLARY. If a law of composition $(f, 1, g)$ on an irreducible projective variety V satisfies $f(f(b, a), g(a)) = b$ for a and b in V , then V is an abelian variety with the law of composition $(f, 1, g)$.

PROOF. Since $(T_{a_r^{-1}} T_a)(x) = (x \circ a) \circ a_r^{-1} = x$, $(a, x \in V)$, T_a is a biregular map of V onto V for every a in V . Hence by virtue of Theorem V is an abelian variety with the law of composition $(f, 1, g)$.

BIBLIOGRAPHY

- [1] MATSUSAKA T., *Polarized varieties, fields of moduli and generalized Kummer varieties of polarized abelian varieties*, Amer. J. of Math. 80 (1958).
- [2] ROSENBLICHT M., *Some basic theorems on algebraic groups*, Amer. J. of Math. 78 (1957).
- [3] WEIL A., *Variétés abéliennes et courbes algébriques*, (1948).