ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

S. SMALE

Stable manifolds for differential equations and diffeomorphisms

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 17, nº 1-2 (1963), p. 97-116

<http://www.numdam.org/item?id=ASNSP_1963_3_17_1-2_97_0>

STABLE MANIFOLDS FOR DIFFERENTIAL EQUATIONS AND DIFFEOMORPHISMS (1)

S. SMALE (2) (New York)

1. Preliminaries.

A (first order) *differential equation* (« autonomous ») may be considered as a C^∞ vector field X on a C^∞ manifold M (for simplicity, for the moment we take the C^∞ point of view; manifolds are assumed not to have a boun. dary, unless so stated). From the fundamental theorem of differential equations, there exist unique C^∞ solutions of X through each point of M. That is, if $x \in M$, there is a curve

$$\varphi_t(x), \ | \ t \ | < \varepsilon \text{ such that, } \quad \varphi_0(x) = x, \ \frac{d\varphi_t}{dt}(x)\Big|_{t = t_0} =$$

$$= x(\varphi_{t_0}(x)) \text{ if } | \ t_0 \ | < \epsilon, \text{ and } \varphi_t(x) \text{ is } C^\infty \text{ on } (t, x)$$

(in a suitable domain).

Moreover if M is compact then $\varphi_t(x)$ is defined for all $t \in R$ (R the real numbers) and X defines a 1-parameter group of transformations of M.

More precisely, a *1-parameter group of transformations* of a manifold M is a C^∞ map $F : R \times M \to M$ such that if $\varphi_t(x) = F(t, x)$, then

$$\text{(a)} \qquad \varphi_0(x) = x$$

$$\text{(b)} \qquad \varphi_{t+s}(x) = \varphi_t \varphi_s(x).$$

---

(1) Lectures given at Urbino, Italy, July 1962, CIME. This work has been partially supported by the National Science Foundation under Grant GP-24.

(2) The author was a Sloan fellow during part of this work.

Then for each t, $\varphi_t : M \longrightarrow M$ is a diffeomorphism (a differentiable homeomorphism with differentiable inverse). A differential equation on a compact manifold defines or *generates* a 1-parameter group of transformations of M. We shall say more generally that a *dynamical system* on a manifold M is a 1-parameter group of transformations of M.

If φ_t is a dynamical system on M, $\left.\dfrac{d\varphi_t(x)}{dt}\right|_{t=0} = X(x)$ defines a C^∞ vector field on M which in turn generates φ_t. We also speak of X as the dynamical system.

Let X, Y be dynamical systems on manifolds M_1, M_2 respectively generating 1-parameter groups φ_t, ψ_t. Then X and Y (or φ_t, ψ_t) are said to be (topologically) equivalent if there is a homeomorphism $h: M_1 \longrightarrow M_2$ with the property that h maps orbits of X into orbits of Y preserving orientation.

The homeomorphism $h: M_1 \longrightarrow M_2$ will be called an equivalence. Often $M_1 = M_2$.

The qualitative study of (1st order) differential equations is the study of properties invariant under this notion of equivalence, and ultimately finding the equivalence classes of dynamical systems on a given manifold (3).

In this paper we are concerned with the problem of topological equivalence. An especially fruitfull concept in this direction is that of *structural stability* due to Andronov and Pontrjagin, see [5]. The definition in our context is as follows.

Assume a fixed manifold M, say compact for simplicity, has some fixed metric on it. An equivalence $h: M \longrightarrow M$ (between two dynamical systems on M) will be called an ε-equivalence if it is pointwise within ε of the identity. One may speak of two vector fields X and Y on M as being C' close (or $d_{C'}(X, Y) < \delta$) if they are pointwise close and in addition, in some fixed finite covering of coordinate systems of M, the maximum of the difference of their 1st derivatives over all these coordinate systems is small. (Similarly one can define a C^r topology, $1 \leq r \leq \infty$, see [7]). Then X is *structurally stable* if given $\varepsilon > 0$, there exists $\delta > 0$ such that if a vector field Y on M satisfies $d_{C'}(X, Y) < \delta$, then X and Y are ε-equivalent.

The problem of structural stability is: given M compact, are the structurally stable vector fields on M, in the above C' topology, dense in all vector fields. If the dimension of M is less than 3, the answer is yes by a theorem of Peixoto [9]; in higher dimensions it remains a fundamental and difficult problem.

---

(3) For a survey of this problem see talk in the Proceedings of the International Congress of Mathematicians, Stockholm 1962.

Although in this paper we are not concerned explicitly with structural stability, this concept lies behind the scenes. Attempts at solving the problem of structural stability, guide one toward the study of the generic or general dynamical systems in contrast to the exceptional ones.

There seems to be no general reduction of the qualitative problems of differential equations. However, there is a problem which has some aspects of a reduction. This is the *topological conjugacy problem for diffeomorphisms* which we proceed to describe.

Two diffeomorphisms $T : M_1 \longrightarrow M_1$, $T' : M_2 \longrightarrow M_2$ are topologically (differentiably) conjugate if there exists a homeomorphism (diffeomorphism) $h : M_1 \longrightarrow M_2$, such that $T'h = hT$. Often $M_1 = M_2$. This topological conjugacy problem is to obtain information on the topological equivalence classes of diffeomorphisms of a single given manifold (4).

When $\dim M = 1$, the problem is solved according to results of Poincaré, Denjoy and others, see [2]. For $\dim M > 1$, there are very few general theorems. We now explain the relevance of this problem to differential equations.

2. Cross-sections.

Suppose X, or φ_t, is a dynamical system on a manifold M. A crosssection for X is a submanifold Σ of codimension 1 of M, closed in M, such that

(*a*) Σ is transversal to X,

(*b*) if $x \in \Sigma$, there is a $t > 0$ with $\varphi_t(x) \in \Sigma$,

(*c*) if $x \in \Sigma$, there is a $t < 0$ with $\varphi_t(x) \in \Sigma$, and

(*d*) Every solution curve passes through Σ.

If X admits a cross section Σ, one can define a map $T : \Sigma \longrightarrow \Sigma$ by $T(x) = \varphi_{t_0}(x)$ where t_0 is the first t greater than zero with $\varphi_t(x) \in \Sigma$. It is not difficult to prove that $T : \Sigma \longrightarrow \Sigma$ is a diffeomorphism, called the *associated diffeomorphism* of Σ.

One can also easily prove that, if M is compact and connected, then conditions (*c*) and (*d*) in the definition of cross-section are consequences of (*a*) and (*b*).

Suppose on the other hand $T_0 : \Sigma_0 \longrightarrow \Sigma_0$ is a diffeomorphism of a manifold. Then on $R \times \Sigma_0$ let (t, x) be considered equivalent to the point $(t + 1, T(x))$. The quotient space under this equivalence relation is a new differentiable manifold say M_0. Let X_0 be the dynamical system on M_0 induced by the constant vector field $(1, 0)$ on $R \times \Sigma_0$, and $\pi : R \times \Sigma_0 \longrightarrow M_0$

(4) See footnote (3).

the quotient map. We say that X_0 on M_0 is the dynamical system *determined* by the diffeomorphism $T_0 : \Sigma_0 \to \Sigma_0$.

2.1. LEMMA.

Let φ_t be dynamical system generated by X on M, which admits a cross section Σ. Then by a C^∞ reparameterization $s_x(t)$ of t, $x \in M$, one can obtain a 1-parameter group φ_s of transformations of M such that if $x \in \Sigma$, $\varphi_1(x) \in \Sigma$ and $\varphi_s(x) \notin \Sigma$ for $0 < s < 1$.

We leave straightforward proof of 2.1 to the reader.

2.2. THEOREM.

Suppose the dynamical system φ_t generated by X on M admits a cross-section Σ with associated diffeomorphism T. Let X_0 on M_0 be the dynamical system determined by the diffeomorphism T. Let X_0 on M_0 le the dynamical system determined by the diffeomorphism $T: \Sigma \to \Sigma$. Then X_0 on M_0 is equivalent to X on M (by a diffeomorphism in fact).

PROOF. First apply 2.1. Then the desired equivalence of 2.2 can be taken as induced by $f: M \to R \times \Sigma$, $f(\varphi_s(x)) = (s, x)$ for $x \in \Sigma$.

2.3. THEOREM.

If $T_0 : \Sigma_0 \to \Sigma_0$ is a diffeomorphism, the dynamical system it determines has a cross-section Σ with the property that the associated diffeomorphism is differentiably equivalent to T_0.

For the proof. of 2.3, one just takes $\pi(0 \times \Sigma_0)$ for Σ, and the equivalence is induced by the map of $\Sigma_0 \to R \times \Sigma_0$ given by $x \to (0, x)$.

2.4 THEOREM.

Let $T_0 : \Sigma_0 \to \Sigma_0$, $T_1 : \Sigma_1 \to \Sigma_1$ be diffeomorphisms which determine respectively dynamical systems X_0 on M_0 and X_1 on M_1. If T_0 and T_1 are topologically (differentiably) equivalent then X_0 and X_1 are topologically equivalent, (equivalent by a diffeomorphism).

The proof is easy and will be left to the reader.

The preceeding theorems show that if a dynamical system admits a cross section, then the problems we are concerned with admit a reduction to a diffeomorphism problem of one lower dimension. Furthermore every diffeomorphism is the associated diffeomorphism of a cross-section of some dynamical system.

REMARK.

The existence of cross-sections in problems of classical mechanics first motivated Poincaré and Birkhoff [1] to study surface diffeomorphisms from the topological point of view.

A local version of the preceeding ideas on cross-sections is especially useful. A *closed or periodic orbit* γ of a dynamical system φ_t on M is a solution $\varphi_t(x)$ with the property $\varphi_t(x) = x$ for some $t \neq 0$. A *periodic point* of a diffeomorphism $T : \Sigma \longrightarrow \Sigma$ is a point $p \in \Sigma$ such that there is an integer $m \neq 0$ with $T^m(p) = p$ (T^m denotes the m^{th} power of T as atransformation). The following is clear.

2.5 Lemma.

Let φ_t be a dynamical system on M with cross-section Σ and associated diffeomorphism T. Then $p \in \Sigma$ is a periodic point of T if and only if the orbit of the dynamical system through p is closed.

A local diffeomorphism about $p \in M$ is a diffeomorphism $T : U \longrightarrow M$, U a neighbourhood of p and $T(p) = p$. Two local diffeomorphisms about $p_1 \in M_1, p_2 \in M_2$, $T_1 : U_1 \longrightarrow M_1$, $T_2 : U_2 \longrightarrow M_2$ are *topologically* (differentiably) *equivalent* if there exists a neighbourhood U of p_1 in U_1 and a homeomor phism (diffeomorphism) $h : U \longrightarrow U_2$ such that $h(p_1) = p_2$ and $T_2 h(x) = h T_1(x)$ for $x \in T_1^{-1}(U) \cap U$. The following is easily proved.

2.6 Lemma.

If X is a vector field on a manifold M, $X(p) \neq 0$, for some $p \in M$, there exists a submanifold Σ of codimension 1 of M containing p and transversal to X.

Let γ be a closed orbit of a dynamical system φ_t generated by X on $M, p \in \gamma$. Let Σ be given by 2.6 containing p, with $(CL\,\Sigma) \cap \gamma = p$. One defines a local diffeomorphism T of Σ about p by $T(x) = \varphi_t(x) \in \Sigma$ where x is in some neighbourhood U of p in Σ and t is the first $t > 0$ with $\varphi_t(x) \in \Sigma$. Call $T : U \longrightarrow \Sigma$, the local diffeomorphism associated to the closed orbit γ, Σ a local cross-section.

2.7. Lemma.

The differentiable equivalence class of T depends only on γ and the vector field X. It is independent of p and Σ.

Proof. Let $p_1, p_2 \in \gamma$ with local cross-sections Σ_1, Σ_2 respectively. Assume first $p_1 \neq p_2$. Then we can assume $\Sigma_1 \cap \Sigma_2 = \emptyset$. Define $h : U \longrightarrow \Sigma_2$, for U a sufficiently small neighbourhood of p in Σ_1, by $h(x) = \varphi_t(x)$ for $x \in U$ by taking $t > 0$ the first t such that $\varphi_t(x) \in \Sigma_2$. Then h acts as a differentiable equivalence. If $p_1 = p_2$, take $p_3 \in \gamma$, distinct from p_1, and with local cross-section Σ_3. Then apply the preceeding to show that the local diffeomorphism of Σ_1 is differentiably equivalent to that of Σ_3 and that of Σ_2 is differentiably equivalent to that of Σ_3. Transitivity finishes the proof.

Now given a local diffeomorphism about $p \in \Sigma$, $T : U \to \Sigma$, one can construct a manifold M_0, with a vector field X_0, containing a closed orbit γ with Σ as a local cross-section. The construction is the same as in the global case. Moreover, and this is a useful fact, the local analogues of 2.2-2.4 are valid.

3. Local Diffeomorphisms.

3.1. THEOREM.

Let $A : E^n \to E^n$ be a linear transformation with eigenvalues satisfying $0 < |\lambda_i| < 1$. Then there exists a Banach space structure on E^n such that $\| A \| = \lambda < 1$.

The proof follows from the fact that every real linear transformation is equivalent to a direct product of the following real canonical forms:

$$\left|\begin{array}{cc:ccc:cc}
\alpha & \beta & & & & & \\
-\beta & \alpha & & & & & \\
\hdashline
\gamma & 0 & & & & & \\
0 & \gamma & & & & & \\
 & & & \ddots & & & \\
\hdashline
 & & & & & & \\
 & & \gamma & 0 & & \alpha & \beta \\
 & & 0 & \gamma & & -\beta & \alpha
\end{array}\right| \quad \text{and} \quad \begin{pmatrix} \gamma & & \delta & & \\ & \ddots & & \ddots & \\ & & \gamma & & \delta \end{pmatrix}.$$

Here γ can be taken arbitrarily small, and $\alpha + i\beta$, $\alpha - i\beta$, δ are the eigenvalues. These canonical forms may be deduced from the usual Jordan canonical form, and the following two easy lemmas.

3.1*a* LEMMA.

The linear transformations given by the following two matrices are equivalent, where α, β are real.

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad \begin{pmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{pmatrix}$$

3.1b LEMMA

The linear transformations given by the following two matrices are equivalent where γ is non-zero, but otherwise arbitrary.

$$\begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ O & & 1 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & & & \\ \gamma & \lambda & & \\ & \ddots & \ddots & \\ O & & \gamma & \lambda \end{pmatrix}$$

The equivalence js given by $\begin{pmatrix} 1 & & & O \\ & \gamma & & \\ & & \ddots & \\ O & & & \gamma^n \end{pmatrix}$

A linear transformation satisfying the conditions of 3.1 will be called a *linear contraction.*

3.2 THEOREM.

Let T be a local diffeomorphism about the origin 0 of E^n whose derivative L at 0 is a linear contraction. Then there is an equivalence R between T and L which is C^∞ except at 0. In fact there is a global diffeomorphism $T' : E^n \longrightarrow E^n$ which agrees with T in some neighbourhood of 0 and a (global) equivalence R between T' and L, C^∞ except at 0.

PROOF.

By 3.1 we can assume $// L // < \theta < 1$, and that $T(x) = Lx + \bar{f}(x)$ where $\frac{// \bar{f}(x) //}{// x //} \longrightarrow 0$ as $// x // \longrightarrow 0$. Choose $r > 0$ so that $// \bar{f}(x) // < (1 - \theta) // x //$ for $// x // < r$. The following is well known.

3.2a LEMMA.

Given $r > 0$, there exists a real C^∞ function φ on E^n which is one on a neighbourhood of 0, $// \varphi(x) // \leq 1$ for all $x \in E^n$, and $\varphi(x) = 0$ for all $// x // \geq r$.

Let $f(x) = \varphi(x) \bar{f}(x)$ where $\varphi(x)$ is given by 3.2a. Then it is sufficient to prove 3.2 for T_0 and L where $T_0(x) = Lx + f(x)$, and T_0 is defined on all of E^n. Observe that for $// x // > r$, $T_0(x) = Lx$. Define $R : E^n \longrightarrow E^n$ by $R(0) = 0$ and $Rx = T_0^N L^{-N} x$ where N is large enough so that $// L^{-N} x // > r$. It is easy to check that R is well-defined, has the equivalence property, and is a C^∞ diffeomorphism except at 0. It remains to check that R is continuous at the origin, or that $\| R(x) \| \to 0$ as $\| x \| \to 0$. First note that there exists $k < 1$, so that for all $x \in E^n$, $\| T_0 x \| < kx$. Also $R(x) = T_0^N L^{-N} x = T_0^N y$ where $y = L^{-N} x$ and we can assume $\| y \| < M$. Then continuity follows from the fact that as $\| y \| \to 0$, the N of defintion of $R(x)$ must go to infinity.

A local diffeomorphism satisfying the condition of 3.2 is called a *local contraction.* A *contraction* of E^q is a diffeomorphism T of E^q onto itself such that there is a differentiably imbedded disk $D \subset E^q$ with $T\ Dc$ interior D, $\bigcap_{i>0} T^i D =$ origin of E^q, $\bigcup_{i<0} T^i D = E^q$. Thus using 3.1, the T constructed in 3.2 is a contraction. If all the eigenvalues of a linear transformation L have absolute value greater than one, then L is called a *linear expansion.* If the derivative at p of a local diffeomorphism T about p is a linear expansion then T is called a *local expansion.* The inverse of a linear (local) expansion is a linear (local) contraction. In this way 3.1 and 3.2 give information about linear and local expansions.

The following theorem was known to Poincaré for $\dim E = 2$. One can find n dimensional versions in Petrovsky [10], D. C. Lewis [6], Coddington and Levinson [2], Sternberg [14] and Hartman [4]. Some of these authors were concerned mainly with the similar theorem for differential equations.

3.3 THEOREM.

Let $T : U \to E$ be a local diffeomorphism about 0 of Euclidean space whose derivative $L : E \to E$ at 0 is a product of L_1, $E_1 \to E_1$, $L_2 : E_2 \to E_2$, $E = E_1 \times E_2$ where $\| L_1 \|$, $\| L^{-1} \| < 1$. Then there is a submanifold V of U with the following properties:

(*a*) $0 \in V$, the tangent space of V at 0 is E_1,

(*b*) $T V \subset V$, and

(*c*) there exists a differentiable equivalence R between a local diffeomorphism T' about 0 of E whose derivative at 0 is L_1 and T restricted to V.

(*d*) $V = \bigcap_{j=0}^{\infty} B_j$ where $B_0 = U \cap TU$ and B_j is defined inductively by $B_j = T^{-1}(B_{j-1} \cap B_0)$.

Due to the previous discussion in this section, the hypothesis of 3.3 is mild, merely that no eigenvalue of L has absolute value 1. One may apply 3.2 to the restriction of T to V. Note by applying 3.3 to T^{-1} one can obtain a submanifold V_2 of U containing 0 whose tangent space at 0 is E_2 and T restricted to V_2 is a local expansion. We call V the *local stable manifold*, V_2 the *local unstable manifold* of T at 0.

Use (x, y) for coordinates of $E = E_1 \times E_2$, so that one can write, using Taylor's expansion,

$$T(x, y) = (L_1 x + g_1(x, y), L_2 y + g_2(x, y)).$$

The proof of 3.3 is based on the following lemma.

3.4 LEMMA.

There exists a unique C^∞ map
$\Phi : U_1 \longrightarrow E_2$, U_1 a neighbourhood of 0 in E_1,
$\Phi(0) = 0$, $\Phi'(0) = 0$ satisfying

3.5 $$\Phi(L_1 x + g_1(x, \Phi(x))) = L_2 \Phi(x) + g_2(x, \Phi(x)).$$

Furthermore $(x, \Phi(x)) \in \bigcap_{j=0}^{\infty} B_j$, B_j as in (d) of 3.3.

To see how 3.3 follows from 3.4, let V be the graph of Φ, i. e. $V = (x, \Phi(x)) \in E_1 \times E_2$ for all x in U_1, where by 3.2 we assume $T' U_1 \subset U_1$. Then letting $R : U_1 \longrightarrow V$ be defined by $R(x) = (x, \Phi(x))$, $T' : U_1 \longrightarrow E_1$ by $T'(x) = L_1 x + g_1(x, \Phi(x))$ and using the equation of 3.4, it is easily verified that V, R, T' satisfy 3.3. Thus it remains to prove 3.4.

This we do not do here, but remark that one solves the functional equation 3.5 by the method of successive approximations.

4. Stable manifolds of a periodic orbit.

The global stable and unstable manifolds we construct in this section were considered by Poincaré and Birkhoff [1] in dimension 1 for a surface diffeomorphism. The analagous stable manifolds for a dynamical system (see section 9) have been considered by Elsgoltz [3], Thom [15]. Reeb [11] and in [12].

Suppose $T : M \longrightarrow M$ is a diffeomorphism and $p \in M$ is a periodic point of T so that $T^m(p) = p$. The derivative L of T^m at p will be a linear automorphism of the tangent space M_p of M at p. The point p will be called an elementary periodic point of T if L has no eigenvalue of absolute value 1, and transversal if no eigenvalue of L is equal to 1.

4.1 THEOREM.

Let p be an elementary fixed point of a diffeomorphism $T : M \longrightarrow M$, and E_1 the subspace of M_p corresponding to the eigenvalues of the derivative of T at p of absolute value less than 1. Then there is a C^∞ map R : $E_1 \longrightarrow M$ which is an immersion (i. e. of rank $=$ dim E_1 everywhere), 1-1, and has the property $TR = RT'$ where $T' : E_1 \longrightarrow E_1$ is a contraction of E_1. Also $R(p) = p$ and the derivative of R at p is the inclusion of E_1 into M_p.

PROOF. One applies 3.3. The map R of 3.3, say R_0, is defined in a neighbourhood U of 0 in E_1 into M. We now extend it to all of E_1 to obtain the map R of 4.1. By 3.2 we can assume T' of 3.3 is a (global) contraction of E_1. If $x \in E_1$, let $Rx = T^{-N} R_0 T'^N x$ where N is large enough so that $T'^N x \in U$. It may be verified with little effort that R is well-defined and satisfies the conditions of 4.1.

The map $R : E_1 \longrightarrow M$, or sometimes the image of R, is called the *stable manifold* of p or T at p. The *unstable manifold* of p or T at p is the stable manifold of T^{-1} at p. These objects seem to be fundamental in the study of the topological conjugacy problem for diffeomorphisms. An (elementary) *periodic orbit* is the finite set $\bigcup_{i \in z} T^i p$ where p is an (elementary) periodic point. The definition of the stable manifold for an elementary periodic orbit (or sometimes elementary periodic point) is as follows. Let $\varphi : E_1 \longrightarrow M$ be the stable manifold of T^m at p where m is the least period of p, p in our periodic orbit. Then $R : E_1^i \longrightarrow M$ is defined by $R = T^i \varphi$ where $0 \leq 1 < m$ and E_1^i is a copy of E_1. Thus the *stable manifold* of a periodic orbit is the $1-1$ immersion R of the disjoint union of m copies of a Euclidean space. The stable manifold of a periodic point B is defined to be the component of the stable manifold of the associated periodic orbit in which B lies. The unstable manifold of a periodic orbit (periodic point) is the stable manifold of the periodic orbit (periodic point) relative to T^{-1}.

5. Elementary periodic points.

Let $\mathcal{D}$ be the set of all diffeomorphisms of classe C^r of a fixed compact C^r manifold M onto itself, $\infty \geq r > 0$. Endow $\mathcal{D}$ with the C^r topology (see [7]). It may be proved that $\mathcal{D}$ is a complete metric space. We recall that in a complete metric space the countable union of open dense sets must be dense.

5.1 THEOREM.

Let M be a compact C^r manifold, $r > 0$, and $\mathcal{D}$ the space of C^r diffeomorphisms of M endowed with the C^r topology. Let $\Xi \subset \mathcal{D}$ be the set of T with the property that every periodic point of T is elementary. Then Ξ is a countable union of open dense sets.

We prove the following stronger theorem which implies 5.1 (since $\Xi = \bigcap_{p \in Z^+} \Xi_p$, Z^+ denotes the positive integers).

5.2 THEOREM.

Let $\mathcal{D}$ be as in 5.1 and Ξ_p be the set of diffeomorphisms T with the property that every periodic point of T of period $\leq p$ is elementary. Then Ξ_p is open and dense in $\mathcal{D}$.

PROOF.

We first show that Ξ_p is open $\mathcal{D}$. Let $T_0 \in \Xi_p$, $T_i \to T_0$ in D, $T_i \in D$. It must be shown that $T_i \in \Xi_p$ for large enough i. Suppose not. Then there exist $p_i\, i = 1, 2, \ldots, |p_i| \leq p$, such that $T_i^{p_i}(x_i) = x_i$ and x_i is not an elementary periodic point of T_i of period p_i. By choosing subsequences, we can assume $x_1 \to x_0 \in M$ and the p_i are constant say p_0. Then $T^{p_0}(x_0) = x_0$. Thus x_0 is a periodic point of T of period $p_0 \leq p$ and elementary since $T \in \Xi_p$. So the derivative of T^{p_0} at x_0 has no eigenvalue of absolute value 1. On the other hand the derivative of $T_i^{p_0}$ at x_i for all i has an eigenvalue of absolute value 1.

This is a contradiction since $T_i \to T$ in the C^r topology, $r > 0$ and $x_i \to x_0$.

We next show that Ξ_p is dense in Ξ_{p-1}, $p \geq 1$, $\Xi_0 = \mathcal{D}$. This will finish the proof of 5.2. Let $\bar{\Xi}_p$ be the analogue of Ξ_p with elementary replaced by transversal. Then it is sufficient to prove: (1) $(\bar{\Xi}_p \cap \Xi_{p-1}$ is dense in Ξ_{p-1}, and (2) Ξ_p is dense in $\bar{\Xi}_p$. We first do the main step, (1).

For $p = 1$, we use the following easily proved lemma.

5.3 LEMMA.

Let $T: M \to M$ be a diffeomorphism. Then $x \in M$ is a transversal periodic point of T of period p if and only if the graph Γ of T^p and the diagonal $\triangle$ in $M \times M$ intersect transversally at (p, p) (i. e. the tangent space of $\triangle$ and Γ at (p, p) span the tangent space of $M \times M$ at (p, p).

Then a general position theorem of differential topology applies to yield that $\bar{\Xi}_1$ is dense in $\mathcal{B}$ (see Thom [16]).

Let $T \in \Xi_{p-1}$ and $\beta_1, \ldots, \beta_k$ be all the periodic points of T of period $\leq p - 1$. Then one can find neighbourhoods N_i of β_i so that any periodic point of T in $N = \bigcup_{i=1}^k N_i$ of period $\leq p$ is one of the β_1 and elementary.

Now let $\delta = \min_{x \in CL(M-N)} d(x, T^i x)$ where $|i| \leq p - i$. Then $\delta > 0$. By possibly choosing δ smaller we can assume that any set U of diameter $\leq 2\delta$ is contained in a coordinate neighbourhood of M and hence that $T(U)$, has the same property. Next for $t \in M$, let $U(x)$, $V(x)$, $W(x)$ be neighbourhoods of radius δ, $1/2^\delta$, $1/3^\delta$ respectively. Let $(U_\alpha, V_\alpha, W_\alpha)$ for $\alpha = 1, \ldots, q$ be a finite set of these such that $\bigcup W_\alpha = M$, for each α choose a coordinate neighbourhood $E_\alpha \supset CLTU_\alpha$.

Then using the linear structure of E_α, $T - S^{-1}: D_\alpha \to E_\alpha$ is a well defined map where $S = T^{p-1}$ and $D_\alpha = \{x \in U_\alpha \mid S^{-1} x \in E_\alpha\}$. By Sard's theorem, see e. g. [16], choose a map $g_\alpha: D_\alpha \to E_\alpha$, small with its first r derivatives so that $T - S^{-1} / g_\alpha: U_\alpha \to E_\alpha$ has 0 has a regular value. Starting with $\alpha = 1$, let $T_1 = T$ outside U_1, $T_1 = T + g_1$ on V_1 using 3.2a. Then T_1

restricted to V_1 has transversel periodic points of period p as can be seen as follows:

If $x \in V_1$ and $T_1^p(x) = x$, then $T_1^p(x) = T^{(p-1)} T_1 x$ and $T_1 x = S^{-1} x$. So $(T_1 - S^{-1}) x = 0$ and since $T_1 - S^{-1}$ has a regular value at 0, the derivative of $T_1^p - I$ at x is non-singular and x is a transversal singular point of T_1 of period p.

One makes the same construction for $\alpha = 2, \dots, q$, making sure that g_α is so small with respect to the « bump function » that the diffeomorphism retains its desirable qualities on N and $W_1, \dots, W_{\alpha-1}$. This proves that $\overline{F}_p$ is dense in F_{p-1}.

We finally show that Ξ_p is dense in $\overline{\Xi}_p$. Let $T_0 \in \overline{\Xi}_p$. Then by 5.3 the periodic points of T_0 of period $\leq p$ are isolated, hence finite in number, say $\beta_1, \dots, \beta_k$. Let $N_1, \dots, N_k$ be disjoint Euclidean neighbourhoods of the β_i. Then it is sufficient to show that given i, $1 \leq i \leq k$, there exists a diffeomorphism $T: M \longrightarrow M$ such that $T = T_0$ outside of N_i, T approximates T_0, and T has β_i as an elementary periodic point. This can be easily done using 3.2a and the fact that linear transformations with no eigenvalue of absolute value 1 are dense in all linear transformations. This finishes the proof of 5.2.

REMARK.

If $T \in \Xi$, then given an integer N, there exists only a finite number of points of period $\leq N$ of T. This follows from 5.3. Hence T has only a countable number of periodic points.

6. Normal intersection.

Two submanifolds W_1, W_2 of a manifold M have normal intersection if for each $x \in W_1 \cap W_2$, the tangent space of W_1 and W_2 at x span the tangent space of M at x. A diffeomorphism $T: M \longrightarrow M$ has the *normal intersection property* if when β_1, β_2 are generic periodic points of T, the stable manifold of β_1 and the unstable manifold of β_2 have normal intersection (this definition is clear even though the stable manifold is not strictly a submanifold).

Let $\mathcal{D}$ and Ξ be as in the previous section and let $\mathcal{E}$ be the subspace of Ξ of diffeomorphisms with the normal intersection property.

6.1. THEOREM.

$\mathcal{E}$ is the countable intersection of open dense subsets of $\mathcal{D}$. (The first theorem of this kind seems to be in [13]).

Let our basic manifold M have some fixed metric and let $N_\epsilon(x)$, for $\epsilon > 0$, $x \in M$, denote the open ϵ neighbourhood of x in M. Let R^+ be the set of positive real numbers.

6.1a LEMMA.

For each $p \in Z^+$, there exists a continuous function $\epsilon : \Xi_p \to R^+$ with the following property. If $T \in \Xi_p$, $x \in M$ is a periodic point of T of period $\leq p$, then $CL\,[N_{\epsilon(T)}(x) \cap W(x)] \subset W(x)$, where $W(x)$ is either the stable or unstable manifold, $W^s(x)$ or $W^u(x)$ respectively, of x with respect to T.

PROOF.

It is clear that on an open neighbourhood N_α of each $T_\alpha \in \Xi_p$ that one can find a constant function ϵ_α with the property of ϵ of 6.1a. Let ϵ_α, N_α be a countable covering of Ξ_p of this type, $\alpha = 1, 2, \dots$. Then let $\epsilon' = \epsilon_1$ on N_1, $\epsilon' = \min(\epsilon_1, \epsilon_2)$ on $N_2 - N_1$, $\min(\epsilon_1, \epsilon_2, \epsilon_3)$ on $N_3 - N_1 - N_2$ etc.. Then ϵ^1 is lower semi continuous on F_p. Finally, for example by Kelly, General Topology, New York 1955 p. 172 one can obtain the ϵ of 6.1a.

Now if x is a periodic point of $T \in F_p$ of period $\leq p$, let $L^\tau(x) = L^\tau(x, T) = CL\,[N_{\epsilon(T)}(x) \cap W^\tau(x)] \cdot \tau = s$ or $\tau = u$. Define ϵ_p^k, $k \in Z^+$, to be the subspace of Ξ_p of diffeomorphisms with the following property. If $x, y \in M$ are periodic points of period $\subseteq p$ of $T \in \Xi_p$, then at each point of $T^k(L^u(x)) \cap T^{-k}(L^s(y))$, $W^u(x)$ and $W^s(y)$ have normal intersection.

6.2. THEOREM.

ϵ_p^k is open and dense in D.

Note that (6.1) follows from (6.2) because $\epsilon = \bigcap_{p, k \in Z^+} \epsilon_p^k$.

For 6.2 we first remark that ϵ_p^k is clearly open in $\mathcal{D}$. Hence in view of 5.2 it is sufficient to show that ϵ_p^k is dense in Ξ_p.

Let $T \in \Xi_\rho$. Denote by $\beta_1, \dots, \beta_r$ the periodic points of T of period $\leq p$ with stable and unstable manifolds $W_i{}^\tau = W^\tau(\beta_i)$, $\tau = s, u$.

We will consider only approximations T' of T which agree with T on some neighbourhood V_0 of the β_i, and so that β_i, $i = 1, \dots, k_0$ are precisely the periodic points of T' of period $\leq p$. Let the corresponding stable manifolds of such a T' be denoted by W_i^{s}, $i = 1, \dots, k_0$, etc.

With T' as above, there is a canonical map φ: $W_i^\tau \to W_i^{\tau'}$, $\tau = s, u$, $i = 1, \dots, k_0$ defined as follows. If $x \in W_i^s$; $m =$ period β_i, there is a positive integer N_0 such that $T^{mn}(x) \in V_0$ for all $N \geq N_0$. Let $\varphi x = T^{-mn_0} T^{mn_0} x$. For $x \in W_i^u$ one takes $n \leq n_0 < 0$. Then φ is a well defined 1-1 immersion.

Now fix i, j, $1 \leq i, j \leq k_0$. Ii is sufficient for 6.2 to approximate T by T' as above such that on the intersection of $T'^{k}(L_i^{u'})$ and $T'^{-k}(L_j^{s'})$, $W_i^{u'}$ and $W_j^{s'}$ have normal intersection where $L_i^{u'} = \varphi L_i^u$, $L_i^u = L^u(\beta_i, T)$ etc.

The first stage of this argument is to replace $T^k(L_i^u)$ and $T^{-k}(L_j^s)$ by submanifolds of M, Y_1 and Y_2 respectively with the following properties:

(6.3) The Yi are diffeomorphic to disks,

$$W_i^u \supset T^{m_1}(Y_1)\rangle\rangle\, Y_1\rangle\rangle\, T^k(L_i^u)$$

$$W_j^s \supset T'^{-m_2}(Y_2)\rangle\rangle\, Y_2\rangle\rangle\, T^{-k}(L_j^s).$$

Here M_1 is the least period of β_i, m_2 that of β_j and $\rangle\rangle$ is interpreted as to mean « contains an open set containing ». Such Y_i clearly exist from 3.1, 3.2 and 4.1.

If T' is an approximation of T agreeing with T on a neighbourhood V_0 of the β_i let $\varphi\, Y_i = Y_i'$, $i = 1, 2$. Then without loss of generality we can assume

$$W_i^{u'} \supset T'^{m_1}(Y_1')\rangle\rangle\, Y_1'\rangle\rangle\, T'^k(L_i^{u'}) \tag{6.3'}$$

$$W_j^{s'} \supset T'^{-m_2}(Y_2')\rangle\rangle\, Y_2'\rangle\rangle\, T'^{-k}(L_j^{s'}).$$

Hence it is sufficient to find such a T' with Y_1' and Y_2' having normal intersection.

The compact subset $CL(T_0^{m_1} Y_1 - Y_1)$ is so to speak a fundamental domain of T^{m_1} restricted to W_i^u. Thus one may find without difficulty connected open sets Z_1, Z_2 in W_i^u with compact closures which are each disjoint from their images under T^{m_1} and in addition $Z_1 \cup Z_2 \supset CL(T^{m_1} Y_1 - Y_1)$.

$$\text{Let } P = CL\{T^l(\bar{Z}_1 \cap Y_2) \mid l \geq 0\}$$

$$Q = CL\{T^{-l}(\bar{Z}_1) \mid l > 1\}$$

The following is easily checked.

6.4 LEMMA.

$$P \cap T^{-1}(\bar{Z}_1 \cap Y_2) = \emptyset$$

$$Q \cap T^{-1}(\bar{Z}_1 \cap Y_2) = \emptyset$$

Let U_p, U_q, V be open sets such that $U_p \supset P$, $U_q \supset Q$, $V \supset T^{-1}(\bar{Z}_1 \cap Y_2)$ and $\bar{U}_p \cap V = \emptyset$, $U_q \cap V = \emptyset$.

By the Thom transversality theorem [16] and a suitable patching by a C^∞ function (similar) to 3.2a) one can find an approximation T' of T with the following properties:

a) $T' = T$ on a neighbourhood V_0 of the β_i and the complement of V in M.

(*b*) $T'[T^{-1}(\bar{Z}_1)]$ and Y_2 have normal intersection (i. e. W_j^s and $T'[T^{-1}(Z_1)]$ have normal intersection on $T'[T^{-1}(\bar{Z}_1)] \cap Y_2$).

Suppose now that $x \in Y_1' \cap Y_2'$, and $T'^m x \in Z_1'$ for some integer m where $Z_1' = q(Z_1)$. We will show that at x, Y_1' and Y_2' have normal intersection. This is a consequence of the following statements.

(*a*) $m \geq 0$ and $T'^m x \in Y_2'$

(*b*) Z_1' is $T'[T^{-1}(Z_1)]$ and so $T'^m x \in T'\ [T^{-1}(Z_1)]$.

(*c*) there exists a neighbourhood of $T'^m x$ in Y_2' which is in Y_2.

It can be shown without difficulty that (*a*), (*b*), and (*c*) are consequences of the choice of V.

Now one carries out exactly the same procedure with $Z_2' = q(Z_2)$ replacing Z_1 in the argument. This gives us an approximation T'' of T' with the desired properties of 6.2.

7. Elementary singularities of a vector field.

We now pass from the diffeomorphism problem to the case of a dynamical system.

Let M be a compact C^r manifold $1 \leq r \leq \infty$ and β the space of all C^r vector fields on M with the C^r topology. One may put a Banach space structure on β if $r < \infty$. In any case β is a complete metric space.

A *singularity* p of X on M is a point at which X vanishes. Let p be a singularity of X on M. Then using some local product structure of the tangent bundle, in a neighbourhood U of p, X is a differentiable map, X: $U \longrightarrow M_p$, whose derivative A at p is a linear transformation of M_p.

We will say that p is an elementary singularity of X on M if the derivative A of X at p has no eigenvalue of real part one, and transversal if A is an automorphism.

Let $\mathcal{C}$ be the subset of β such that if $X \in \mathcal{C}$ X has only elementary singularities.

7.2 Theorem.

$\mathcal{C}$ is an open dense set of β.

To see thas, one first checks the following lemma.

7.3 Lemma.

Let X be a vector field on M. Then $x \in M$ is a transversal singular point of X if and only if X, as a cross section in the tangent bundle meets the zero cross-section over M transversally.

From this and the transversality theorem of Thom [16] one concludes

7.4 Lemma.

Let $\mathcal{C}$ be the subset of β of vector fields on M which have only transversal singular points. Then $\mathcal{C}'$ is an open dense subset of β.

Now 7.2 follows from 7.4 as in the proof of 5.2 where Ξ_p was shown to be dense in $\overline{\Xi}_p$.

Note that if $X \in \mathcal{C}$, or even $\mathcal{C}'$, by 7.3, the singular points of X are isolated and hence finite in number.

8. Elementary closed orbits.

Let γ be a closed orbit of a vector field X on a manifold with associated local diffeomorphism $T: U \longrightarrow \Sigma$ about $p \in \gamma \cap \Sigma$. Then γ will be called an *elementary* (transversal) closed orbit of X if T has p as an elementary (transversal) fixed point.

8.1 Theorem.

Let $\mathcal{C}_0$ be the subspace of $\mathcal{C}$ (of section 7) of vector fields X on M such that every closed orbit of X is elementary. Then $\mathcal{C}_0$ is the countable intersection of open dense sets of β. L. Marcus [18] has a theorem in this direction. Also R. Abraham has an independent proof of 8.1 [17].

If γ is a closed orbit of X on M, then one can assign a positive real number, the period of γ as follows. Let $x \in \gamma$, $\varphi_{t_0}(x) = x$ where $t_0 > 0$, $\psi_t(x) \neq x$, $0 < t < t_0$. Then t_0 is an e invariant of γ, the *period* of γ.

For a positive real number L, let $\mathcal{C}_L \subset \mathcal{C}$ consist of X on M such that, if γ is a closed orbit of length $\leq L$, then γ is elementary.

Since $\mathcal{C}_0 = \bigcap_{L \in Z^+} \mathcal{C}_L$, with 7.2, 8.1 is a consequence of the following.

8.2 Theorem.

For every positive L, $\mathcal{C}_L$ is open and dense in $\mathcal{C}$.

The proof is somewhat similar to the proof of 5.2.

First that $\mathcal{C}_L$ is open in $\mathcal{C}$ follows from a similar argument to that of 5.2 used in showing that Ξ_p is open in Ξ. We leave this for the reader.

It remains to show: $\mathcal{C}_L$ is dense in $\mathcal{C}$. Let $X \in \mathcal{C}$. The first step is to construct a finite number of open cells U_α of M of codimension 1, transversal to X such that $U_\alpha \supset W_\alpha$ where W_α is a closed sub-disk of U_α such that every trajectory of X passes through some W_α. It is a straightforward matter to show that such a set of (U_α, W_α) exists.

Fixing α now, the next step is to approximate X by X', a vector field on M equal to X outside a neighbourhood of W_α so that if γ is a closed orbit of X' of length $\leq L$, intersecting some fixed neighbourhood of W_α in U_α, then γ is elementary. The existence of such an approximation is sufficient for the proof of 8.2.

The construction of the approximation X' of the preceeding paragraph is based on the methods of Section 2 and 5. We outline how this is done. Let V_α be a compact neighbourhood of W_α in U_α. Then let $D_\alpha \subset U_\alpha$ be the set of points x of U_α such that $\varphi_t(x) \in V_\alpha$ for some t, $0 \leq t \leq 2L$, and $T : D_\alpha \to V_\alpha$ the associated diffeomorphism, say really defined on some neighbourhood of D_α in U_α. Now apply the methods of 5.2 to approximate T by T' such that T' is defined in a neighbourhood of D_α and that T' has only generic periodic points. Now using the construction of Section 2 and 3.2a one defines the above X' using T'.

9. Stable manifolds for a differential equation.

The following is the global stable manifold theorem for singularities of a vector field.

9.1. THEOREM

Let X be a C^∞ vector field on a C^∞ manifold generating a 1-parameter group φ_t, with an elementary singularity at $x_0 \in M$. Let $E_1 \subset M_{x_0}$ be the subspace of the tangent space of M at x_0 corresponding to the eigenvalues with real part negative. Then there is a $1-1$ C^∞ immersion $\psi : E_1 \to M$ with the following properties:

(a) X is everywhere tangent to $\psi(E_1)$ and as $t \to \infty$, $\varphi_t(x) \to X_0$ for all $x \in \psi(E_1)$.

(b) $\psi(0) = x_0$ and the derivative of ψ at x_0 is the inclusion of E_1 into M_{x_0}.

PROOF

It can be checked that the map R of 4.1 satisfies 9.1 using φ_1 for T of 4.1.

Of course there is a local version of this theorem which can be found for example in [2]. One may also derive 9.1 directly from this. The map ψ of 9.1 or its image is called the *stable manifold* of x_0.

One has a stable manifold associated to an elementary closed orbit of a differential equation by the following theorem.

9.2. THEOREM

Let γ be an elementary closed orbit of a differential equation X on M generating a 1-parameter group φ_t. Let $x \in \gamma$, $T : \Sigma \to \Sigma$ be an associated local diffeomorphism of γ at x with derivative L at X, and E_1 the linear subspace of M_x tangent to Σ corresponding to the eigenvalues of L with absolute value < 1. Then there exists a contraction $T_1 : E_1 \to E_1$ with the following true. The construction preceeding 2.1 applied to T_1 defines a manifold M_0 with a vector field X_0 on M_0. Then there is a $1 - 1$ immersion $\psi : M_0 \to M$ mapping X_0 into X up to a scalar factor and $\psi(p) = x$ where p is the point of M_0 corresponding to $(0, 0)$ of $E_1 \times R$ (in the definition of M_0).

For the proof we only need to note that ψ is defined first in a neighbourhood of $0 \times R$ and then extended to M_0 by the device used in the proof of 4.1.

Then ψ or its image is called the ***stable manifold of*** γ. The ***unstable manifold*** of a singularity or closed orbit of X on M is the respective stable manifold with respect to $-X$.

In general M_0 is either $S' \times E$, or the twisted product.

If X is a dynamical system on a manifold M, we say that X has the *normal intersection property* if the stable and unstable manifolds of X have normal intersection with each other. Fixing compact M, let $\mathcal{C}_0$, β be as in the previous section and $\mathcal{A}_0$ be the set of X in $\mathcal{C}_0$ with the normal intersection property.

9.3. THEOREM

$\mathcal{A}_0$ is the countable intersection of open and dense sets of β.

This theorem and its proof are somewhat analagous to (6.1).

For the proof of 9.3, let $\epsilon : \mathcal{C}_L \to R$ be defined in a completely analogous fashion to the ε of (6.1a) where $\mathcal{C}_L$ is defined in section 8. Let $X \in \mathcal{C}_L$ and x be a singular point of X or a closed orbit of period $\leq L$ of X. Let $W^\tau(x)$, $\tau = u, s$ be the unstable manifold, stable manifold respectively of x and $L^\tau(x) = \mathcal{A}(N_{\epsilon(X)}(x) \cap W^\tau(x))$ similar to the proof of 6.1. Next let $\mathcal{A}_{Lr}$ be the subspace of X of $\mathcal{C}_L$ with the following property: If x, y are singular points or periodic orbits of X of period $\leq L$, then at each point of $\varphi_r(L^u(x)) \cap \varphi_{-r}(L^s(y))$, $W^u(x)$ and $W^s(y)$ have normal intersection. Here φ_t is generated by X and $r > 0$. Then 9.3 is implied by the following.

9.4 THEOREM.

$\mathcal{A}_{L^r}$ is open and dense in $\mathcal{D}$.

As in section 6, for the proof of 9.4, it is sufficient to approximate a given $X \in \mathcal{C}_L$ by a vector field in $\mathcal{A}_{L^r}$.

Also just as in section 6, one defines maps $\mathcal{q}$ and submanifolds Y_1 of M. The only difference in the proof from that of 6.1 is in the details of the construction of the approximation itself. One uses here exactly the approximation in [13] page 202. We will not repeat it here, but only remark that one can do it a little simpler than in [13] by changing X on a finite sequence of Euclidean cells one at a time.

This completes the proof of 9.3.

We conclude by remarking that if one takes for M, the 2-sphere, then $\mathcal{A}_0$ is open as well as dense in $\mathcal{L}$ that each $X \in \mathcal{A}_0$ has only a finite number of closed orbits, and by a theorem first stated essentially by Andronov and Pontrjagin, X is structurally stable. In this case, i. e., $M = S^2$, density of $\mathcal{A}_0$ in β was first proved by M. Peixoto [8].

REFERENCES

1. G. D. BIRKHOFF, *Collected Mathematical Papers* New York 1950.
2. CODDINGTON and LEVINSON, *Theory of Ordinary differential equations,* McGraw-Hill, New York 1955.
3. L. E. ELSGOLTZ, *An estimate for the number of singular points of a dynamical system defined on a manifold,* Amer. Math. Soc. Translation No. 68, 1952.
4. P. HARTMAN, *On local homeomorphisms of Euclidean space, Proceedings of the Symposium on Ordinary Differential Equations,* Mexico City, 1959.
5. S. LEFSCHETZ, *Differential Equations, Geometric Theory,* New York 1957.
6. D. C. LEWIS, *Invariant manifolds near an invariant point of unstable type,* Amer. Journal Math. Vol. 60 (1938) pp. 577-587.
7. R. S. PALAIS, *Local Triviality of the restriction map for embeddings,* Comm. Math Helv. Vol. 34 (1960) pp. 305-312.
8. M. PEIXOTO, *On structural stability* Ann. of Math. Vol. 69 (1959) pp. 199-222.
9. M. PEIXOTO, *Structural stability on 2-dimensional manifolds,* Topology Vol. 2 (1962) pp. 101-121.
10. I. PETROVSKY, *On the behavior of the integral curves of a system of differential equations in the neighbourhood of singular point.* Rec. Math. (Mat. Sbornik) N. S. Vol. 41-(1934) pp. 107-155.
11. G. REEB, *Sur certaines propriétés topologiques des projectoires des systemes dynamiques,* Acad. Roy. Belg. Cl. Sci. Mem. Coll. 8⁰ 27 N⁰ 9, (1952).
12. S. SMALE, *Morse inequalities for a dynamical system,* Bull Amer. Math. Soc. Vol. 48 (1940) pp. 883-890.
13. S. SMALE, *On Gradient Dynamical Systems,* Ann. of Math. Vol. 74 (1961) pp. 199-206.
14. S. STERNBERG, *Local contractions and a theorem of Poincaré,* Amer. Journ. Math, Vol. 79 (1957) pp. 809-824.
15. R. THOM, *Sur une partition en cellules associee à une function sur une variété,* C. R. Acad. Sci. Paris Vol. 228 (1949) pp. 973-975.
16. R. THOM, *Quelques propriétés globoles des variétés differentiables,* Comm. Math. Helv., 28 (1954), pp. 17-86.
17. R. ABRAHAM, *Transversality of manifolds of mappings to appear.*
18. L. MARCUS, *Structurally stable differential systems,* Ann, of Math. Vol. 73 (1961) pp. 1-19.