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REGULAR POINTS FOR ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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Introduction.

The main purpose of this paper is to investigate the notion of regular boundary points for Dirichlet's problem with respect to uniformly elliptic equations in divergence form

$$(*) \qquad Lu = -(a_{ij}(x)\, u_{x_i})_{x_j} = 0$$

when the coefficients are only supposed to be bounded and measurable. The boundary values are given by an arbitrarily assigned continuous function.

When L is the Laplace operator the question of determining whether the solution attains its boundary value continuously at a particular boundary point (regularity) is classical. The regular points were characterized by Wiener [30.31]. The Wiener criterion has been stated and proved in several ways by Kellogg [8], Vasilesco [9]. De la Vallée Poussin [10], Frostman [5], and Brelot [2]. Püschel [17] showed that a boundary point is regular for the equation $(*)$ if and only if it is regular for the Laplace operator. Püschel has to assume that the coefficients are twice continuously differentiable. Tautz [27,28] and Olĕinik [16] extended this result to equations with lower order terms. These proofs use the smoothness of the coefficients in a very essential way.

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Using an axiomatic approach, R. M. Hervé [7] has recently extended these results to equations of the form

$$a_{ij} u_{x_i x_j} + b_i u_{x_i} + cu = 0$$

with locally Lipschitz continuous coefficients.

In recent years it has been found that solutions of uniformly elliptic equations (*) have some continuity properties even when the coefficients are only bounded and measurable.

We will show (§ 3) that a locally continuous solution of (*) may be associated with any continuous boundary values. Therefore the question of regularity of a boundary point can again be posed.

In § 9 we prove that even in this general case the regular points are the same as those for Laplace's equation.

In order to prove this result we make extensive use of those recent results on uniformly elliptic equations with discontinuous coefficients described in § 2.

We also establish some new results which are of interest in themselves.

In § 5 we prove the existence of solutions vanishing at the boundary of $Lu = \mu$ for any measure μ of bounded variation. In particular, the Dirac measure gives rise to the Green's function $g(x, y)$ (§ 6). This Green's function enjoys the fundamental properties of the classical one.

The solution of $Lu = \mu$ can be represented by $u = \int g \, d\mu$ (§ 6). This allows us to prove that for non-negative μ, u is lower semi-continuous and has a mean value property (§ 8).

The capacitary potential u of a set E is defined by a classical variational problem (§ 4). The capacitary distribution μ arises naturally from the variational problem. The capacitary potential is a solution of the equation $Lu = \mu$.

An important role is played by the fact that the ratio $g(x,y)/\bar{g}(x,y)$ of the Green's functions corresponding to two equations of the form (*) is bounded in any compact subset by a constant depending only on the ellipticity constant (§ 7).

The results are extended to unbounded domains in § 10.

1. Some Notation and Preliminaries.

Let Ω be a bounded domain in Euclidean n-space, let $\partial\Omega$ denote its boundary, and $\bar{\Omega}$ its closure.

We denote the class of real continuous functions in $\bar{\Omega}$ by $C^0(\bar{\Omega})$. We let $C^1(\bar{\Omega})$ be the subclass of $C^0(\bar{\Omega})$ functions having continuous first partial derivatives in Ω which can be extended continuously to $\bar{\Omega}$.

The completion of $C^1(\bar{\Omega})$ with respect to the norm

$$\| u \|_{H^{1,p}(\Omega)} = \| u \|_{L^p(\Omega)} + \sum_{i=1}^{n} \| u_{x_i} \|_{L^p(\Omega)} \qquad (p \geq 1)$$

will be denoted by $H^{1,p}(\Omega)$.

We shall say that $u \in H^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in H^{1,p}(\Omega')$ for every Ω' with closure in Ω.

The closure in $H^{1,p}(\Omega)$ of the subclass $C_0^1(\bar{\Omega})$ of the $C^1(\bar{\Omega})$ functions vanishing near $\partial\Omega$ is $H_0^{1,p}(\Omega)$.

The dual of $H_0^{1,p}(\Omega)$ (for $p > 1$) is called $H^{-1,p'}(\Omega)$, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$. It is well known [11, p. 225; 20] that $H^{-1,p'}(\Omega)$ consist of distributions on Ω which are first derivatives of functions in $L^{p'}(\Omega)$. $H_0^{1,p}(\Omega)$ is a reflexive Banach space.

In the following we shall denote by Σ a sphere and consider the space $H_0^{1,p}(\Sigma)$. Of this space we have to use some properties which will be explained here.

To begin with we make the following:

Remark (1.1). Let M_0 be the space of Lipschitz functions in $\bar{\Sigma}$ vanishing near $\partial\Sigma$. Any function of M_0 can be approximated by functions of $C^1(\bar{\Sigma})$ vanishing near $\partial\Sigma$ in the norm of $H_0^{1,p}(\Sigma)$ for any $p \geq 1$. Therefore $C_0^1(\bar{\Sigma}) \subset M_0 \subset H_0^{1,p}(\Sigma)$, and $H_0^{1,p}(\Sigma)$ is also the completion of M_0 with respect to the norm of $H_0^{1,p}(\Sigma)$.

Let $u(x)$ be any element of $H_0^{1,p}(\Sigma)$. Since $u(x)$ belongs to $L^p(\Sigma)$, we have to distinguish two different definitions of positivity of u on a subset of Σ.

Definition (1.1). A function $u(x)$ of $H_0^{1,p}(\Sigma)$ will be said to be *non-negative on a set E in the sense of $L^p(\Sigma)$ or almost everywhere* (a. e.) if

$$\operatorname{meas} [\{x \mid u < 0\} \cap E] = 0.$$

Definition (1.2). A function $u(x)$ of $H_0^{1,p}(\Sigma)$ will be said to be *non-negative on a set E in the sense of $H_0^{1,p}(\Sigma)$* if there exists a sequence u_m of functions of M_0 such that: i) $u_m \geq 0$ on E; ii) $u_m \to u$ in $H_0^{1,p}(\Sigma)$.

Obviously the two definitions are quite different, and while the first one imposes no restriction on u for a set E of zero measure, the second may.

The following properties of inequalities are easily proved.

PROP. (1.1). *If $u(x)$ belongs to $H_0^{1,p}(\Sigma)$ and $u(x)$ is non-negative on a set E in the sense of $H_0^{1,p}(\Sigma)$ then $u(x)$ is non-negative almost everywhere on E.* (1)

PROP. (1.2). *If $u(x)$ belongs to $H_0^{1,p}(\Sigma)$ and $u(x)$ is non-negative almost everywhere on a set E, then $u(x)$ is non-negative in the sense of $H_0^{1,p}(\Sigma)$ on any subset of E which is bounded away from ∂E.*

The first property is a consequence of the well known theorem on the quasi-uniform convergence of a subsequence of functions converging in L^p. The second property follows from the regularity theorem (mollification) for the functions of $H_0^{1,p}(\Sigma)$.

The following property shows that weak convergence may be used instead of strong convergence in the definition (1.2).

PROP. (1.3). *A function $u(x)$ of $H_0^{1,p}(\Sigma)$ is non-negative on a set E if there exists a sequence u_m of functions of M_0 such that* i) $u_m \geq 0$ *on E,* ii) $u_m \to u$ *weakly in $H_0^{1,p}(\Sigma)$.*

In fact, from a well known theorem by Banach and Saks [18, p. 80] there is a sequence u'_m of means of u_m which converges to u in $H_0^{1,p}(\Sigma)$. This sequence satisfies the properties i) and ii) of the definition (1.2).

If $u(x) \in L^p(\Sigma)$ and k is a positive number we define the truncated function

$$\{u\}^k = \begin{cases} u & \text{if} \quad u \leq k \\ k & \text{if} \quad u \geq k. \end{cases}$$

We prove the following.

LEMMA (1.1). *If $u(x) \in H_0^{1,p}(\Sigma)$ and $k \geq 0$ then $\{u(x)\}^k \in H_0^{1,p}(\Sigma)$.*

Let u_m be a sequence in M_0 converging to $u(x)$ in $H_0^{1,p}(\Sigma)$. Consider the sequence $\{u_m\}^k$ of functions still in M_0. Since

$$\lim_{m\to\infty} \| \{u_m\}^k - \{u\}^k \|_{L^p(\Sigma)} \leq \lim_{m\to\infty} \| u_m - u \|_{L^p(\Sigma)} = 0$$

and

$$\limsup_{m\to\infty} \| \{u_m\}^k \|_{H_0^{1,p}(\Sigma)} \leq \| u \|_{H_0^{1,p}(\Sigma)}$$

(1) A more precise result holds if instead of Lebesgue measure one considers a suitable capacity [1, 4, 6].

there exists a subsequence, which we still call $\{u_m\}^k$, which converges weakly in $H_0^{1,p}(\Sigma)$ to $\{u\}^k$. Then by the Banach-Saks theorem [18] a sequence u'_m of means of the $\{u_m\}^k$ converges strongly in $H_0^{1,p}(\Sigma)$ to $\{u\}^k$ (²). The u'_m are clearly in M_0.

It is easy to define more general inequalities in the sense of $H_0^{1,p}(\Sigma)$, such as $u \geq$ const., $u \leq 0$, or $u \leq$ const. on a set E. In particular a function $u(x)$ of $H_0^{1,p}(\Sigma)$ is equal to 1 on a set E in the sense of $H_0^{1,p}(\Sigma)$ if $u(x)$ is both ≤ 1 and ≥ 1 in the sense of $H_0^{1,p}(\Sigma)$ (see definition (1.2)). It is easy to see that if $u = 1$ on E, there exists a sequence $u_m \in M_0$ such that $u_m = 1$ on E and $u_m \longrightarrow u$ in $H_0^{1,p}(\Sigma)$. We mention the following.

LEMMA (1.2). *If $u(x)$ belongs to $H_0^{1,p}(\Sigma)$ and $u(x) \geq 1$ on a set E in the sense of $H_0^{1,p}(\Sigma)$ then $\{u(x)\}^1 = 1$ on the same set E in the same sense.*

In fact if u_m is a sequence converging to $u(x)$ in $H_0^{1,p}(\Sigma)$ with $u_m \geq 1$ on E, a subsequence of $\{u_m\}^1$ converges weakly. Then by the Banach-Saks theorem [18] a sequence of means of $\{u_m\}^1$ converges strongly to $\{u\}^1$, and is equal to 1 on E.

LEMMA (1.3). *The set of functions of $H_0^{1,p}(\Sigma)$ which satisfy the condition*

$$u(x) \geq 1$$

on a set E in the sense of $H_0^{1,p}(\Sigma)$ is a closed convex set.

This is clear from the definition. We also have to consider some inequalities for functions of $H^{1,p}(\Omega)$ on $\partial\Omega$.

DEFINITION (1.2′). A function $u(x) \in H^{1,p}(\Omega)$ will be said to be *non-negative on $\partial\Omega$ (in the sense of $H^{1,p}(\Omega)$)* if there exists a sequence u_m of functions of $C^1(\overline{\Omega})$ such that i) $u_m \geq 0$ on $\partial\Omega$, ii) $u_m \longrightarrow u$ in $H^{1,p}(\Omega)$. By the same proof as that of Lemma (1.1) we have the following,

LEMMA (1.4). *If $u(x) \in H^{1,p}(\Omega)$ is non-positive on $\partial\Omega$ in the sense of $H^{1,p}(\Omega)$, then the function*

$$u(x) - \{u(x)\}^k$$

for any positive k, belongs to $H_0^{1,p}(\Omega)$.

REMARK (1.2). In the same way as in definition (1.2′) we can give a meaning to functions non-negative on a subset E of $\partial\Omega$ or bigger than a constant on a subset E of $\partial\Omega$.

(²) In fact, N. Meyers has shown that $\{u_m\}^k$ converges to $\{u\}^k$ in $H_0^{1,p}(\Sigma)$. (Oral communication).

We consider the differential operator

$$Lu \equiv -(a_{ij}(x)\, u_{x_i})_{x_j}$$

where $a_{ij}(x) = a_{ji}(x)$ are real bounded measurable functions defined in Ω.

Throughout, we use the summation convention. We assume L is uniformly elliptic. That is, there is a constant $\lambda \geq 1$, such that

$$\lambda^{-1} \, |\, \xi \,|^2 \leq a_{ij}(x)\, \xi_i\, \xi_j \leq \lambda \, |\, \xi \,|^2 \tag{1.1}$$

for all x in Ω and all real Euclidean n vectors ξ. We shall suppose that the coefficients a_{ij} are defined and satisfy (1.1) in all E^n. This can always be done by putting $a_{ij} = \delta_{ij}$ outside of Ω.

Definition (1.3). Given $n+1$ functions $f_0, f_1, \ldots, f_n$ in $L^2(\Omega)$, the function $u(x) \in H^{1,2}(\Omega)$ is said to be a *solution of the equation*

$$Lu = f_0 - \sum_{i=1}^{n} (f_i)_{x_i} \tag{1.2}$$

if

$$\int_\Omega a_{ij}\, u_{x_i}\, \Phi_{x_j}\, dx = \int_\Omega \left(f_0\, \Phi + \sum_{i=1}^{n} f_i\, \Phi_{x_i} \right) dx \tag{1.3}$$

for all $\Phi \in H_0^{1,2}(\Omega)$. If moreover $u \in H_0^{1,2}(\Omega)$, then u will be said to vanish on the boundary $\partial \Omega$.

Definition (1.4). A function $u(x) \in H^{1,2}_{loc}(\Omega)$ will be called a *local solution in Ω of the equation $Lu = 0$*, if

$$\int_\Omega a_{ij}\, u_{x_i}\, \Phi_{x_j}\, dx = 0 \tag{1.4}$$

for all Φ in $C^1(\overline{\Omega})$ with compact support in Ω.

Definition (1.5). A function $u(x) \in H^{1,2}(\Omega)$ is called an *L-subsolution in Ω* if

$$\int_\Omega a_{ij}\, u_{x_i}\, \Phi_{x_j}\, dx \leq 0 \tag{1.5}$$

for all non-negative functions Φ of $H_0^{1,2}(\Omega)$. $u(x)$ is called an *L-supersolution in Ω* if $-u$ is an L-subsolution.

2. Some Known Results.

We recall some results on uniformly elliptic operators which we shall use later. The following results are related to local solutions of the equation $Lu = 0$. To begin with, we recall the following standard

LEMMA (2.1). *For any local solution of $Lu = 0$ in Ω the following inequality holds.*

$$\sum_i \int\limits_{S(y,\varrho)} u_{x_i}^2 \, dx \le \frac{4\lambda^2}{(R-\varrho)^2} \int\limits_{S(y,R)} u^2 \, dx$$

where λ is the constant of (1.1) *and $S(y, r)$ denotes the sphere with center at y and radius r; $\varrho < R$ and $S(y, R) \subset \Omega$.*

It is enough to take in (1.4) $\Phi = \Psi^2 u$ where Ψ is an M_0-function equal to 1 in $S(y, \varrho)$ and to 0 outside of $S(y, R)$.

THEOREM (2.1). *Any local solution of $Lu = 0$ in Ω is Hölder continuous in any compact subdomain of Ω. More precisely, for any Ω' such that $\overline{\Omega'} \subset \Omega$ there are constants* (3) $K = K(\lambda, \Omega', \Omega)$ $\alpha = \alpha(\lambda, \Omega', \Omega)$ *such that*

$$| u(x') - u(x'') | \le K \, \| u \|_{L^2(\Omega)} \, | x' - x'' |^{\alpha}$$

for $x', x'' \in \Omega'$.

This theorem is due to De Giorgi [3] and Nash [15]. A simpler proof was given by Moser [13].

THEOREM (2.2). *For any positive local solution of the equation $Lu = 0$ in Ω and for any Ω' such that $\overline{\Omega'} \subset \Omega$*

$$\max_{x \in \Omega'} u(x) \le c \min_{x \in \Omega'} u(x)$$

where c is a constant depending only on λ, Ω, and *Ω'.*

This extension of the classical Harnack inequality is due to Moser [14]. As Moser has shown [14], theorem (2.1) can be deduced from this theorem. The following results are related to solutions of the equation (1.3). The following theorem is proved by the classical Hilbert space approach.

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(3) Throughout this paper λ will denote the ellipticity constant in (1.1).

Theorem (2.3). *Given* $f_i \in L^2(\Omega)$ $(i = 1, 2, \ldots, n)$ *and* $h \in H^{1,2}(\Omega)$, *there exists one and only one solution of the equation*

$$Lu = (f_i)_{x_i} \qquad (i = 1, 2, \ldots, n) \tag{2.1}$$

such that

$$u - h \in H_0^{1,2}(\Omega). \tag{2.2}$$

The following theorem is true only for a domain whose boundary satisfies certain smoothness assumptions. For our present purposes it is sufficient to take as domain a sphere Σ.

Theorem (2.4). *If* $f_i \in L^p(\Sigma)$ $(i = 1, 2, \ldots, n)$ *and* $h \in H^{1,p}(\Sigma)$ *with* $p > n$, *then the solution of the equation* (2.1) *satisfying* (2.2) *is Hölder continuous in* Σ.

This theorem is a special case of a theorem proved by Morrey [12] and Stampacchia [24]. The following two theorems are maximum principles.

Theorem (2.5). *If* $u(x) \in H^{1,2}(\Omega)$ *is an L-subsolution* (def. (1.5)) *which is non-positive in the sense of* $H^{1,2}(\Omega)$ *on* $\partial\Omega$ (def. (1.2′)) *then* $u(x)$ *is non-positive almost everywhere in* Ω.

For this theorem see Stampacchia [25, 26]. We repeat here the easy proof.

Let $u(x)$ be an L-subsolution in Ω which is non-positive on $\partial\Omega$. For a fixed $\varepsilon > 0$ the function $u(x) - \{u(x)\}^\varepsilon$ is non-negative in Ω and moreover, by Lemma (1.4), belongs to $H_0^{1,2}(\Omega)$. Taking in (1.5) $\Phi(x) = u(x) - \{u(x)\}^\varepsilon$ we get

$$\int_\Omega a_{ij}\, \Phi_{x_i}\, \Phi_{x_j}\, dx \leq 0$$

and because of the ellipticity of L, $\Phi_{x_i} = 0$ $(i = 1, 2, \ldots, n)$ i. e. $u(x) = \{u(x)\}^\varepsilon$. Since ε is an arbitrary positive number we get $u(x) \leq 0$ a. e.

Theorem (2.6). *If* $u(x)$ *is a solution of the equation* (2.1) *vanishing on* $\partial\Omega$, *where* $f_i \in L^p(\Omega)$ $(p > n)$ *then*

$$\max_\Omega |u(x)| \leq c\,\lambda\,[\text{meas}\ \Omega]^{1/n - 1/p}\, \|f_i\|_{L^p(\Omega)}$$

where c *depends only on* p (and n).

This inequality is due to Stampacchia [22, 23]. A different proof giving the best value of c is given by Weinberger [29].

3. The boundary value problem and regularity.

Given a continuous function $h(x)$ on the boundary $\partial\Omega$ we seek to solve the Dirichlet problem

$$Lu = 0 \text{ in } \Omega \tag{3.1}$$

$$u = h \text{ on } \partial\Omega. \tag{3.2}$$

If h is the trace on $\partial\Omega$ of a function $\overline{h}(x)$ in $H^{1,2}(\Omega)$, theorem (2.3) gives a solution $u \in H^{1,2}(\Omega)$ which, by theorem (2.1) is locally Hölder continuous. The boundary values h are attained in the sense that $u - \overline{h} \in H_0^{1,2}(\Omega)$. Obviously, the solution u is the same if instead of $\overline{h}(x)$ we consider a new function $\overline{\overline{h}}(x)$ such that $\overline{h} - \overline{\overline{h}} \in H_0^{1,2}(\Omega)$. We denote by $\tau^{1,2}$ the quotient space $H^{1,2}(\Omega)/H_0^{1,2}(\Omega)$ where

$$\| h \|_{\tau^{1,2}} = \inf_{\overline{\overline{h}} - \overline{h} \in H_0^{1,2}(\Omega)} \| \overline{\overline{h}} \|_{H^{1,2}(\Omega)}, \tag{3.3}$$

and we consider the continuous linear mapping of $\tau^{1,2}$ into $H^{1,2}(\Omega)$ just defined. We denote it by

$$u = Bh. \tag{3.4}$$

By theorem (2.5) if h is bounded on $\partial\Omega$ in the sense of $H^{1,2}(\Omega)$, then

$$\min_{\partial\Omega} h \leq \min_{\Omega} Bh \leq \max_{\Omega} Bh \leq \max_{\partial\Omega} h \tag{3.5}$$

where $\max_{\partial\Omega} h$ $[\min_{\partial\Omega} h]$ means the minimum [maximum] of the numbers Φ such that $\overline{h} \leq \Phi$ $[\overline{h} \geq \Phi]$ on $\partial\Omega$ in the sense of $H^{1,2}(\Omega)$ (4).

By lemma (2.1) and (3.5) we get

$$|||\, Bh \,||| \leq C(\lambda, \Omega) \max_{\partial\Omega} |\, h \,| \tag{3.6}$$

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(4) $\max_{\Omega}$ and $\min_{\Omega}$ mean the essential max and min, as usual.

where

$$\||| g \||| = \underset{\overline{\Omega}' \subset \Omega}{\text{l.u.b}}\ \delta \left(\sum_i \int_{\Omega'} g_{x_i}^2\, dx \right)^{1/2} + \max_{\Omega} | g |,$$

δ being the distance between $\overline{\Omega}'$ and $\partial\Omega$.

We conclude that Bh is a linear mapping of the subset $B^{1,2}$ of functions of $\tau^{1,2}$ which are bounded on $\partial\Omega$ (in the sense of $H^{1,2}(\Omega)$), into the space of functions u such that $\||| u \||| < \infty$.

Since any continuous function h on $\partial\Omega$ can be approximated in the norm $\max_{\partial\Omega} |\quad|$ by functions smooth in any set containing $\overline{\Omega}$ (for instance by polynomials), we conclude that the set $B^{1,2}$ is dense in the space of continuous functions h on $\partial\Omega$ with the norm $\max_{\partial\Omega} | h |$.

Therefore, the linear mapping $u = B(h)$, restricted to the space of continuous functions (on $\partial\Omega$) of $\tau^{1,2}$, can be extended to the space of continuous functions on $\partial\Omega$. The mapping so obtained is still denoted by

$$u = Bh.$$

It associates to any continuous function h on $\partial\Omega$ a unique function u locally Hölder continuous such that

$$\||| u \||| < \infty.$$

It is easily proved that the function $u(x)$ is a local solution of the equation $Lu = 0$ (see Def. 1.4).

We have thus proved the following theorem.

THEOREM (3.1). *There exists a mapping Bh which to any continuous function h on $\partial\Omega$ associates a local solution u of the equation $Lu = 0$ (which is locally Hölder continuous by theorem* (2.1)) *in such a way that if h is the trace of a $C^1(\overline{\Omega})$ function, then $u = Bh$ coincides with the solution in $H^{1,2}(\Omega)$ obtained by the variational approach* (*theorem* (2.3)). *Moreover* (3.5) *and* (3.6) *hold.*

In the case of smooth coefficients it can be shown that $u = Bh$ coincides with the Perron-Wiener solution [2] of the boundary value problem.

Our problem is to investigate whether $u = Bh$, just defined, approaches the boundary values h. For this purpose we consider the following.

DEFINITION (3.1). A point $y \in \partial\Omega$ is said to be *regular* if for any continuous function $h(x)$ on $\partial\Omega$ the generalized solution $u = Bh$ satisfies

$$\lim_{\substack{x \to y \\ x \in \Omega}} u(x) = h(y). \tag{3.7}$$

If there is at least one continuous function h on $\partial\Omega$ for which (3.7) is not satisfied, the point y is said to be *irregular.*

We first show that irregularity at a point, if it exists, already arises in the variational problem, without extending B to all continuous functions.

LEMMA (3.1). *If at a point $y \in \partial\Omega$ (3.7) holds for $u = Bh$ for every continuous $h \in \tau^{1,2}$, then y is regular.*

PROOF. Let h be continuous on $\partial\Omega$. Given any $\varepsilon > 0$ there is a continuous function $h_\varepsilon \in \tau^{1,2}$ such that

$$|h - h_\varepsilon| < \frac{1}{3}\varepsilon \quad \text{on} \quad \partial\Omega.$$

Let $u = Bh$, $u_\varepsilon = Bh_\varepsilon$. By the maximum principle (3.5) $|u - u_\varepsilon| < \frac{1}{3}\varepsilon$ in Ω. By (3.7) there is a neighborhood N_ε of y in Ω such that

$$|h_\varepsilon(y) - u_\varepsilon(x)| < \frac{1}{3}\varepsilon \quad \text{for} \quad x \in N_\varepsilon.$$

Then

$$\begin{aligned} |h(y) - u(x)| &\leq |h(y) - h_\varepsilon(y)| + |h_\varepsilon(y) - u_\varepsilon(x)| + |u_\varepsilon(x) - u(x)| \\ &< \varepsilon \qquad\qquad \text{for} \quad x \in N_\varepsilon. \end{aligned}$$

so that y is regular.

In order to characterize regular points, we define a barrier at a point y of $\partial\Omega$.

DEFINITION (3.2). *A function $V_y(x) \in H^{1,2}(\Omega)$ is called a barrier at the point $y \in \partial\Omega$* if

i) $L(V_y) = 0$ in Ω in the sense of def. (1.4)

ii) For any $\varrho > 0$ there is a number $m > 0$ such that $V_y(x) \geq m$ in the sense of $H^{1,2}(\Omega)$ on the set

$$\{x \mid x \in \partial\Omega; \quad |x - y| \geq \varrho\}.$$

iii) $V_y(x)$ is continuous at the point y, and

$$\lim_{\substack{x \to y \\ x \in \Omega}} V_y(x) = V_y(y) = 0.$$

The condition (iii) has meaning because from (i) and theorem (2.1) $V_y(x)$ can be defined in any point of Ω and is continuous in Ω.

We now prove the following lemma:

LEMMA (3.2). *A point $y \in \partial\Omega$ is regular if and only if there exists a barrier V_y at y.*

PROOF. If y is a regular point, the function

$$V_y = B(|x - y|)$$

is a barrier because on $\partial\Omega$ $|x - y| \in \tau^{1,2}$ and is continuous.

Suppose now that a barrier V_y exists. Let $h(x)$ be any continuous function in $\tau^{1,2}$, and $u = Bh$. Given any $\varepsilon > 0$ there is a $\varrho > 0$ such that

$$|h(x) - h(y)| < \frac{1}{2}\varepsilon$$

for $|x - y| < \varrho$, $x \in \partial\Omega$. Moreover h is bounded, so that $|h(x)| \leq M$ on $\partial\Omega$. Since $V_y \geq m$ for $|x - y| \geq \varrho$, we find that

$$h(y) + \frac{1}{2}\varepsilon + \frac{2M}{m} V_y(x) - h(x) \geq 0$$

on $\partial\Omega$ in the sense of $H^{1,2}(\Omega)$. By theorem (2.5)

$$u(x) \leq h(y) + \frac{1}{2}\varepsilon + \frac{2M}{m} V_y(x).$$

Similarly

$$u(x) \geq h(y) - \frac{1}{2}\varepsilon - \frac{2M}{m} V_y(x).$$

Since $V_y(x) \longrightarrow 0$ as $x \longrightarrow y$, we can find a neighborhood N_ε of y in Ω such that $2M\, V_y(x) < \frac{1}{2}\varepsilon m$ for $x \in N_\varepsilon$. Then

$$|u(x) - h(y)| < \varepsilon \qquad \text{for} \quad x \in N_\varepsilon,$$

and (3.7) holds for continuous $h \in \tau^{1,2}$. Therefore, by lemma (3.1) y is regular.

Next we show that regularity is a local property.

LEMMA (3.3). *Let Ω' be a subdomain of Ω. Let y be a boundary point of both Ω and Ω', and for some sphere $S(y, \varrho) \equiv \{x \mid |x - y| < \varrho\}$ let*

$\partial\Omega \cap S(y, \varrho) = \partial\Omega' \cap S(y, \varrho)$. *Then y is a regular boundary point of Ω if and only if it is a regular point of Ω'.*

Suppose first that y is regular with respect to Ω'. We wish to show that $V_y = B(|x - y|)$ is a barrier in Ω. This function is clearly bounded. Therefore, $V_y(x) \leq c\,|x - y|$ on $\partial\Omega' - (\partial\Omega \cap \partial\Omega')$ for some constant c. By the maximum principle (3.5) $0 \leq V_y(x) \leq c V'_y(x)$ where V'_y is the corresponding function $B'(|x - y|)$ in Ω'. By the regularity of y whith respect to Ω', $V'_y(x) \longrightarrow 0$ as $x \longrightarrow y$.

Hence $V_y(x) \longrightarrow 0$ as $x \longrightarrow y$, and V_y is a barrier.

We now suppose that y is regular with respect to Ω. Let Σ be a sphere containing $\bar{\Omega}$. We define the sets

$$E_\varrho = [\Sigma - \Omega] \cap \overline{S(y, \varrho)}$$

and

$$\Omega_\varrho = \Sigma - E_\varrho .$$

(We suppose ϱ so small that $\overline{S(y, \varrho)} \subset \Sigma$).

We note that $\partial\Omega_\varrho$ coincides with $\partial\Omega$ in $S(y, \varrho)$ and $\Omega_\varrho \supset \Omega$. Hence by the first part of the theorem y is a regular point for Ω_ϱ. Therefore the solution $u^\varrho(x) \in H^{1,2}(\Omega_\varrho)$ of

$$\begin{aligned} Lu^\varrho &= 0 \qquad \text{in } \Omega_\varrho \\ u^\varrho &= 0 \qquad \text{on } \partial\Sigma \\ u^\varrho &= 1 \qquad \text{on } \partial E_\varrho \end{aligned}$$

is continuous at y.

We define

$$V'_y(x) = \sum_{k=2}^{\infty} 2^{-k}(1 - u^{\frac{\varrho}{k}}(x)).$$

This series converges uniformly, and hence is continuous at y. Moreover, the strong maximum principle, which follows from theorem (2.2), shows that for each k there is an $m_k > 0$ such that

$$1 - u^{\varrho/k}(x) \geq 2^k m_k \qquad \text{on} \quad \partial\Omega' \quad \text{for} \quad |x - y| \geq \frac{2\varrho}{k} .$$

It follows that

$$V'_y(x) > m_k \qquad \text{for} \qquad |x - y| \geq 2\varrho/k.$$

Thus V'_y is a barrier for Ω', and y is regular with respect to Ω'.

REMARK: It is clear from the above proof that y is regular with respect to Ω if and only if the functions $u^\varrho(x)$ are continuous at y.

4. Capacity.

Let Σ be a fixed open sphere and let E be a closed subset of Σ. We define the capacity of E (with respect to the operator L and the sphere Σ) as the infimum of

$$D_L(\Phi) = \int_\Sigma a_{ij}\, \Phi_{x_i}\, \Phi_{x_j}\, dx \tag{4.1}$$

among functions $\Phi \in H_0^{1,2}(\Sigma)$ satisfying $\Phi \geq 1$ on E in the sense of $H_0^{1,2}(\Sigma)$. This infimum will be denoted by *cap. E*.

THEOREM (4.1). *There exists one and only one function $u(x) \in H_0^{1,2}(\Sigma)$ such that $u(x) \geq 1$ on E in the sense of $H_0^{1,2}(\Sigma)$ and*

$$cap\ E = D_L(u). \tag{4.2}$$

Moreover $u = 1$ on E in the sense of $H_0^{1,2}(\Sigma)$.

PROOF. By lemma (1.3) the set of functions $\Phi \in H_0^{1,2}(\Sigma)$ satisfying $\Phi \geq 1$ on E in the sense of $H_0^{1,2}(\Sigma)$ is a closed convex set. Since $H_0^{1,2}$ is a Hilbert space, there exists one and only one function $u \in H_0^{1,2}(\Sigma)$ assuming the infimum of $D_L(\Phi)$. Moreover since,

$$D_L(\{u\}^1) \leq D_L(u)$$

it follows from lemma 1.2 that $u(x) = 1$ on E in the sense of $H_0^{1,2}(\Sigma)$, and the statement is proved.

DEFINITION (4.1). The function $u(x)$ giving the minimum to $D_L(\Phi)$ among the $H_0^{1,2}(\Sigma)$-functions such that $\Phi \geq 1$ on E in the sense of $H^{1,2}(\Sigma)$ is called *the capacitary potential of the set E* (with respect to Σ and L).

Using a well known variational technique, we find that the capacitary potential u of the set E satisfies

$$(4.3) \qquad \int_{\Sigma} a_{ij}\, u_{x_i}\, \Phi_{x_j}\, dx \geq 0$$

for any $\Phi \in H_0^{1,2}(\Sigma)$ which is non negative on E in the sense of $H_0^{1,2}(\Sigma)$.

In fact $u + \varepsilon\,\Phi$ is, for $\varepsilon \geq 0$, an admissible function for the considered problem. Then $D_L(u + \varepsilon\Phi) \geq D_L(u)$; i.e. $\frac{d}{d\varepsilon} D_L(u + \varepsilon\Phi)\,|_{\varepsilon=0} \geq 0$.

The property (4.3) and the fact that $u = 1$ on E in the sense of $H^{1,2}(\Sigma)$ characterize the capacitary potential of the set E.

The inequality (4.3) means that the capacitary potential u of the set E is an L-supersolution in Σ which is equal to 1 on E in the sense of $H^{1,2}(\Sigma)$. Moreover, since any $C^1(\bar{\Sigma})$ function Φ with compact support in $\Sigma - E$ is admissible in (4.3) we deduce that $u \in H^{1,2}(\Sigma - E)$ and is the solution of the equation $Lu = 0$ in $\Sigma - E$ which equals 1 on ∂E and vanishes on $\partial\Sigma$. From the maximum principle (theorem (2.5)) it follows that $0 \leq u(x) \leq 1$ a.e. on $\Sigma - E$.

We have proved the following.

THEOREM (4.2). *The capacitary potential u of the compact subset E of Ω is the function of $H^{1,2}(\Sigma)$ which is equal to 1 on E in the sense of $H^{1,2}(\Sigma)$, satisfies the equation $Lu = 0$ in $\Sigma - E$, and has the boundary values 1 on ∂E and 0 on $\partial\Sigma$ in the sense of theorem* (2.3). *Moreover $u(x)$ is an L-supersolution in Σ.*

Further properties of the capacitary potentials will be proved in sections 5 and 6.

The functions u^{ϱ} introduced in the preceding section are the capacitary potentials of the sets E_{ϱ}. Therefore the remark at the end of this section becomes:

REMARK. y is regular with respect to Ω if and only if the capacitary potentials for all the sets E_{ϱ} are continuous at y.

5. Weak solutions.

We again let Σ be an (open) sphere in Euclidean n-space.

DEFINITION (5.1). For a measure μ of bounded variation on Σ we say that $u \in L^1(\Sigma)$ is *a weak solution of the equation*

$$Lu = \mu$$

vanishing at the boundary $\partial\Sigma$ if it satisfies

$$\int_{\Sigma} u\, L\Phi\, dx = \int_{\Sigma} \Phi\, d\mu$$

for every $\Phi \in H_0^{1,2}(\Sigma) \cap C^0(\bar{\Sigma})$ such that $L\Phi \in C^0(\bar{\Sigma})$.

By theorem (2.3) with $h = 0$ there is a continuous linear operator G from $H^{-1,2}(\Sigma)$ to $H_0^{1,2}(\Sigma)$ such that for $T \in H^{-1,2}(\Sigma)$ $u = G(T)$ is the unique solution in $H_0^{1,2}(\Sigma)$ of

$$Lu = T.$$

We call G the *Green's operator.*

By theorem (2.4), this operator takes $H^{-1,p}(\Sigma)$ with $p > n$ into $C^0(\bar{\Sigma})$. By theorem (2.6) the restriction of G maps $H^{-1,p}(\Sigma)$ with $p > n$ into $C^0(\bar{\Sigma})$ continuously. In particular, for any $\psi \in C^0(\bar{\Sigma})$ we have

$$\max_{\bar{\Sigma}} |\, G\psi \,| \leq c\lambda\, [\text{meas}\ \Sigma]^{\frac{1}{n}-\frac{1}{p}}\, \|\, \psi\, \|_{H^{-1,p}}, \quad p > n \tag{5.1}$$

where c depends only on p.

It is clear that u is a weak solution vanishing on $\partial\Sigma$ of the equation $Lu = \mu$ in Σ if and only if

$$\int_{\Sigma} u\, \psi\, dx = \int_{\Sigma} G(\psi)\, d\mu$$

for every $\psi \in C^0(\bar{\Sigma})$.

Obviously there is at most one solution to this problem.

From (5.1) we find that

$$\int_{\Sigma} u\psi dx \leq c\lambda\, [\text{meas}\ \Sigma]^{\frac{1}{n}-\frac{1}{p}} \int_{\Sigma} |\, d\mu\, |\; \|\, \psi\, \|_{H^{-1,p}(\Sigma)}$$

for any $\psi \in C^0(\Sigma)$. Since $C^0(\Sigma)$ is dense in $H^{-1,p}(\Sigma)$, we obtain

$$\|\, u\, \|_{H_0^{1,p'}(\Sigma)} \leq c\lambda\, [\text{meas}\ \Sigma]^{\frac{1}{n}-\frac{1}{p}} \int_{\Sigma} |\, d\mu\, |. \tag{5.2}$$

The transformation $\mu \longrightarrow u$ is the adjoint operator G^* of G. That is, $u = G^*(\mu)$. Since $G(H^{-1,p}(\Sigma)) \subset C^0(\Sigma)$ and G^* is defined on the dual space of $G(H^{-1,p}(\Sigma))$, it is certainly defined for all measures μ of bounded variation.

The range of G^* is in the dual space of $H^{-1,p}(\Sigma)$, which is just $H_0^{1,p'}(\Sigma)\left(\frac{1}{p}+\frac{1}{p'}=1\right)$. Thus we have

THEOREM (5.1). *A unique weak solution u of the equation $Lu=\mu$ vanishing on $\partial\Sigma$ exists for every μ, and lies in $H_0^{1,p'}(\Sigma)$ for every $p'<n/(n-1)$.*

REMARK (5.1). If $u \in H_0^{1,2}(\Sigma)$ satisfies

$$\int_\Sigma a_{ij}\, u_{x_i}\, \Phi_{x_j}\, dx = \int_\Sigma \Phi\, d\mu,$$

it is the weak solution vanishing on $\partial\Sigma$ of the problem

$$Lu = \mu.$$

For, if we take $\Phi = G(\psi)$, $\psi \in C^0(\overline{\Sigma})$, then by definition

$$\int_\Sigma u\,\psi\, dx = \int_\Sigma a_{ij}\, \Phi_{x_i}\, u_{x_j}\, dx = \int_\Sigma \Phi\, d\mu\,.$$

In this case $\mu \in H^{-1,2}$ in the sense that

$$\int_\Sigma \Phi\, d\mu \leq c\, \|\, \Phi\, \|_{H_0^{1,2}}$$

for all $\Phi \in H_0^{1,2}(\Sigma) \cap C^0(\overline{\Sigma})$. Conversely, if $\mu \in H^{-1,2}$, it can be represented by $-\sum_{i=1}^{n}(f_i)_{x_i}$ with $f_i \in L^2(\Sigma)$ in the sense that

$$\int_\Sigma \Phi\, d\mu = \int_\Sigma f_i\, \Phi_{x_i}\, dx$$

for all $\Phi \in H_0^{1,2}(\Sigma) \cap C^0(\overline{\Sigma})$. In this case theorem (2.3) shows that there is a solution $u \in H_0^{1,2}(\Sigma)$ of $Lu=\mu$. Thus we have proved

THEOREM (5.2). *The weak solution u of $Lu=\mu$ is in $H_0^{1,2}(\Sigma)$ if and only if $\mu \in H^{-1,2}$.*

Suppose now we have a non-negative $\mu \in H_0^{-1,2}(\Sigma)$. Let μ' be another measure such that $0 \leq \mu'(e) \leq \mu(e)$ for all subsets e of Σ. We note that if $\Phi \in H_0^{1,2}(\Sigma) \cap C^0(\bar{\Sigma})$, the same is true of $|\Phi|$. Then

$$\left|\int_\Sigma \Phi \, d\mu'\right| \leq \int_\Sigma |\Phi| \, d\mu'$$

$$\leq \int_\Sigma |\Phi| \, d\mu$$

$$\leq \|\Phi\|_{H_0^{1,2}} \, \|\mu\|_{H^{-1,2}} .$$

Thus, $\mu' \in H^{-1,2}$ and $\|\mu'\|_{H^{-1,2}} \leq \|\mu\|_{H^{-1,2}}$.

We have thus proved:

THEOREM (5.3). *If the solution u of $Lu = \mu$ is in $H_0^{1,2}$ and $0 \leq \mu' \leq \mu$, then the solution v of $Lv = \mu'$ is in $H_0^{1,2}$.*

We shall now show that a weak solution can be approximated by weak solutions of equations with continuous coefficients.

We take any family of mollifiers $\alpha_s(x)$ having the properties:

(1) $\alpha_s(x) \in C^\infty(E^n)$

(2) $\alpha_s(x) \geq 0$

(3) $\alpha_s(x) = 0$ for $|x| > \dfrac{1}{s}$

(4) $\displaystyle\int_{E^n} \alpha_s \, dx = 1.$

We let $L^s u = -(a_{ij}^{(s)} u_{x_i})_{x_i}$ with

$$a_{ij}^{(s)}(x) = \int_{E_n} a_{ij}(y) \, \alpha_s(x-y) \, dy,$$

where a_{ij} is extended as δ_{ij} outside Σ. It is clear that L^s is uniformly elliptic with the same constant λ, since

$$a_{ij}^{(s)} \xi_i \xi_j = \int_{E_n} a_{ij}(y) \, \xi_i \xi_j \, \alpha_s(x-y) \, dy .$$

Moreover, $a_{ij}^{(s)} \in C^\infty(E^n)$.

THEOREM (5.4). *For any measure μ of bounded variation on Σ the weak solutions $u^{(s)}$ of $L^s u^{(s)} = \mu$ converge to the weak solution u of $Lu = \mu$ weakly in $H_0^{1,p'}(\Sigma)$ for any $p' < n/(n-1)$ and therefore strongly in $L^q(\Sigma)$ for any $q < n/(n-2)$.*

PROOF. We first prove the theorem in the case when $d\mu = \Psi\, dx$ where $\Psi \in C^0(\bar{\Sigma})$. Then the $u^{(s)}$ are actually solutions in $H_0^{1,2}(\Sigma)$. They satisfy

$$\int_{\Sigma} a_{ij}^s\, u_{x_i}^{(s)}\, \Phi_{x_j}\, dx = \int_{\Sigma} \Psi\, \Phi\, dx,$$

for all $\Phi \in H_0^{1,2}(\Sigma)$.

By theorem (2.3) $\| u^{(s)} \|_{H^{1,2}(\Sigma)}$ is uniformly bounded. By theorem (2.4) the $u^{(s)}$ are uniformly Hölder continuous in $\bar{\Sigma}$. Hence there exists a sequence $s_\nu \longrightarrow \infty$ such that $u^{(s_\nu)}$ converges weakly in $H_0^{1,2}$ and uniformly in $\bar{\Sigma}$ to a Hölder continuous function u. Since $a_{ij}^{(s)}$ converges to a_{ij} in every $L^r(\Sigma)$, we have

$$\int a_{ij}\, u_{x_i}\, \Phi_{x_j}\, dx = \int \Psi\, \Phi\, dx$$

for all $\Phi \in C^\infty(\bar{\Sigma})$ of compact support and therefore also for all $\Phi \in H_0^{1,2}(\Sigma)$. By definition, $u \in H_0^{1,2}(\Sigma)$ is the solution of $Lu = \Psi$. Since this solution is unique, the whole sequence $u^{(s)}$ converges to u weakly in $H_0^{1,2}(\Sigma)$ and uniformly in $\bar{\Sigma}$.

We now consider the general case. We have the weak solutions $u^{(s)}$ of $L^s u^{(s)} = \mu$, which satisfy for any $\Psi \in C^0(\bar{\Sigma})$

$$\int_{\Sigma} u^{(s)}\, \Psi\, dx = \int_{\Sigma} \Phi^{(s)}\, d\mu \tag{5.3}$$

where $\Phi^{(s)} \in H_0^{1,2}(\Sigma)$ is the solution of $L^s\, \Phi^{(s)} = \Psi$. By the above consideration $\Phi^s \longrightarrow \Phi$ uniformly, where Φ is the solution of $L\Phi = \Psi$.

On the other hand, by inequality (5.2), $\| u^{(s)} \|_{H^{1,p'}(\Sigma)}$ is uniformly bounded. Therefore there exists a subsequence $s_i \longrightarrow \infty$ such that $u^{(s_i)}$ converges weakly to a function $u \in H_0^{1,p'}(\Sigma)$. Taking limits in (5.3) we find

$$\int_{\Sigma} u\, \Psi\, dx = \int_{\Sigma} \Phi\, d\mu$$

for any $\Psi \in C^0(\bar{\Sigma})$ and $\Phi \in H_0^{1,2}(\Sigma)$ such that $L\Phi = \Psi$. Hence u is the weak solution of $Lu = \mu$.

It is well known (for references see [11, p. 253]) that if the sequence $u^{(s_i)}$ converges weakly in $H_0^{1,p'}(\Sigma)$ to u, a subsequence converges strongly to u in $L^q(\Sigma)$ with $\frac{1}{q} > \frac{1}{p'} - \frac{1}{n}$. Since the weak solution u is unique, the whole sequence $u^{(s)}$ converges to u weakly in $H_0^{1,p'}(\Sigma)$ and strongly in $L^q(\Sigma)$.

6. Green's function and representation of the capacitary potential.

We define the Green's function $g(x, y)$ of the operator L on Σ as the weak solution, vanishing on $\partial\Sigma$, of the equation

$$Lg = \delta_y$$

where δ_y is the Dirac measure at y. Hence, by definition (5.1) the solution Φ in $H_0^{1,2}(\Sigma)$ of

$$L\Phi = \Psi$$

for an arbitrary $\Psi \in C^0(\bar{\Sigma})$ is given by

$$\Phi(y) = \int_{\Sigma} g(x, y)\, \Psi(x)\, dx. \tag{6.1}$$

It is well-known that for an operator L with C^∞ coefficients

$$g(x, y) \geq 0$$

and

$$g(x, y) = g(y, x).$$

By the limiting process of the preceding section, these results are easily extended to any uniformly elliptic operator L.

We now prove

THEOREM (6.1). *For every measure μ of bounded variation the integral*

$$u(x) \equiv \int g(x, y)\, d\mu(y)$$

exists and is finite a. e., and is the weak solution vanishing on $\partial\Sigma$ of the quation $Lu = \mu$.

PROOF. We can assume without loss of generality that μ is non-negative. Let Ψ be a non-negative function in $C^0(\Sigma)$ and let Φ be the solution in $H_0^{1,2}(\Sigma)$ of the equation $L\Phi = \Psi$. Then $\Phi \in C^0(\bar{\Sigma})$ by theorem (2.4) and is non-negative by theorem (2.5). Moreover

$$\Phi(y) = \int g(x, y)\, \Psi(x)\, dx.$$

Therefore, by Fubini's theorem, the integral $\int g(x, z)\, d\mu(y)$ exists a. e., and

$$\int \Phi(y)\, d\mu(y) = \iint g(x, y)\, \Psi(x)\, dx\, d\mu(y) = \int \Psi(x)\, u(x)\, dx.$$

Since this identity holds for any continuous Ψ we find that u satisfies the definition (5.1), which proves our theorem.

From the above and another application of Fubini's theorem follows:

THEOREM (6.2). *If u and v are weak solution, vanishing on $\partial\Sigma$, of the equations $Lu = \mu$ and $Lv = \nu$ respectively, then*

$$\int_{\Sigma} u\, d\nu = \int_{\Sigma} v\, d\mu = \iint g(x, y)\, d\mu(x)\, d\nu(y) \tag{6.2}$$

in the sense that if one exists, so does the other and they are equal.

We now consider the capacitary potential u of a closed set E as defined in § 4. We saw that it satisfies (4.3) for any $\Phi \in C^\infty(\bar{\Sigma})$ with compact support in Σ such that $\Phi \geq 0$ on E. By a theorem of L. Schwartz [21, t. 1, p. 29] there exists a non-negative measure μ on E such that

$$\int_{\Sigma} a_{ij}\, u_{x_i}\, \Phi_{x_j}\, dx = \int_{\Sigma} \Phi\, d\mu. \tag{6.3}$$

Furthermore, since $u = 1$ in the sense of $H_0^{1,2}(\Sigma)$ on E, the support of μ is on ∂E. The measure μ just found will be called the *capacitary distribution* of E.

The capacitary potential $u(x)$ of a set E is, by remark (5.1), the weak solution, vanishing on $\partial\Sigma$, of the equation $Lu = \mu$, and can be represented, by theorem (6.1), as

$$u(x) = \int_{\Sigma} g(x, y)\, d\mu(y). \tag{6.4}$$

Since $u \in H_0^{1,p}(\Sigma)$ and $u = 1$ on E in the sense of $H_0^{1,2}(\Sigma)$ we can find a sequence $\Phi^s \longrightarrow u$ in $H_0^{1,2}(\Sigma)$ such that $\Phi^s \in M_0$ and $\Phi^s = 1$ on E.

We consider (6.3) with $\Phi = \Phi^s$. Since the support of μ is on E, the right-hand side equals $\mu(E)$ for any s, while the left-hand side converges to the capacity of E. Thus we have proved that

$$\mu(E) = \text{cap } E. \tag{6.5}$$

7. Properties of Green's function.

Since we can approximate the Green's function for any uniformly elliptic L, by theorem (5.4), we first consider only an L with coefficients in $C^\infty(\bar{\Sigma})$.

Then it is well-known that for $x \neq y$, $g(x, y)$ is continuous in x and y, and that

$$\lim_{x \to y} g(x, y) = +\infty.$$

Consequently y is an interior point of the set

$$J_a = \{x \mid g(x, y) \geq a\}.$$

Therefore, the capacitary potential corresponding to this set as represented by (6.4), is continuous and hence equal to 1 at y. The representation (6.4) then gives

$$1 = \int_\Sigma g(x, y)\, d\nu_a(x)$$

where ν_a is the capacitary distribution of J_a. Since the support of ν_a is on ∂J_a where $g(x, y) = a$, (6.5) gives

$$\text{cap } J_a = \frac{1}{a}. \tag{7.1}$$

We now let Σ_γ be the sphere obtained from Σ by a uniform dilatiation with the factor γ $(0 < \gamma < 1)$ about the point $y \in \Sigma$.

Let

$$a = \min_{x \in \partial\Sigma_\gamma} g(x, y).$$

Then by the maximum principle

$$\bar{\Sigma}_\gamma \subset J_a.$$

It is clear from the definition that the capacity is a non-decreasing function of the set E. Hence

$$\operatorname{cap} \bar{\Sigma}_\gamma \leq \operatorname{cap} J_a = \frac{1}{a} = \frac{1}{\min\limits_{\partial \Sigma_\gamma} g} .$$

Similarly, if we let

$$b = \max_{\partial \Sigma_\gamma} g$$

we find

$$\operatorname{cap} \bar{\Sigma}_\gamma \geq \operatorname{cap} J_b = \frac{1}{b} = \frac{1}{\max\limits_{\partial \Sigma_\gamma} g} .$$

Thus

$$\min_{\partial \Sigma_\gamma} g \leq [\operatorname{cap} \bar{\Sigma}_\gamma]^{-1} \leq \max_{\partial \Sigma_\gamma} g.$$

We now note that g is a positive solution of $Lg = 0$ in $\Sigma - y$. Therefore by Moser's Harnack inequality (theorem (2.2)) there is a constant c depending only on λ and γ such that

$$\max_{\partial \Sigma_\gamma} g \leq c \min_{\partial \Sigma_\gamma} g.$$

Hence [5] on $\partial \Sigma_\gamma$

$$c^{-1} [\operatorname{cap} \bar{\Sigma}_\gamma]^{-1} \leq g(x, y) \leq c\, [\operatorname{cap} \bar{\Sigma}_\gamma]^{-1}. \tag{7.2}$$

Since the ellipticity constant λ is invariant under a uniform dilation, c can be chosen as a non-decreasing function of γ. Hence for $\gamma \leq \gamma_0 < 1$ we can choose c depending only on λ.

Now let $\bar{L}$ be any other uniformly elliptic operator with the same λ and smooth coefficients. Let $\bar{g}(x, y)$ be its Green's function. We can define a capacity $\overline{\operatorname{cap}}(E)$ with the operator $\bar{L}$.

By the above proof

$$c^{-1} [\overline{\operatorname{cap}}\, \bar{\Sigma}_\gamma]^{-1} \leq \bar{g}(x, y) \leq c\, [\overline{\operatorname{cap}}\, \bar{\Sigma}_\gamma]^{-1} \tag{7.3}$$

for $\gamma \leq \gamma_0$ with the same c.

It is clear from the definition of capacity that for any L and $\bar{L}$ with ellipticity constant λ and any set E

$$\lambda^{-2}\, \overline{\operatorname{cap}}(E) \leq \operatorname{cap}(E) \leq \lambda^2\, \overline{\operatorname{cap}}(E).$$

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(5) A similar result was found independently by Royden [19]. See also [14, p. 590].

Therefore, it follows from (7.1) and (7.3) that

$$\lambda^{-2} c^{-2} \bar{g}(x, y) \leq g(x, y) \leq \lambda^2 c^2 \bar{g}(x, y) \tag{7.4}$$

for all $x \in \bar{\Sigma}_{r_0}$.

If L does not have C^∞ coefficients, we approximate it by $L^{(s)}$ as in section 5. Then all the $g^{(s)}$ satisfy (7.4) with a fixed function $\bar{g}(x, y)$. Since $g^{(s)} \rightarrow g$ in $L^q(\Sigma)$, the limit $g(x, y)$ again satisfies (7.4) for $\bar{L}$ with C^∞ coefficients. We can now fix L and approximate any $\bar{L}$ by $\bar{L}^{(s)}$ with C^∞ coefficients. Then we find that (7.4) holds for any L and $\bar{L}$ with ellipticity constant λ. Thus we have proved:

THEOREM (7.1). *Let $g(x, y)$ and $\bar{g}(x, y)$ be the Green's functions for any uniformly elliptic operators L and $\bar{L}$ with ellipticity constant λ on a sphere Σ. Then for any compact subset C of Σ there exists a constant K depending only on C, Σ and λ such that*

$$K^{-1} \bar{g}(x, y) \leq g(x, y) \leq K \bar{g}(x, y) \quad \text{for } x, y \in C. \tag{7.5}$$

REMARKS. 1. This theorem can easily be transferred to any simply connected domain Σ' which can be mapped smoothly onto the sphere.

2. If $\bar{L}$ is taken to be the Laplace operator, Σ the sphere $|x| < R$, and $n > 2$,

$$\bar{g}(x, y) = (n-2)^{-1} \omega_n^{-1} \{ |x - y|^{2-n} - |x - R^2 |y|^{-2} y|^{2-n} \}, \tag{7.6}$$

where ω_n is the area of the unit sphere, so that (7.4) leads to bounds of order $|x - y|^{2-n}$ for $|x - y|$ small.

The following is an obvious corollary of theorem (7.1).

COROLLARY (7.1). *If, in the notation of theorem (7.1), μ is a non negative measure with support on C, and u and $\bar{u}$ are the weak solutions vanishing on $\partial\Sigma$ of $Lu = \mu$ and $\bar{L}\bar{u} = \mu$, respectively, then*

$$K^{-1} \bar{u}(x) \leq u(x) \leq K \bar{u}(x) \qquad \text{a. e. on } C. \tag{7.7}$$

If $\mu \in H^{-1,2}(\Sigma)$, this inequality also is valid in the sense of $H_0^{1,2}(\Sigma)$.

The inequality (7.7) is an immediate consequence of (7.5) and the representation theorem (6.1). The inequality in the sense of $H_0^{1,2}(\Sigma)$ follows from property (1.2) by applying (7.7) to an open set $\Omega \supset C$ such that $\bar{\Omega} \subset \Sigma$.

We now return to the approximation of $g(x, y)$ by Green's functions $g^{(s)}(x, y)$ of operators L^s with coefficients in $C^\infty(\Sigma)$. We let $\bar{L}$ be the Laplace operator, so that $\bar{g}$ is given by (7.6). It now follows from (7.5) and the maximum principle that $g^{(s)}(x, y)$ is uniformly bounded in any compact subset of $\Sigma - y$. Furthermore $g^{(s)} \in C^\infty$ and $L^{(s)} g^{(s)} = 0$ in this set. That is,

$$\int_\Sigma a_{ij}^{(s)} g_{x_i}^{(s)} \Phi_{x_j} \, dx = 0 \tag{7.8}$$

for any $\Phi \in C'(\bar{\Sigma})$ with compact support in $\Sigma - y$. Therefore by lemma 2.1 $\| g^{(s)} \|_{H^{1,2}(C)}$ over any compact subset C of $\Sigma - y$ is uniformly bounded. It follows that a subsequence of $g^{(s)}$, and hence by convergence in $H^{1,p'}(\Sigma)$ the whole sequence $g^{(s)}$, converges to g weakly in $H^{1,2}(C)$. Since $a_{ij}^{(s)} \longrightarrow a_{ij}$ in $L^2(\Sigma)$, we find from (7.7) that

$$\int_\Sigma a_{ij} \, g_{x_i} \, \Phi_{x_j} \, dx = 0$$

for any Φ with compact support in $\Sigma - y$.

By theorem (2.4) the functions $g^{(s)}$ are equicontinuous in any compact subset of $\Sigma - y$ and hence $g^{(s)}(x, y) \longrightarrow g(x, y)$ uniformly in any compact subset of $\Sigma - y$. Moreover, $g(x, y)$ is Hölder continuous in $\Sigma - y$.

It is easily seen that as the radius of Σ goes to infinity, the Green's function increases to a Hölder continuous function. This function has all the basic properties of Green's function. It is not in $H^{1,p'}(E^n)$, but is in $H^{1,p'}_{\mathrm{loc}}(E^n) \cap H^{1,2}_{\mathrm{loc}}(E^n - y)$.

The bounds (7.5) now hold uniformly in E^n. In particular if we take the Laplace operator for $\bar{L}$ and $n \geq 3$, we have

$$K^{-1} \, | x - y |^{2-n} \leq g(x, y) \leq K \, | x - y |^{2-n} \tag{7.9}$$

uniformly in E^n.

8. **Properties of weak solutions.**

We consider a weak solution $u \in H_0^{1,p'}(\Sigma)$, $p' < n/(n-1)$, as defined in section 5, of

$$Lu = \mu$$

for a non-negative measure μ with support in the sphere Σ.

The representation theorem (6.4) gives

$$u(y)=\int_{\Sigma} g(x,y)\,d\mu(x). \tag{8.1}$$

This was previously interpreted in the sense of $L^1(\Sigma)$, since g was only defined almost everywhere. We now agree to use the function $g(x,y)$ which is continuous for $x \neq y$, whose existence was established in the preceding section. Then (8.1) defines $u(y)$ at each point of Σ. (It may, of course, be infinite at some points).

Since g becomes infinite for $x=y$, we must define the integral in (8.1) as the limit of integrals of an increasing sequence of continuous functions converging to g. Let

$$F_a(\xi)=\begin{cases} \xi & \xi \leq a \\ \xi-\dfrac{1}{4a}(\xi-a)^2 & a \leq \xi \leq 3a \\ 2a & \xi \geq 3a \end{cases}$$

Then $F_a(g(x,y))$ is continuous in x and y. We define

$$\widehat{u}(y)=\lim_{a\to\infty}\int_{\Sigma} F_a(g(x,y))\,d\mu(x). \tag{8.2}$$

Thus $\widehat{u}$ is a particular function in the L^1 equivalence class of the solution u of $Lu=\mu$. The function $\widehat{u}$ is the limit of a non-decreasing sequence of continuous functions. Thus it is lower semi-continuous; that is,

$$\widehat{u}(y)\leq \liminf_{x\to y}\widehat{u}(x).$$

It is easily seen that $F_a(g)$ is in $H_0^{1,2}(\Sigma)$, and is the solution vanishing on $\partial\Omega$ of

$$L(F_a(g))=-F_a''(g)\,a_{ij}\,g_{x_i}\,g_{x_j}=\begin{cases} \dfrac{1}{2a}\,a_{ij}\,g_{x_i}\,g_{x_j} & a\leq g\leq 3a \\ 0 & \text{elsewhere.}\end{cases}$$

Therefore, by the reciprocity relation (6.2)

$$\widehat{u}(y)=\lim_{a\to\infty}\frac{1}{2a}\int_{a\leq g\leq 3a} u(x)\,a_{ij}\,g_{x_i}(x,y)\,g_{x_j}(x,y)\,dx \tag{8.3}$$

where the integral on the right exists for all y.

By the results of section 7 $a_{ij}\, g_{x_i}\, g_{x_j}$ is locally integrable for $x \neq y$ and

$$\frac{1}{2a} \int\limits_{a \leq g \leq 3a} a_{ij}\, g_{x_i}\, g_{x_j}\, dx = 1.$$

Moreover by (7.5) with $\bar{L}$ the Laplacian, the set $a \leq g \leq 3a$ shrinks to the point y as $a \longrightarrow \infty$. Thus (8.3) represents $\widehat{u}\,(y)$ as the limit of averages of $\widehat{u}$ over a set shrinking to y. Therefore,

$$\widehat{u}\,(y) \geq \liminf_{x \to \infty} u,$$

where u is any member of the equivalence class $u = \widehat{u}$ a. e. Since $\widehat{u}$ is lower semi-continuous, we find that

$$\widehat{u}\,(y) = \liminf_{x \to y} \widehat{u}\,(x) = \operatorname{ess} \liminf_{x \to y} u\,(x). \tag{8.4}$$

By the essential limit inferior we mean the largest limit inferior obtained among functions u such that $u = \widehat{u}$ a. e.

In particular if $\widehat{u}$ is the capacitary potential of a closed set E, we know that $\widehat{u} = 1$ almost everywhere on E. Moreover, since $\widehat{u}$ is continuous in $\Sigma - E$, it coincides with the continuous solution u of the boundary value problem $Lu = 0$ in $\Sigma - E$, $u = 0$ on $\partial \Sigma$, $u = 1$ on ∂E. Hence $\widehat{u} = u \leq 1$ in $\Sigma - E$. It follows that if y is a boundary point of E

$$\widehat{u}\,(y) = \liminf_{\substack{x \to y \\ x \in \Sigma - E}} \widehat{u}\,(x).$$

Then $u\,(y) = 1$ if and only if the capacitary potential is continuous at y. Thus the remark at the end of § 4 becomes:

LEMMA (8.1) *The boundary point y is regular with respect to Ω if and only if $\widehat{u^\varrho}\,(y) = 1$ for all ϱ, where*

$$\widehat{u^\varrho}\,(y) = \lim_{a \to \infty} \int\limits_{\Sigma} F_a\,(g\,(x, y))\, d\mu_\varrho\,(y)$$

and μ_ϱ is the capacitary distribution of the set

$$E_\varrho = \{x \mid x \in \Sigma - \Omega, \mid x - y \mid \leq \varrho\}.$$

We note that if $\widehat{u}(x)$ is finite, the limit (8.2) may be replaced by

$$\widehat{u}(x) = \lim_{\sigma \to 0} \int_{|x-y| > \sigma} g(x, y)\, d\mu. \tag{8.6}$$

Since $F_a(g) \leq g$, we have

$$\widehat{u}(x) - \int_{|x-y|>\sigma} g\, d\mu \leq \left[u(x) - \int_{\Sigma} F_a(g)\, d\mu \right] + \int_{|x-y| \leq \sigma} F_a(g)\, d\mu. \tag{8.7}$$

The bracket can be made less that $\frac{1}{2}\,\varepsilon$ by choosing a sufficiently large independently of σ because of (8.2). The second integral is bounded by

$$2a \int_{|x-y| \leq \sigma} du.$$

By (7.5) with $\overline{L}$ the Laplace operator

$$\widehat{u}(x) \geq \int_{|x-y| \leq \sigma} g\, d\mu \geq c\sigma^{2-n} \int_{|x-y| \leq \sigma} d\mu,$$

where c is a constant. Hence the second integral in (8.7) is bounded by

$$\frac{a}{c}\, \sigma^{n-2}\, u(x),$$

which can be made less than $\frac{1}{2}\,\varepsilon$ for σ sufficiently small. The result follows.

9. The uniformity of regular points.

We are now in a position to prove our main result. Let L and $\overline{L}$ be any two uniformly elliptic operators with ellipticity constant λ defined on the sphere Σ. Let Ω be a domain such that $\overline{\Omega} \subset \Sigma$. Let y be a boundary point of Ω. Our theorem is:

THEOREM (9.1). *If y is an irregular point with respect to the uniformly elliptic operator L, it is also irregular with respect to any other uniformly elliptic operator $\overline{L}$.*

PROOF. Let

$$E_\varrho = \{x \mid x \in \Sigma - \Omega, \mid x - y \mid \leq \varrho\}.$$

(We suppose ϱ so small that $\{ \mid x - y \mid \leq \varrho \} \subset \Sigma$).

Let $u^\varrho(x)$ and $\bar{u}^\varrho(x)$ be the capacitary potentials of E_ϱ with respect to Σ and the operators L and $\bar{L}$, respectively. Let $\widehat{u^\varrho}$ and $\widehat{\bar{u}^\varrho}$ be their representations of the form (8.2). By lemma (8.1) y is irregular with respect to L if and only if for some $\varrho > 0$ $\widehat{u^\varrho}(y) < 1$. It is irregular with respect to $\bar{L}$ if $\widehat{\bar{u}^\tau}(y) < 1$ for some $\tau > 0$.

Thus to prove our theorem, we need only show that if $\widehat{u^\varrho}(y) < 1$, there is a τ such that $\widehat{\bar{u}^\tau}(y) < 1$. We show this as follows.

Let μ_ϱ be the capacitary distribution corresponding to u^ϱ.
Then

$$\widehat{u^\varrho}(y) = \int g(y, z)\, d\mu_\varrho(z).$$

By (8.6) we can find a $\sigma < \varrho$ such that

$$\int_{|z - y| \leq \sigma} g(y, z)\, d\mu_\varrho(z) < \varepsilon$$

for any preassigned $\varepsilon > 0$. Let

$$v(x) = \int_{|x - z| \leq \sigma} g(x, z)\, d\mu_\varrho(z)$$

$$w(x) = \int_{|x - z| > \sigma} g(x, z)\, d\mu_\varrho(z),$$

so that

$$\widehat{u^\varrho}(x) = v(x) + w(x)$$

and

$$v(y) < \varepsilon.$$

(The integrals are defined as in (8.2)). By theorem (5.3) v and w are in $H_0^{1,2}(\Sigma)$. Moreover $w(x)$ is continuous at y, and $w(y) \leq \widehat{u^\varrho}(y) < 1$. Therefore, there is a $\tau < \sigma$ such that

$$w(x) \leq \frac{1}{2}\,[1 + \widehat{u^\varrho}(y)] \quad \text{on} \quad E_\tau.$$

Now $u^{\varrho}(x) = 1$ on E_{ϱ}, and hence on E_{τ}, in the sense of $H_0^{1,2}(\Sigma)$. Therefore.

$$v(x) = 1 - w(x)$$

$$\geq \frac{1}{2}[1 - \widehat{u^{\varrho}}(y)] \quad \text{on} \quad E_{\tau}$$

in the sense of $H_0^{1,2}(\Sigma)$.

We now define

$$\bar{v}(x) = \int\limits_{|x-y| \leqq \sigma} \bar{g}(x, z)\, d\mu_{\varrho}(z) \tag{9.1}$$

where $\bar{g}$ is the Green's function corresponding to $\bar{L}$. The integral is again defined as in (8.2). Then by (7.5)

$$\bar{v}(y) < \varepsilon K \tag{9.2}$$

while by corollary (7.1)

$$\bar{v}(x) \geq K^{-1} v(x) \quad \text{on} \quad |x - y| \leq \varrho \tag{9.3}$$

$$\geq \frac{1}{2} K^{-1}[1 - \widehat{u^{\varrho}}(y)] \quad \text{on} \quad E_{\tau}$$

in the sense of $H_0^{1,2}(\Sigma)$. The constant K depends only on λ and ϱ.

If we let $\bar{u}^{\tau}(x)$ be the capacitary potential of E_{τ} with respect to the operator $\bar{L}$, the last inequality becomes

$$\bar{v}(x) \geq \frac{1}{2} K^{-1}[1 - \widehat{u^{\varrho}}(y)]\, \bar{u}^{\tau}(x) \quad \text{on} \quad E_{\tau}$$

in the sense of $H_0^{1,2}(\Sigma)$.

In $\Sigma - E_{\tau}$ $\bar{v}(x) - \frac{1}{2} K^{-1}[1 - \widehat{u^{\varrho}}(y)]\, \bar{u}^{\tau}(x)$ is a supersolution for $\bar{L}$ and hence by the maximum principle

$$\bar{v}(x) \geq \frac{1}{2} K^{-1}[1 - \widehat{u^{\varrho}}(y)]\, \bar{u}^{\tau}(x) \quad \text{in} \quad \Sigma \tag{9.4}$$

in the sense of $H_0^{1,2}$.

Since $\overline{u}^\tau$ is a capacitary potential,

$$\widehat{\overline{u}}^\tau(y) = \liminf_{\substack{x \to y \\ x \in \Sigma - E_\tau}} \widehat{\overline{u}}^\tau(x).$$

Since $\overline{v}$ is given by the representation (9.1), we have

$$\overline{v}(y) = \operatorname{ess}\liminf_{x \to y} \overline{v}(x).$$

We choose

$$\varepsilon < \frac{1}{2} K^{-2} [1 - \widehat{u^\varrho}(y)]. \tag{9.5}$$

Then by (9.2) and (9.3) $\overline{v}(x)$ is bounded away from $\overline{v}(y)$ almost everywhere on E_τ. Hence

$$\overline{v}(y) = \liminf_{\Sigma - E_\tau} v(x).$$

Now $\overline{L}\,\overline{v} = \overline{L}\,\overline{u}^\tau = 0$ in $\Sigma - E_\sigma$. Hence by theorem (2.1) $\overline{v}$ and $\overline{u}^\tau$ are continuous in $\Sigma - E_\sigma$, so that the inequality (9.4) holds at every point. It follows that

$$\begin{aligned} \overline{v}(y) = \liminf_{\substack{x \to y \\ x \in \Sigma - E_\tau}} \overline{v}(x) &\geq \frac{1}{2} K^{-1} [1 - \widehat{u^\varrho}(y)] \liminf_{\substack{x \to y \\ x \in \Sigma - E_\tau}} \widehat{\overline{u}}^\tau(x) \\ &= \frac{1}{2} K^{-1} [1 - \widehat{u^\varrho}(y)]\, \widehat{\overline{u}}^\tau(y). \end{aligned}$$

(Note that the intersections of $\Sigma - E_\tau$ and $\Sigma - E_\sigma$ with the neighborhood $|x - y| < \tau$ of y coincide).

By (9.2) and (9.5) we have

$$\widehat{\overline{u}}^\tau(y) < 1,$$

and the theorem is proved.

By choosing first L, then $\overline{L}$ to be the Laplace operator, we have the immediate corollary.

COROLLARY (9.1). *A point $y \in \partial\Omega$ is regular with respect to any uniformly elliptic operator L if and only if it is regular with respect to the Laplace operator.*

By taking $\bar{L} = L$ in the proof of theorem (9.1) and noting that ε is arbitrary, we find the following generalization of a result of de la Vallée Poussin [10].

COROLLARY (9.2). *A point y of $\partial\Omega$ is irregular if and only if $\widehat{u^\varrho}(y) \longrightarrow 0$ as $\varrho \longrightarrow 0$.*

For the sake of completeness we give a formulation and proof of the Wiener criterion due to Frostman [5].

THEOREM (9.2). *Let $c(\varrho)$ be the capacity of the set*

$$E_\varrho = \{x \mid x \in \Sigma - \Omega, \ \mid x - y \mid \leq \varrho\}.$$

The point $y \in \partial\Omega$ is irregular if and only if

$$\int_0 c(r)\, r^{1-n}\, dr < \infty. \tag{9.6}$$

PROOF. We let L be the Laplace operator and Σ an infinite sphere, so that

$$g(x, z) = \alpha \mid x - z \mid^{2-n},$$

where $\alpha = (n-2)^{-1}\, \omega_n^{-1}$.

By integration by parts we have [6]

$$\begin{aligned} \widehat{u^\varrho}(y) &= \int_{E_\varrho} g(x, y)\, d\mu_\varrho(y) = \alpha \int_0^\varrho r^{2-n}\, d\mu_\varrho(E_r) \\ &= \alpha \varrho^{2-n} \mu_\varrho(E_\varrho) + (n-2)\, \alpha \int_0^\varrho r^{1-n}\, \mu_\varrho(E_r)\, dr, \end{aligned} \tag{9.7}$$

where $\widehat{u^\varrho}$ is the capacitary potential of E_ϱ, and μ_ϱ is its capacitary distribution.

By the reciprocity theorem (6.2) we have for $r \leq \varrho$

$$\mu_r(E_r) = \mu_\varrho(E_r) + \int_{E_\varrho - E_r} \widehat{u^r}\, d\mu_\varrho\,. \tag{9.8}$$

Suppose first that (9.6) holds, and note that $\mu_r(E_r) = c(r)$. By (9.8) we find that $\mu_\varrho(E_r) \leq \mu_r(E_r)$, so that the integral on the right of (9.7) is bounded

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(6) Note that $\varepsilon^{2-n}\, \mu_\varrho(E_\varepsilon) \geq \int_0^\varepsilon r^{2-n}\, d\mu(E_r) \to 0$ as $\varepsilon \to 0$ by (8.6).

by $\int_0^{\varrho} r^{1-n} c(r)\, dr$. This can be made arbitrarily small by choosing ϱ small.

It also follows that from (9.6) that $\varrho^{2-n} \mu_\varrho (E_\varrho)$ cannot be bounded away from zero as $\varrho \longrightarrow 0$. Hence, the right-hand side of (9.7) can be made arbitrarily small. Hence there is a ϱ for which $\widehat{u^\varrho}(y) < 1$ and y is irregular.

Suppose, on the other hand, that y is irregular. By (9.7) and the fact that $\widehat{u^r} \leq 1$ we have

$$\mu_{r/2} (E_{r/2}) \leq \mu_\varrho (E_r) + \int_{E_\varrho - E_r} u^{r/2}(x)\, d\mu_\varrho (x). \tag{9.9}$$

Now

$$\widehat{u^{r/2}}(x) = \alpha \int_{E_{r/2}} |x - z|^{2-n} d\mu_{r/2}(z).$$

For $x \in E_\varrho - E_r$ and $z \in E_{r/2}$ we have

$$|x - z| \geq |x - y| - \frac{1}{2} r \geq \frac{1}{2} |x - y|,$$

and hence

$$\widehat{u^{r/2}}(x) \leq \alpha\, 2^{n-2} \,|x - y|^{2-n} \mu_{r/2}(E_{r/2}).$$

Thus from (9.9)

$$\mu_{r/2}(E_{r/2}) \leq \mu_\varrho (E_r) + 2^{n-2}\, \widehat{u^\varrho}(y)\, \mu_{r/2}(E_{r/2}). \tag{9.10}$$

By corollary (9.2) we can choose ϱ so small that $\widehat{u^\varrho}(y) < 2^{1-n}$. Then (9.10) gives

$$c\left(\frac{1}{2} r\right) = \mu_{r/2}(E_{r/2}) \leq 2\, \mu_\varrho (E_r).$$

Since the integral on the right of (9.7) is finite, we find that $\int_0^{\varrho} c\left(\frac{1}{2} r\right) r^{1-n}\, dr$ is finite, so that (9.6) holds.

10. Unbounded domains.

It is easily seen that lemma (1.4) and theorems (2.3) and (2.5) are still valid when Ω is unbounded. Thus we have a solution $u = Bh$ in $H^{1,2}(\Omega)$ of the problem (3.1), (3.2) for any $h \in \tau^{1,2}$, and the solution satisfies (3.5).

If $\partial\Omega$ is bounded (exterior problem), we restrict B to continuous functions in $\tau^{1,2}$ and then extend it to all continuous functions h on $\partial\Omega$ exactly as in § 3.

If $\partial\Omega$ extends to infinity, the operator B can be continued to the closure in the maximum norm of $\tau^{1,2}$. This closure certainly contains those functions h which are continuous on $\partial\Omega$ and vanish at infinity. By putting $B(1) = 1$, we define B for continuous h which have a limit at infinity. The function $u = Bh$ is a local solution in $H^{1,2}_{loc}(\Omega)$ of $Lu = 0$ and satisfies the maximum principle (3.5).

The proof of the localization lemma (3.3) can be extended to the case where Ω or Ω' or both are unbounded simply by replacing the function $|x - y|$ by the truncated function $\{|x - y|\}^1$.

Thus the question of the regularity of y with respect to an unbounded domain Ω can be reduced to that of its regularity with respect to a bounded subdomain Ω'. Therefore theorems (9.1) and (9.2) remain valid.

The Green's function $g(x, y)$ for an infinite sphere constructed at the end of § 7 satisfies the inequality (7.9) uniformly. Therefore it serves as a barrier at infinity if $\partial\Omega$ extends to infinity. Thus the point at infinity is always regular.

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