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ON FUNCTIONAL REPRESENTATION BY A CERTAIN TYPE OF GENERALIZED POWER SERIES (1)

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I. T^{α} — Functions

1. DEFINITION 1. $f(x)$ is said to belong to T^{α}, i. e., $f(x) \in T^{\alpha}$, if and only if there exist a sequence of numbers $\{a_n\}$ and $\alpha > 0$ (2) such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - a)^{n\alpha}, \tag{1.1}$$

on L: $0 \leqq a \leqq x \leqq b < \varrho$, where ϱ is the radius of convergence of (1.1).

It is clear that the series (1.1) is absolutely and uniformly convergent on L, and that when $\alpha = 1$, $f(x)$ is a Taylor's function. If z is a complex variable such that $R(z) \leq \ln (x_0 - a)$, $a < x_0 \leq b$, then

$$\Phi(z) = \sum_{n=0}^{\infty} a_n \exp n\alpha z$$

is an analytic function for $R(z) \leq \ln (x_0 - a)$ since it is uniformly convergent there; and consequently $\Phi[\ln (x - a)]$ is an analytic function on $[a, b]$, and therefore $f(x)$ is analytic.

THEOREM 1.1. If $g(x) = \sum_{n=0}^{\infty} b_n (x - a)^{n\alpha}$ such that

(1) Presented to the Mathematical Association of America during the annual Texas meeting at Rice University, April 6, 1962.

(2) This definition together with the properties which will follow can be extended to include the cases where α is a complex number such that $R\alpha > 0$.

$\lim\limits_{n\to\infty} \sup | b_n |^{1/n\alpha} = 1/r$ or $\lim\limits_{n\to\infty} | b_{n+1}/b_n | = 1/r^\alpha$, $r > 0$, then $g(x) \in T^\alpha$.

Proof. For $g(x)$ to belong to T^α, the series

$$\sum_{n=0}^{\infty} b_n (x-a)^{n\alpha}$$

must be absolutely and uniformely convergent on $[a, b]$ where $a \leqq x \leqq b < r$. This can be shown easily as the proof of the absolute and uniform convergence of this series is similar to that in the case when $\alpha = 1$.

Theorem 1.2 If $f_1(x), f_2(x) \in T^\alpha$ on L, then $[f_1 + f_2] \in T^\alpha$ and $(f_1 f_2) \in T^\alpha$ on L.

Proof. The proof follows from Definition 1, as both functions f_1 and f_2 are represented by absolutely and uniformly convergent series on L.

Theorem 1.3. If $f(x) \in T^\alpha$ on L, then $f(x)$ is continuous at $x = a$.

Proof. Suppose that $0 < \varrho < r$. Then $\sum_{n=0}^{\infty} | a_n | \varrho^{n\alpha}$ and $\sum_{n=1}^{\infty} |a_n| \varrho^{(n-1)\alpha} = k > 0$ converge together, and for every x such that $| x - a | \leqq \varrho$, we have

$$| f(x) - a_0 | = | (x-a)^\alpha | \left| \sum_{n=1}^{\infty} a_n (x-a)^{\overline{n-1}\,\alpha} \right| \leqq k \varrho^\alpha .$$

If $\mu > 0$, there exists $\delta_\mu \leqq \min(\varrho^\alpha, \mu/k)$ such that $|f(x) - a_0| < \mu$ or $|f(x) - f(a)| < \mu$, whenever $| x - a | < \delta$.

Corollary 1.3. If $f(x) \in T^\alpha$, then as $x \to a$

$$\left[f(x) - \sum_{p=0}^{p=n} a_p (x-a)^{p\alpha} \right] / (x-a)^{\overline{n+1}\alpha} \to a_{n+1}$$

for every $n = 0, 1, 2, \ldots$; hence in particular

$$f(x) - \sum_{p=1}^{p=n} a_p (x-a)^{p\alpha} = O\left((x-a)^{\overline{n+1}\alpha}\right)$$

Lemma 1.1. If $f(x) \in T^\alpha$ on L then $g(z) = \left[\sum_{n=0}^{\infty} a_n z^n \right] \in T$ on L_0: $0 \leqq \leqq z \leqq b < \varrho^\alpha$.

PROOF. The proof follows from the fact that $g(z) = f(z^{1/\alpha} + a)$.

THEOREM 1.4. If $f(x) \in T^{\alpha}$, then $f(x)$ is continuous at every point interior to the interval of convergence.

PROOF. The two series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} a_n (x-a)^{n\alpha}$ converge together, $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$ represents a Taylor's function which is continuous at every point interior to its interval of convergence, and by Lemma 1.1

$$\Phi[(x-a)^{\alpha}] \equiv f(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n\alpha}$$

on L. Therefore $f(x)$ is continuous at every point interior to L.

THEOREM 1.5. If $f(x) \in T^{\alpha}$, then $f^{(n)}(x)$, $n = 1, 2, \dots$, exist and are continuous at every point x on $L_1 : a < A \leq x \leq b$.

PROOF. By Lemma 1.1

$$\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$$

is continuous together with $\Phi^{(n)}(z)$, $n = 1, 2, \dots$, on $0 \leq z \leq b < r^{\alpha}$. Now since

$$f(x) = \Phi[(x-a)^{\alpha}],$$

then

$$f'(x) = \alpha (x-a)^{\alpha-1} \Phi'$$

a relation which shows clearly that $f'(x)$ is continuous at every point $x \neq a$ on L. Similarly and by the same process of derivation, it can be shown easily that $f^{(n)}(x)$ are continuous at every point $x \neq a$ in L, i. e., in L_1.

Theorem 1.6. If $f(x) \in T^{\alpha}$ and

(i) $$f_k(x) = \sum_{p=k}^{\infty} \frac{\Gamma(p\alpha + 1)}{\Gamma(\overline{p + k\alpha - 1})} a_p (x-a)^{\overline{p-k\alpha}}$$

(ii) $$g_k(x) = \sum_{p=k}^{\infty} \frac{\Gamma(p+k)}{\Gamma(p-k+1)} a_p (x-a)^{\overline{p-k\alpha}},$$

$(k = 1, 2, \dots)$, then $f_k(x)$ and $g_k(x) \in T^{\alpha}$ for each value of k.

Proof. (i). Let $k = K$ be a fixed positive integer, and suppose $U_{n,k}$ is the coefficient of the nth term of the series, then

$$\limsup_{n\to\infty} |U_{n,K}|^{1/n\alpha} = \limsup_{n\to\infty} |a_n|^{1/n\alpha},$$

since for large n we can write

$$\frac{\Gamma(n\alpha+1)}{\Gamma(\overline{n-K}\,\alpha+1)} \sim (\overline{n-K}\alpha+1)^{K\alpha},$$

$$\text{and hence } \left[\frac{\Gamma(n\alpha+1)}{\Gamma(\overline{n-K}\alpha+1}\right]^{1/n\alpha} \longrightarrow 1 \text{ as } n \longrightarrow \infty .$$

Consequently, by Theorem 1.1 $f_K(x) \in T^\alpha$, inasmuch as $f(x) \in T^\alpha$. Since the last limit holds independent of $k, f_k(x) \in T^\alpha$ for all $k = 1, 2, \dots$.

(ii) Let $k = 1$. Then

$$g_1(x) = \sum_{p=0}^{\infty} p a_p (x-a)^{\overline{p-1}\alpha}$$

But

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|n a_n|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$$

since $\sqrt[n]{n} \longrightarrow 1$ as $n \longrightarrow \infty$. Consequently $g_1(x) \in T^\alpha$. By induction it can be shown that $g_k(x) \in T^\alpha$ for $(k = 1, 2, \dots)$.

Corollary 1.6. The functions $f_k(x)$ and $g_k(x), k = 1, 2, \dots,$ are continuous at every point interior to L.

Theorem 1.7. If $F(x)$ and $G(x) \in T^\alpha$ such that

$$F(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n\alpha} \quad \text{and} \quad G(x) = \sum_{n=0}^{\infty} b_n (x-a)^{n\alpha},$$

and the two series have a radius of convergence $\geq r_0 > 0$, and if $F(x)$ and $G(x)$ have the same values in a neighborhood of $x = a$ or their values coincide only at points x_i of a certain sequence $\{x_i\}$ with $|x_i - a| < r_0$, then $F(x) \equiv G(x)$, i. e., $a_n = b_n$.

PROOF. Since $F(x_i) = G(x_i)$ for $i = 0, 1, 2, \ldots$ and the functions are continuous at $x = a$, by Theorem 1.4

$$F(a) = \lim_{i \to \infty} F(x_i) = \lim_{i \to \infty} G(x_i) = G(a), \quad \text{or} \quad a_0 = b_0 .$$

Now let

$$F_1 = a_1 + a_2 (x - a)^{\alpha} + a_3 (x - a)^{2\alpha} + \ldots$$

$$G_1 = b_1 + b_2 (x - b)^{\alpha} + b_3 (x - b)^{2\alpha} + \ldots ,$$

where

$$F_1 = \frac{F(x) - a_0}{(x - a)^{\alpha}} \quad \text{and} \quad G_1 = \frac{G(x) - b_0}{(x - a)^{\alpha}} .$$

Then by the same reasoning we conclude that $a_1 = b_1$. Similarly by induction we have $a_n = b_n$.

THEOREM 1.8. If $f(x) \in T^{\alpha}$, then $[f'(x)]^{(n\alpha - 1)}$ and $f_{\alpha}^{(n)}(x)$ (3), $n = 1, 2, \ldots$, are continuous at every point in L.

PROOF. It is sufficient to show that

$$[f'(x)]^{(n\alpha - 1)} = f_n(x)$$

and

$$f_{\alpha}^{(n)}(x) = g_n(x), \quad n = 1, 2, \ldots ,$$

where f_n and g_n are the functions defined in theorem 1.6, since these functions are continuous in L by Corollary 1.6.

Now we have

$$f(x) = \sum_{n=0}^{\infty} a_n (x - a)^{n\alpha}$$

then

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - a)^{n\alpha - 1} . \tag{1.81}$$

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(3) $f_{\alpha}^{(n)}(x)$ denotes the nth derivative with respect to $(x - a)^{\alpha}$ and $[f'(x)]^{(n\alpha - 1)} = \overset{x}{\underset{a}{I}}{}^{1 - \alpha n} f'$, where f' is the derivative of $f(x)$ with respect to x (for details about the operational properties of this transform, which will be used later in this paper, see the writer's work [1]). The transform of such indices may be called the derivatives of $(n\alpha - 1)$-order.

The term by term differentiation is permissible since the derived function of each term is continuous at every interior point of L and the resulting series is uniformly convergent. Hence from (1.81) we obtain

$$[f'(x)]^{(\alpha-1)} = f_1(x) = \sum_{n=1}^{\infty} \frac{\Gamma(n\alpha+1)}{\Gamma(\overline{n-1}\,\alpha+1)} a_n (x-a)^{\overline{n-1}\alpha} . \tag{1.82}$$

Taking the derivative with respect to x of both sides of (1.82) we get

$$[f'(x)]^{(\alpha)} = \sum_{n=2}^{\infty} \frac{\Gamma(n\alpha+1)}{\Gamma(\overline{n-1}\alpha)} a_n (x-a)^{\overline{n-1}\alpha-1} ,$$

and hence

$$[f'(x)]^{(2\alpha-1)} = \sum_{n=2}^{\infty} \frac{\Gamma(n\alpha+1)}{\Gamma(\overline{n-2}\,\alpha+1)} a_n (x-a)^{\overline{n-2}\,\alpha} = f_2(x).$$

By continuing this process in the same manner we find that

$$[f'(x)]^{(n\alpha-1)} = f_n(x), \quad n = 3, 4, \ldots$$

The other relation can be established easily by taking the nth derivative of $f(x)$ with respect to $(x-a)^{\alpha}$.

Theorem 1.9. If $f(x) \in T^{\alpha}$, then

$$(i) \qquad a_n = \frac{[f'(a)]^{(n\alpha-1)}}{\Gamma(n\alpha+1)} , \qquad n = 1, 2, \ldots , \quad a_0 = f(a) ;$$

or

$$(ii) \qquad a_n = \frac{f_{\alpha}^{(n)}(a)}{\Gamma(n+1)}, \quad n = 0, 1, 2, \ldots$$

Proof. From Theorem 1.6 we have

$$f_k(a) = \Gamma(k\alpha+1)\, a_k$$

$$g_k(a) = \Gamma(k+1)\, a_k .$$

But by Theorem 1.8 we find that

$$[f'(a)]^{(k\alpha-1)} = f_k(a)$$

and

$$f_{\alpha}^{(k)}(a) \qquad = g_k(a),$$

and thus relations (*i*) and (*ii*) are established.

This theorem shows that $f(x)$ may be represented by any one of the following expansions:

$$(1.91) \qquad f(x) = \sum_{n=0}^{\infty} \frac{[f'(a)]^{(n\alpha-1)}}{\Gamma(n\alpha+1)} (x-a)^{n\alpha}, \quad [f'(a)]^{-1} = f(a),$$

or

$$(1.92) \qquad f(x) = \sum_{n=0}^{\infty} \frac{f_a^{(n)}(a)}{\Gamma(n+1)} (x-a)^{n\alpha}.$$

Thus by Theorem 1.7 we have the following

COROLLARY 1.9. If $f(x) \in T^\alpha$, then

$$\frac{[f'(a)]^{(n\alpha-1)}}{\Gamma(n\alpha+1)} = \frac{f_a^{(n)}(a)}{\Gamma(n+1)}$$

THEOREM 1.10. If $f(x) \in T^\alpha$, then (*i*) $\overset{x}{\underset{a}{I}}{}^{p\alpha} f \in T^\alpha$, $(p = 1, 2, \ldots)$, and if λ is a parameter and $\{b_n\}$ is an infinite sequence of numbers such that the ratio $|b_{n+1}/b_n|$ is bounded independent of n, and if $F(x) = \sum_{p=0}^{\infty} b_p \lambda^p \overset{x}{\underset{a}{I}}{}^{p\alpha} f$, then (*ii*) $F(x) \in T^\alpha$.

PROOF. (*i*) Let p be a fixed positive integer such that $p = P$. Since

$$(1.101) \qquad f(x) = a_0 + a_1 (x-a)^\alpha + a_2 (x-a)^{2\alpha} + \ldots,$$

and by similar reasoning given in the proof of Theorem 1.6, the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(n\alpha+1)}{\Gamma(\overline{n+P}\,\alpha+1)} a_n (x-a)^{\overline{n+P}\alpha}$$

is absolutely and uniformly convergent, and hence term by term integration of (1.101) is permissible and consequently

$$\overset{x}{\underset{a}{I}}{}^{P\alpha} f = \sum_{n=0}^{\infty} \frac{\Gamma(n\alpha+1)}{1(\overline{n+P}\,\alpha+1)} a_n (x-a)^{\overline{n+P}\alpha}.$$

Therefore $\overset{x}{\underset{a}{I}}{}^{p\alpha} f \in T^\alpha$.

(*ii*) Since $f(x) \in T^{\alpha}$ then by theorem 1.4 it is continuous and consequently bounded on L, i. e., there exists a number $M > 0$, such that $|f(x)| < M$. Accordingly we have

$$\left| b_n \lambda^n \overset{x}{\underset{a}{I}}{}^{n\alpha} f \right| < |b_n| \, |\lambda^n| \, M \frac{(x-a)^{n\alpha}}{\Gamma(n\alpha)}.$$

But the convergence ratio of the dominant series ΣA_n, where A_n is defined by the right hand side of the inequality may be given by

$$\left| \frac{C_{n+1}}{C_n} \right| = \left| \frac{b_{n+1}}{b_n} \right| |\lambda| \frac{\Gamma(n\alpha)}{\Gamma(\overline{n+1}\,\alpha)} < K |\lambda| \frac{\Gamma(n\alpha)}{\Gamma(\overline{n+1}\,\alpha)} \backsim \frac{K|\lambda|}{(n\alpha)^{\alpha}},$$

where $|b_{n+1}/b_n| < K > 0$, and $A_n = C_n (x-a)^{n\alpha}$. Thus we have

$$\lim_{n \to \infty} \left| \frac{C_{n+1}}{C_n} \right| = 0,$$

and accordingly the series $\sum\limits_{p=0}^{\infty} b_p \lambda^p \overset{x}{\underset{a}{I}}{}^{p\alpha} f$ is absolutely and uniformly convergent and $F(x) \in T^{\alpha}$, where

$$F(x) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\Gamma(q\alpha+1)}{\Gamma(\overline{p+q}\,\alpha+1)} a_p b_q \lambda^p (x-a)^{\overline{p+q}\alpha}.$$

Example. Let us consider the function

$$f(x) = \exp bx, \quad (b \text{ is real} \neq 0).$$

For $\alpha > 0$ we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \exp bx = \frac{1}{\Gamma(\alpha)} \int_a^x \exp bt \, (x-t)^{\alpha-1} \, dt.$$

Let $t = x - \dfrac{x-a}{u+1}$; then we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \exp bx = b^{-\alpha} \exp bx / \Gamma(\alpha) \int_0^{\infty} z^{\alpha-1} \exp(-z) \, dz = b^{-\alpha} \exp bx,$$

where $z = b \dfrac{x-a}{u+1}$.

Now if $\alpha + n > 0$, then we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \exp bx = D_x^n \overset{x}{\underset{a}{I}}{}^{\alpha+n} \exp bx = b^{-n-\alpha} D_x^n \exp bx = b^{-\alpha} \exp bx.$$

Therefore for all values of α, $\overset{x}{\underset{a}{I}}{}^{\alpha} \exp bx = b^{-\alpha} \exp bx$.

Consequently

$$[f'(x)]^{(n\alpha-1)} = b^{n\alpha} \exp bx,$$

and by (1.91) we have

$$\exp bx = \exp ba \cdot \sum_{n=0}^{\infty} \frac{b^{n\alpha}}{\Gamma(n\alpha+1)} (x-a)^{n\alpha}$$

II. Some elementary T^{α} — functions.

2. In this section we will discuss some properties of certain elementary T^{α}-functions which may represent, as we will see later, the generalized exponential, trigonometric and hyperbolic functions.

(*i*) Let us define

$$E_{\alpha}(m\,;x) = \sum_{n=1}^{\infty} \frac{m^{n-1}}{\Gamma(n\alpha)} x^{n\alpha-1} \tag{2.1}$$

where m is a number and $\alpha > 0$. This series converges for $x \neq 0$ and converges everywhere for $\alpha \geqq 1$. If $\alpha = 1$, then (2.1) is reduced to the exponential function, i. e.

$$E_1(m\,;x) = \exp mx = e^{mx}. \tag{2.2}$$

Also the function is α-differentiable and its derivative of α-order may be given by

$$E_{\alpha}^{(\dot{\alpha})}(m\,;x) = \sum_{n=2}^{\infty} \frac{m^{n-1}}{\Gamma(\overline{n-1}\,\alpha)} x^{\overline{n-1}\,\alpha-1} = mE_{\dot{\alpha}}(m\,;x),$$

so that for all derived functions of higher α-order we may have

$$E_{\alpha}^{(n\alpha)}(m\,;x) = m^n E_{\alpha}(m\,;x). \tag{2.3}$$

This clearly shows that $y = E_\alpha(\mp m; x)$ is a solution of the differential equation of non-integer order

$$y^{(\dot{\alpha})} \pm my = 0.$$

Moreover, from the following two relations

$$\overset{x}{\underset{0}{I}} E_\alpha(m; x) = \sum_{n=1}^{\infty} \frac{m^{n-1}}{\Gamma(n\alpha+1)} x^{n\alpha}$$

$$\overset{x}{\underset{0}{I}}{}^{1-\alpha} E_\alpha(m; x) = \sum_{n=0}^{\infty} \frac{m^{n}}{\Gamma(n\alpha+1)} x^{n\alpha},$$

we obtain

$$\overset{x}{\underset{0}{I}}{}^{1-\alpha} E_\alpha(m; x) - m \overset{x}{\underset{0}{I}} E_\alpha(m; x) = 1. \tag{2.4}$$

For $m \neq 0$

$$E_\alpha(m; x) = 1/m\, E_\alpha(1; m^{1/\alpha} x). \tag{2.5}$$

Also we have

$$E_\alpha(m; 0) = \begin{cases} 0 & \text{when} \quad \alpha > 1 \\ 1 & \text{»} \quad \alpha = 1. \end{cases} \tag{2.6}$$

The properties of $E_\alpha(0; x)$ can be easily discussed for different values of α since we have

$$E_\alpha(0; x) = x^{\alpha-1}/\Gamma(\alpha).$$

We shall attempt to deduce further properties of the function (2.1) from the series itself. According to Theorem 1.9 and (1.91) we may write

$$E_\alpha(m; x) = E_\alpha(m; x_1) + \frac{1}{\Gamma(\alpha+1)} [E'_\alpha(m; x_1)]^{(\alpha-1)} (x - x_1)^\alpha + \tag{2.71}$$

$$+ \frac{1}{\Gamma(2\alpha+1)} [E'_\alpha(m; x_1)]^{(2\alpha-1)} (x - x_1)^{2\alpha} + \dots$$

But we have

$$[E'(m; x)]^{(n\alpha-1)} = \sum_{p=n+1}^{\infty} \frac{m^{p-1}}{\Gamma(\overline{p - n\alpha})} x^{\overline{p-n\alpha}-1}$$

$$= m^n \sum_{q=1}^{\infty} \frac{m^{q-1}}{\Gamma(q\alpha)} x^{q\alpha-1}$$

$$= m^n E_\alpha(m; x).$$

Hence (2.71) may be written in the form

$$(2.7) \qquad E_\alpha(m\,;x) = E_\alpha(m\,;x_1)\left[1 + \frac{m}{\Gamma(\alpha+1)}(x-x_1)^\alpha + \right.$$

$$\left. + \frac{m^2}{\Gamma(2\alpha+1)}(x-x_1)^{2\alpha} + \dots\right].$$

From this relation we obtain

$$E_\alpha(m\,;x_1+x_2) = E_\alpha(m\,;x_1)\left[1 + \frac{m}{\Gamma(\alpha+1)}x_2^\alpha + \frac{m^2}{\Gamma(2\alpha+1)}x_2^{2\alpha} + \dots\right]$$

$$= E_\alpha(m\,;x_1)\overset{x_2}{\underset{0}{I}}{}^{1-\alpha} E_\alpha(m\,;x_2),$$

so that in general we have

$$(2.8) \qquad E(m\,;x_1+x_2+\dots+x_n) = E_\alpha(m\,;x_1)\prod_{k=2}^{n}\overset{x_k}{\underset{0}{I}}{}^{1-\alpha} E_\alpha(m\,;x_k).$$

In particular for $\alpha = 1$ this relation yields

$$E_1(m\,;x_1+x_2+\dots+x_n) = \prod_{p=1}^{n} E_1(m\,;x_p)$$

or

$$\exp m(x_1+x_2+\dots+x_n) = \prod_{p=1}^{n}\exp\, mx_p.$$

Now let $H_\alpha(m\,;x) = \overset{x}{\underset{0}{I}}{}^{1-\alpha} E_\alpha(m\,;x)$; then we have

$$E_\alpha(m\,;x_1+x_2+\dots+x_n)/E_\alpha(m;x_1) = \prod_{p=2}^{n} H_\alpha(m\,;x_p),$$

and if $x_1 = x_2 = \dots = x_n = 1$, then

$$E_\alpha(m\,;n)/E_\alpha(m\,;1) = H_\alpha^{n-1}(m\,;1),$$

and when $x_1 = x_2 = \dots = x_n = x$ we find that

$$(2.9) \qquad E_\alpha(m\,;nx)/E_\alpha(m\,;x) = H_\alpha^{n-1}(m\,;x).$$

It is interesting to note that for $|m| < 1$

$$H_0(m; x) = \sum_{p=0}^{\infty} m^p = 1/1 - m,$$

and from this relation we obtain

$$\int_0^{\infty} E_\alpha(m; t) \exp(-t)\, dt = H_0(m; x). \tag{2.10}$$

(ii) Let

$$S_\alpha(m; x) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1} m^{2p-1}}{\Gamma(2p\alpha)} x^{2p\alpha-1} \tag{2.11}$$

$$C_\alpha(m; x) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1} m^{2(p-1)}}{\Gamma(\overline{2p-1}\,\alpha)} x^{\overline{2p-1}\alpha-1}. \tag{2.12}$$

Each of these series represents a continuous functions for $x \neq 0$, and $\alpha > 0$. They are every where continuous for $\alpha \geqq 1$. It is to be noted that for $\alpha = 1$, $C_1(m; x) = \cos mx$ and $S_1(m; x) = \sin mx$. From (2.11) and (2.12) the following relations can be established easily :

$$\begin{aligned} S_\alpha(m; x) &= 1/2i\,[E_\alpha(\mathrm{i}m; x) - E_\alpha(-im; x)] \\ C_\alpha(m; x) &= 1/2\,[E_\alpha(\mathrm{i}m; x) + E_\alpha(-im; x)], \end{aligned} \tag{213}$$

where $i = \sqrt{-1}$,

$$\begin{aligned} S_\alpha^{(2n\alpha)}(m; x) &= (-1)^n m^{2n} S_\alpha(m; x) \\ S_\alpha^{(\overline{2n-1}\alpha)}(m; x) &= (-1)^{n-1} m^{2n-1} C_\alpha(m; x) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} C_\alpha^{(2n\alpha)}(m; x) &= (-1)^n m^{2n} C_\alpha(m; x) \\ C_\alpha^{(\overline{2n-1}\alpha)}(m; x) &= (-1)^n m^{2n-1} S_\alpha(m; x) \end{aligned} \tag{2.15}$$

where $n = 1, 2, \ldots$

Formulae (2.14) and (2.15) represent the fractional derivatives of $n\alpha$ — order for these functions.

Also it may pointed out that $C_\alpha(0; x) = x^{\alpha-1}/\Gamma(\alpha)$, $S_\alpha(0; x) = 0$, $S_\alpha(-m; x) = -S_\alpha(m; x)$ and $C_\alpha(-m; x) = C_\alpha(m; x)$.

From (2.13) we have

$$S_\alpha^2(m; x) + C_\alpha^2(m; x) = E_\alpha(im; x)\, E_\alpha(-im; x), \tag{2.16}$$

and from the definitions of S_α and C_α we find that

$$(2.17)\qquad \int_0^\infty S_\alpha(m\,;x)\ \exp(-x)\,dx = m\sum_{p=0}^{\infty}(-1)^p\, m^{2p} = m/1+m^2,$$

and

$$(2\ 18)\qquad \int_0^\infty C_\alpha(m\,;x)\exp(-x)\,dx = 1/1+m^2.$$

It is clear that (2.16), for $\alpha = 1$, becomes $\sin^2 mx + \cos^2 mx = 1$.

In addition to the above mentioned properties we have the two formulae:

$$(2.19)\ S_\alpha(m\,;x+z) = m^{-1}\,S_\alpha(m\,;x)\,\overset{z}{\underset{0}{I}}{}^{1-2\alpha}\,S_\alpha(m\,;z) + m\,C_\alpha(m\,;x)\,\overset{z}{\underset{0}{I}}\,C_\alpha(m;z),$$

$$(2.20)\ C_\alpha(m\,;x+z) = m^{-1}\,C_\alpha(m\,;x)\,\overset{z}{\underset{0}{I}}{}^{1-2\alpha}\,S_\alpha(m\,;z) - m\,S_\alpha(m\,;x)\,\overset{z}{\underset{0}{I}}\,C_\alpha(m\,;z).$$

To establish (2.19), we have by Theorem 1.9 and (1.91)

$$(2.191)\qquad S_\alpha(m\,;y) = S_\alpha(m\,;x) + \frac{[S_\alpha'(m\,;x)]^{\alpha-1}}{\Gamma(\alpha+1)}(y-x)^\alpha + \frac{[S_\alpha'(m\,;x)]^{(2\alpha-1)}}{\Gamma(2\alpha+1)}(y-x)^{2\alpha} + \dots$$

But we have

$$[S_\alpha'(m\,;x)]^{(\alpha-1)} = \sum_{p=1}^{\infty}\frac{(-1)^{p+1}\,m^{2p-1}}{\Gamma(\overline{2p-1}\,\alpha)}\,x^{\overline{2p-1}\alpha-1} = m\,C_\alpha(m\,;x)\,;$$

and by (2.14)

$$[S_\alpha'(m\,;x)]^{(2\alpha-1)} = m\,C_\alpha^{(\alpha)}(m\,;x) = -\,m^2\,S_\alpha(m\,;x),$$

$$[S_\alpha'(m\,;x)]^{(3\alpha-1)} = -\,m^3\,C_\alpha(m\,;x),$$

$$[S_\alpha'(m\,;x)]^{(4\alpha-1)} = m^4\,C_\alpha(m\,;x),\ \dots\ \text{etc.}$$

Therefore if we let $y = x+z$ in (2.191) we would have

$$S_\alpha(m\,;x+z) = S_\alpha(m\,;x)\left[1 - \frac{m^2}{\Gamma(2\alpha+1)}z^{2\alpha} + \frac{m^4}{\Gamma(4\alpha+1)}z^{4\alpha} - \dots\right] +$$

$$+ C_\alpha(m;x)\left[\frac{m}{\Gamma(\alpha+1)}z^{\alpha} - \frac{m^3}{\Gamma(3\alpha+1)}z^{3\alpha} + \frac{m^5}{\Gamma(5\alpha+1)}z^{5\alpha} - \dots\right] =$$

$$= m^{-1} S_\alpha(m;x) \overset{z}{\underset{0}{I}}{}^{1-2\alpha} S_\alpha(m;z) + m\, C_\alpha(m;x) \overset{z}{\underset{0}{I}} C_\alpha(m;z).$$

By a similar method (2.20) can be established (4).

Other functions may be defined in terms of S_α and C_α and their properties may be studied by considering the relations :

$$TN_\alpha(m;x) = S_\alpha(m;x)/C_\alpha(m;x)$$

$$CT_\alpha(m;x) = C_\alpha(m;x)/S_\alpha(m;x)$$

(2.21)

$$SC_\alpha(m;x) = 1/C_\alpha(m;x)$$

$$CS_\alpha(m;x) = 1/S_\alpha(m;x).$$

(iii) We define the functions SH_α and CH_α as follows :

(2.22) $$SH_\alpha(m;x) = \sum_{p=1}^{\infty} \frac{m^{2p-1}}{\Gamma(2p\alpha)} x^{2p\alpha-1}$$

(2.23) $$CH_\alpha(m;x) = \sum_{p=1}^{\infty} \frac{m^{2(p-1)}}{\Gamma(\overline{2p-1}\alpha)} x^{\overline{2p-1}\alpha-1}.$$

These functions are continuous for $\alpha > 0$ and $x \neq 0$, and when $\alpha \geqq 1$ they are continuous at $x = 1$. It may be noted that $SH_1(m;x) = \sinh mx$ and $CH_1(m;x) = \cosh mx$.

From (2.22) and (2.23) the following relations can be derived easily:

$$SH_\alpha(m;x) = 1/2\,[E_\alpha(m;x) - E_\alpha(-m;x)]$$

(2.24)

$$CH_\alpha(m;x) = 1/2\,[E_\alpha(m;x) + E_\alpha(-m;x)],$$

and

(2.25) $$CH^2(m;x) - SH^2(m;x) = E_\alpha(m;x)\,E_\alpha(-m;x).$$

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(4) Again we notice that (2.19) and (2.20), for $\alpha = 1$, assume the forms:

$$\sin m(x+z) = \sin mx \cos mz + \cos mx \sin mz$$

and

$$\cos m(x+z) = \cos mx \cos mz - \sin mx \sin mz$$

respectively.

The formulae for derivatives of non-integer order, i. e. $n\alpha$ — order may be given by

(2.26)
$$SH_{\alpha}^{(2n\alpha)}(m\,;x) = m^{2n}\, SH_{\alpha}(m\,;x)$$
$$SH_{\alpha}^{(\overline{2n-1}\alpha)}(m\,;x) = m^{2n-1}\, CH_{\alpha}(m\,;x)$$

(2.27)
$$CH_{\alpha}^{(2n\alpha)}(m\,;x) = m^{2n}\, CH_{\alpha}(m\,;x)$$
$$CH_{\alpha}^{(\overline{2n-1}\alpha)}(m\,;x) = m^{2n-1}\, SH_{\alpha}(m\,;x)$$

where $(n = 0, 1, 2, \ldots)$.

It is to be noted that $SH_{\alpha}(m\,;x) = -\,SH_{\alpha}(-\,m\,;x)$, $CH_{\alpha}(m\,;x) = CH_{\alpha}(-\,m\,;x)$, $SH_{\alpha}(0\,;x) = S_{\alpha}(0\,;x)$ and $CH_{\alpha}(0\,;x) = C_{\alpha}(0\,;x)$. Furthermore for $|\,m\,| < 1$

(2.28)
$$\int_0^{\infty} SH_{\alpha}(m\,;x)\exp(-\,x)\,dx = \sum_{p=1}^{\infty} m^{2p-1} = m/1 - m$$
$$\int_0^{\infty} CH_{\alpha}(m\,;x)\exp(-\,x)\,dx = \sum_{p=1}^{\infty} m^{2(p-1)} = 1/1 - m^2$$

By a similar method already used for establishing formulae (2.19) and (2.20) the following formulae can be derived easily:

(2.29) $$SH_{\alpha}(m;x+z) = m^{-1}\, SH_{\alpha}(m;x)\overset{z}{\underset{0}{I}}{}^{1-2\alpha}\, SH_{\alpha}(m\,;z) + m\,CH_{\alpha}(m;x)\overset{z}{\underset{0}{I}}CH_{\alpha}(m;z)$$

(2.30) $$CH_{\alpha}(m;x+z) = m^{-1}\, CH_{\alpha}(m;x)\overset{z}{\underset{0}{I}}{}^{1-2\alpha}\, SH_{\alpha}(m;z) + m\,SH_{\alpha}(m;x)\overset{z}{\underset{0}{I}}\,CH_{\alpha}(m;z).$$

For $\alpha = 1$ (2.29) and (2.30) take the forms

$$SH_1(m\,;x+z) = \sinh m\,(x+z) \text{ and } CH_1(m\,;x+z) = \cosh m\,(x+z)$$

respectively.

The fuctions TH_{α}. CTH_{α}, SCH_{α}, and CSH_{α} may be defined in terms of SH_{α} and CH_{α} by a similar way the functions (2.21) were defined, and their properties may be derived and studied accordingly.

REMARK. This writer must point out that only few properties of these functions, as this article reveals, have been studied so far and further investigations of these and related functions may yield more interesting results.

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