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On the pseudo-rigidity of Stein manifolds


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ON THE PSEUDO-RIGIDITY OF STEIN MANIFOLDS(*)

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Suppose we have a family of domains \( \{D_t\} \) in \( \mathbb{C}^n \) depending continuously on a parameter \( t \in \mathbb{C} \) for \( |t| < \epsilon \). Given a compact subset \( K \subset D_0 \), we can find an \( \epsilon > 0 \) such that \( K \subset D_t \) for every \( t \) with \( |t| < \epsilon \).

This fact can be formulated in a more general setting and leads to the notion of pseudo-trivial classes of local deformations of a complex space. The precise definition is given here in § 1.

The present paper is devoted to proving that any family of Stein manifolds whose parameter space is an open set in some numerical space \( \mathbb{C}^n \) gives a class of pseudo-trivial local deformations.

For Stein manifolds of dimension 1, i.e., for non-compact connected Riemann surfaces, this result was proved, using potential theory, by M. S. Narasimhan [3]. Our proof is a straightforward application of the theory of deformations developed by K. Kodaira and D. C. Spencer [2] modulo some minor changes to adapt it to the case of deformations of non-compact spaces.

The theorem given here is a particular case of an analogous theorem concerning 1-convex spaces (cf. [1]), but the proof of it is technically more involved. For this reason we believe it not useless to have a simple-minded proof for the particular case we have considered.

§ 1. FAMILIES OF COMPLEX SPACES.

1. Definitions. a) Let \( V_0 \) be a complex space (4). A deformation of \( V_0 \) is the set of the following data:

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(4) All complex spaces will be assumed to have a countable basis for open sets.
A punctured complex space \((M, m_0)\)

a complex space \(\mathcal{V}\)

two holomorphic maps

\[ \omega: \mathcal{V} \to M, \quad i: V_0 \to \mathcal{V} \]
satisfying the following conditions:

i) the map \(i\) is an isomorphism of \(V_0\) onto \(\omega^{-1}(m_0)\)

ii) for every \(x \in \mathcal{V}\) there exist

a neighbourhood \(W\) of \(x\) in \(\mathcal{V}\)
a neighbourhood \(U\) of \(\omega(x)\) in \(M\)
an analytic set \(S\) in an open set of some space \(\mathcal{C}^N\)
an isomorphism \(\varphi: U \times S \to W\)

such that \(\omega \circ \varphi = \text{natural projection of } U \times S \text{ onto } U\).

By condition ii) the map \(\omega\) is open. If \(\mathcal{V}\) and \(M\) are complex manifolds and \(\omega\) is of maximal rank at every point of \(\mathcal{V}\), then condition ii) is always satisfied.

We will usually identify \(V_0\) with \(i(V_0) = \omega^{-1}(m_0)\).

We will say that \((\mathcal{V}, \omega, M)\) defines a \textit{differentiably trivial} deformation of \(V_0\) if

iii) there exists a \(C^\infty\) homeomorphism \(f: M \times V_0 \to \mathcal{V}\) such that \(\omega \circ f = \text{natural projection of } M \times V_0 \text{ onto } M\).

b) Two deformations \((\mathcal{V}, \omega, M), (\mathcal{V}', \omega', M)\) of the same space \(V_0\) over the same base \((M, m_0)\) are said to be \textit{equivalent} if there exists an isomorphism \(\psi: \mathcal{V} \to \mathcal{V}'\) such that the following diagram is commutative:

Two deformations \((\mathcal{V}, \omega, M), (\mathcal{V}', \omega', M)\) of the same space \(V_0\) over the same base space \((M, m_0)\) are said to be \textit{locally equivalent} if there exists a neighbourhood \(U\) of \(m_0\) in \(M\) such that the deformations \((\omega^{-1}(U), \omega, U)\)
and \( (\omega'^{-1}(U), \omega', U) \) are equivalent. This enables us to consider classes of local deformations of \( V_0 \) over \((M, m_0)\).

A deformation \( (\mathcal{D}, \omega, M) \) of \( V_0 \) is said to be (locally) trivial if it is (locally) equivalent to the deformation \( (M \times V_0, \text{pr}_M, M) \).

c) Let \( (\mathcal{D}, \omega, M) \) be a deformation of \( V_0 \) over \((M, m_0)\). Let \( A \) be an open subset of \( V_0 \). Any open subset \( A \) of \( \mathcal{D} \) such that \( A \cap V_0 = A \) defines a deformation of the complex space \( A \) over \((M, m_0)\).

We will say that the deformation \( (\mathcal{D}, \omega, M) \) of \( V_0 \) over \((M, m_0)\) defines a locally pseudo-trivial deformation of \( V_0 \) if for every relatively compact open subset \( A \subset V_0 \) we can find an open subset \( \mathcal{A} \subset \mathcal{D} \) such that \( \mathcal{A} \cap V_0 = A \), which defines a trivial deformation of \( A \).

2. Families of complex manifolds. a) Given a deformation \( (\mathcal{D}, \omega, M) \) of a complex space \( V_0 \) and a sheaf of commutative groups \( \mathcal{F} \) on \( \mathcal{D} \), one can consider the \( q \)-th direct image sheaf \( R^q \omega \mathcal{F} \) on \( M \). This is the sheaf defined by the presheaf on \( M \) which associates to every open subset \( U \subset M \) the group \( H^q(\omega^{-1}(U), \mathcal{F}) \), the restriction homomorphism being defined in an obvious way.

If \( \mathcal{F} \) is an analytic sheaf on \( \mathcal{D} \), then the sheaves \( R^q \omega \mathcal{F} \) are analytic sheaves on \( M \).

If \( \mathcal{A} \subset \mathcal{D} \) is an open subset of \( \mathcal{D} \), we can consider the sheaf \( \mathcal{F} \mathcal{A} = \mathcal{F}|_{\mathcal{A}} \).

By transposition of the injection \( \mathcal{A} \subset \mathcal{D} \) one obtains a homomorphism

\[
\alpha : R^q \omega \mathcal{F} \rightarrow R^q \omega |_{\mathcal{A}} (\mathcal{F} |_{\mathcal{A}})
\]

which is a homomorphism of analytic sheaves if \( \mathcal{F} \) is an analytic sheaf on \( \mathcal{D} \).

b) Let us now assume that \( \mathcal{D} \) and \( M \) are complex manifolds and \( \omega \) a holomorphic map of maximal rank at each point of \( \mathcal{D} \).

Since we are interested only in the local deformations of \( V_0 \), we may assume that \( M \) is a polycylinder \( M_0 = 0 \) in \( \mathbb{C}^n \) with center \( m_0 = 0 \) and radius \( r_0 \):

\[
M_0 = \{ t = (t_1, \ldots, t_m) \in \mathbb{C}^m \mid |t^a| < r_0, \ a = 1, \ldots, m \}.
\]

By definition of a deformation (condition ii)) we may find a locally finite coordinate covering of \( \mathcal{D} \), \( \mathcal{U} = \{ U_i \}_{i \in I} \) with the following properties: the coordinates \( (z_i^1, \ldots, z_i^{m+n}) \) in the coordinate patch \( U_i \) are so chosen that

a) the restriction \( \omega|_{U_i} \) of \( \omega \) to \( U_i \) is given by

\[
\omega|_{U_i} : (z_i^1, \ldots, z_i^{m+n}) \rightarrow (t^1 = z_i^{n+1}, \ldots, t^m = z_i^{m+n})
\]
\( \beta \) for any \( x \in U_i, (x^1, \ldots, x^n) \) are local coordinates at \( x \) on the manifold \( \omega^{-1} (\omega (x)) \).

We will denote the coordinates on the coordinate patch \( U_i \) by \((x^1, \ldots, x^n, t^i, \ldots, t^m) = (z_i, t)\). If

\[
\begin{aligned}
\begin{cases}
  z^\alpha_i = h^\alpha_{ij} (x_j, t) ; t = t \\
  1 \leq \alpha \leq n
\end{cases}
\end{aligned}
\]

are the coordinate transformations in \( U_i \cap U_j \) and if

\[
\psi = \sum_1^m \phi^i (t) \frac{\partial}{\partial t^i}
\]

is a holomorphic vector field on \( M_{r_0} \), then

\[
\Theta_{ij} (x_i, t) = \psi h^{\alpha}_{ij} (x_j, t) = \sum_1^m \phi^i (t) \frac{\partial h^{\alpha}_{ij} (x_j, t)}{\partial t^i}
\]

are the components of a holomorphic vector field along the fibres in \( U_i \cap U_j \).

Let \( \Theta \) be the sheaf of germs of holomorphic vector fields on \( \mathcal{V} \) along the fibres. One verifies that \( \{ \Theta_{ij} \} \) is a cocycle on the covering \( \mathcal{U} \) with values in \( \Theta \), i.e.,

\[
\psi (\Theta) = \{ \Theta_{ij} \} \in Z^1 (\mathcal{U}, \Theta).
\]

A new choice of coordinates on the covering \( \mathcal{U} \) changes the above cocycle by a coboundary. Hence if \( T \) is the sheaf of germs of holomorphic tangent vectors to \( M \), we obtain a map:

\[
\tilde{\varphi} : H^0 (M_{r_0}, T) \rightarrow H^1 (\mathcal{V}, \Theta)
\]

which is linear over \( H^0 (M_{r_0}, \mathcal{O}) \), \( \mathcal{O} \) being the sheaf of germs of holomorphic functions on \( M_{r_0} \).

If \( 0 < r \leq r_0 \) and \( M_r = \{ t \in M_{r_0} \mid |t^\alpha| < r \} \), \( \mathcal{V}_r = \omega^{-1} (M_r) \), the same argument can be repeated with \( M_r \) and \( \mathcal{V}_r \) in the place of \( M_{r_0} \) and \( \mathcal{V} \) respectively. For \( 0 < r' < r \leq r_0 \) we have an obvious commutative diagram:

\[
\begin{array}{ccc}
H^0 (M_r, T) & \rightarrow & H^1 (\mathcal{V}_r, \Theta) \\
\downarrow \sim \downarrow \sim & \sim \downarrow \sim \\
H^0 (M_{r'}, T) & \rightarrow & H^1 (\mathcal{V}_{r'}, \Theta)
\end{array}
\]
By passing to the limit with \( r \rightarrow 0 \) we obtain a map:

\[
\tilde{\varphi} : T_{[0]} \rightarrow \mathcal{O}_{[0]} \omega (\Theta)_{[0]}
\]

which is linear over \( \mathcal{O}_{[0]} \). This is the homomorphism of Kodaira and Spencer [2].

c) We want now to prove the following

**Proposition 1.** Let \((\mathcal{C}, \omega, M)\) be a deformation of the complex manifold \(V_0\). If \( \tilde{\varphi} = 0 \), then \((\mathcal{C}, \omega, M)\) defines a locally pseudo-trivial deformation of \(V_0\).

**Proof.**

\( a) \) Every element \( \varphi \in T_{[0]} \) is of type

\[
\varphi = \sum_1^m \varphi^\mu \frac{\partial}{\partial t^\mu}
\]

with \( \varphi^\mu \in \mathcal{O}_{[0]} \). By the assumption \( \tilde{\varphi} = 0 \) there exists \( r, 0 < r < r_0 \) and on each \( U_i \cap \mathcal{V}_r \) holomorphic vector fields along the fibres:

\[
\theta_{\mu i} (z_i, t) = (\theta_{\mu i}^1 (z_i, t), \ldots, \theta_{\mu i}^m (z_i, t)) \quad 1 \leq \mu \leq m
\]

such that, for \( \theta_{\mu i} (z_i, t) = \frac{\partial h_{ij} (z_i, t)}{\partial \bar{t}^\mu} \), one has

\[
\theta_{\mu i} (p) = \theta_{\mu i} (p) - \theta_{\mu i} (p) \quad 1 \leq \mu \leq m
\]

for any \( p \in U_i \cap U_j \cap \mathcal{V}_r \). This is expressed by the formulas:

\[
\frac{\partial h_{ij} (z_j, t)}{\partial \bar{t}^\mu} = \sum_\beta \theta_{\mu i}^\beta (z_j, t) \frac{\partial h_{ij} (z_j, t)}{\partial z_j^\beta} - \theta_{\mu i}^\alpha (h_{ij} (z_j, t), t).
\]

\( \beta) \) Let \((\xi, t)\) be a new system of coordinates on \( U_i \cap \mathcal{V}_r \) and let

\[
\left\{
\begin{array}{l}
\xi^\alpha_i = h_{ij}^\alpha (\xi_j, t); \quad t = t \\
1 \leq \alpha \leq n
\end{array}
\right.
\]

(\(1\) Note that if a 1-cocycle on a covering \( \mathcal{U} \) of a space \( X \) with values in a sheaf of commutative groups induces a coboundary on a refinement of the covering \( \mathcal{U} \), then it is also a coboundary on \( \mathcal{U} \) (\( \mathcal{U} \) locally finite).
be the corresponding coordinate transformations. Let

\[ z^a_i = g^a_i (\xi_i, t) \]

be the expression of the old coordinates in terms of the new in \( U_i \cap \mathcal{V}_r \).

If \( \mathcal{V} \) defines a locally trivial deformation of \( V_0 \), then the new coordinates \( \xi_i \) can be so chosen that

i) for \( t = 0 \) then

\[ g^a_i (\xi_i, 0) = \xi^a_i \]

ii) \( \frac{\partial k^a_{ij}}{\partial t^\mu} = 0 \) for \( 1 \leq \alpha \leq n \) and \( 1 \leq \mu \leq m \)

provided \( r \) is sufficiently small.

\( \gamma \) From the identity in \( U_i \cap U_j \cap \mathcal{V}_r \)

\[ g^a_i (k^a_{ij}(\xi_j, t), t) = h^a_{ij}(g^a_j(\xi_j, t), t) \]

we obtain by differentiation with respect to \( t^\mu \):

\[ 0 = \frac{\partial}{\partial t^\mu} [g^a_i (k^a_{ij}(\xi_j, t), t) - h^a_{ij}(g^a_j(\xi_j, t), t)] = \]

\[ = \sum_{\beta} \frac{\partial g^a_i}{\partial \xi^\beta} \frac{\partial k^a_{ij}}{\partial t^\mu} + \frac{\partial g^a_i}{\partial t^\mu} \sum_{\beta} \frac{\partial k^a_{ij}}{\partial \xi^\beta} - \frac{\partial h^a_{ij}}{\partial t^\mu} \cdot \]

Hence if condition ii) of \( \beta \) is satisfied, we obtain a relation of type (1) with \( \theta^a_{\mu i} \) replaced by \( -\frac{\partial g^a_i}{\partial t^\mu} \).

This shows that \( \theta^a_{\mu i} \) will be a global holomorphic vector field \( \theta^a_{\mu i} \) along the fibres of \( \mathcal{V}_r \), for every \( \mu \).

\( \delta \) We introduce the following notations:

\[ M_r (s) = \{(t^1, \ldots, t^n) \in \mathbb{C}^n \mid t^a < r, 1 \leq \alpha \leq n \} \]

\[ I_r (\delta) = \{t^h \in \mathbb{C} \mid t^h < \delta \}. \]

Let \( \mathcal{V}_r (s) = \omega^{-1} (M_r (s)) \).

Let \( \mathcal{U}_0 = \{ U_i \}_{i \in I_0} \) be the set of those \( U_i \) such that \( U_i \cap V_0 = \emptyset \).

Let \( \mathcal{U}_0 = \{ U_i \}_{i \in I_0} \), \( \mathcal{U}_0 = \{ U^*_i \}_{i \in I_0} \) be two other coverings of \( V_0 \) in \( \mathcal{V} \) with open sets such that:

\[ U^*_i \subset U_i \subset U_i \text{ for every } i \in I_0. \]
For every \( i \in I_0 \) we can find an \( \varepsilon_i > 0 \) and a solution of the system of ordinary differential equations

\[
\begin{cases}
\frac{\partial g_i^\alpha(\xi_i, t)}{\partial t^m} + \Theta_{i\alpha}(g_i(\xi_i, t), t) = 0 \\
1 \leq \alpha \leq n
\end{cases}
\]

defined for \( t \in M_{n_1}(m-1) \times I_{i}(m) \), where \( r_i = \frac{1}{2} r_0 \), with initial values

\[
\begin{cases}
g_i^\alpha(\xi_i, t^1, \ldots, t^{m-1}, 0) = \xi_i^\alpha \\
1 \leq \alpha \leq n
\end{cases}
\]

where \( \xi_i^\alpha \in U_i^\ast \cap V_0 \) and contained in \( U_i \).

We may also assume that the \( n \) functions \( g_i^\alpha \) thus obtained define holomorphic coordinates in \( U_i \cap \omega^{-1}(M_{n_1}(m-1) \times I_{i}(m)) = U_i^\ast \).

By virtue of \( \gamma \) these new coordinate patches will satisfy the condition

\[
\sum \frac{\partial g_i^\alpha}{\partial \xi_i^\beta} \frac{\partial h_j^\beta}{\partial t^m} = 0 \quad \text{in} \quad U_i^\ast \cap U_j^\ast.
\]

Therefore the coordinate transformations \( h_{ij} \) will be independent of \( t^m \).

It follows that in the open set \( \bigcup_{i \in I_0} U_i^\ast \) there is a neihbourhood \( \mathcal{A} \) of \( V_0 \) in \( \mathcal{V} \) which can be isomorphically imbedded in the product \( \mathcal{V}_{n_1}(m-1) \times \mathbb{C} \), the isomorphism being the identity on \( \mathcal{V}_{n_1}(m-1) \).

Finally, now replace the family \( \mathcal{V} \) with \( \mathcal{A} \). Then the deformation-cocycle \( \varrho \left( \frac{\partial}{\partial t^{m-1}} \right) \) with respect to the new coordinates considered on \( \mathcal{A} \) will again be a coboundary. The same will be true for the restriction of this cocycle to \( \mathcal{V}_{n_1}(m-1) \). By the above argument we can find a neighbourhood of \( V_0 \) in \( \mathcal{V}_{n_1}(m-1) \) which can be isomorphically imbedded in the product \( \mathcal{V}_{n_1}(m-2) \times \mathbb{C} \), where \( r_2 = \frac{1}{2} r_1 \), the isomorphism being the identity on \( \mathcal{V}_{n_1}(m-2) \).

Continuing in this way we see that a neighbourhood of \( V_0 \) in \( \mathcal{V} \) can be isomorphically imbedded in the product \( V_0 \times \mathbb{C}^{m} \), the isomorphism being the identity on \( V_0 \). This proves our statement.

**Remark 1.** Actually we have proved a little more, i.e., that in the hypothesis specified above, if \( \tilde{\varrho} = 0 \), there exists a neighbourhood of \( V_0 \) in \( \mathcal{V} \) which can be isomorphically imbedded into the product \( V_0 \times \mathbb{C}^{m} \), the isomorphism being the identity on \( V_0 \).
Remark 2. An analogous argument applies to differentiable families of complex or differentiable manifolds. In this last case the sheaf \( \Theta \) is a fine sheaf. Hence given a complex deformation \((\mathcal{V}, \omega, M)\) of the complex manifold \(V_0\), a neighborhood of \(V_0\) in \(\mathcal{V}\) can always be differentially imbedded in the product \(V_0 \times \mathbb{C}^m (m = \text{dim}_\mathbb{C} M)\) (with a fiber-preserving imbedding which is the identity on \(V_0\)).

§ 2. Deformation of Stein manifolds.

3. a) Let us now assume that \((\mathcal{V}, \omega, M)\) is a local deformation of a holomorphically complete manifold \(V_0\) over the polycylinder

\[
M = M_{r_0} = \{ t = (t^1, \ldots, t^m) \in \mathbb{C}^m \mid |t^\alpha| < r_0, \ 1 \leq \alpha \leq m \}.
\]

We can now prove the following

**Proposition 2.** Let \(A\) be a relatively compact open subset of \(V_0\). There exists a neighborhood \(\mathcal{U}\) of \(A\) in \(\mathcal{V}\) with \(\mathcal{U} \cap V_0 = A\) such that for any coherent sheaf \(\mathcal{F}\) on \(\mathcal{V}\) the natural homomorphism

\[
r : \mathcal{R}^q \omega (\mathcal{F})_0 \to \mathcal{R}^q \omega |_{A^\prime} (\mathcal{F} |_{\mathcal{A}})_0
\]

is the 0-homomorphism, when \(q \geq 1\).

**Proof.** a) Since we are interested only in relatively compact open subsets of \(V_0\), by the remark 2 at the end of proposition 1 we see that it is not restrictive to assume that \(\mathcal{V}\) is differentially trivial. Let \(f : M \times V_0 \to \mathcal{V}\) be the fiber-preserving differentiable homeomorphism which gives the differentiable triviality of \(\mathcal{V}\).

Since \(V_0\) is a Stein manifold, there exists on \(V_0\) a \(C^\infty\) function \(g : V_0 \to \mathbb{R}\) such that

i) the sets \(B_c = \{ x \in V_0 \mid g(x) < c \}\) are relatively compact in \(V_0\) for every \(c \in \mathbb{R}\)

ii) the function \(g\) is strongly plurisubharmonic on \(V_0\), i.e., at each point \(x \in V_0\) the Levi form expressed in local coordinates \(z^\alpha\) by

\[
\mathcal{L}(g) = \sum \frac{\partial^2 g}{\partial z^\alpha \partial \overline{z}^\beta} \overline{u^\alpha u^\beta}
\]

is a positive definite hermitian form (cf. [4]).

Consider on \(\mathcal{V}\) the following function:

\[
\tilde{g}(\xi) = g \circ pr_{V_0} \circ f^{-1}(\xi).
\]
This is a $C^\infty$ function and if, as is permitted, we assume that $f|_{V_0}$ is the identity map, the function $\tilde{g}|_{V_0}$ coincides with the function $g$.

Given a compact set $K \subset V_0$ we can find a constant $a_0(K) > 0$ such that for any $a > a_0(K)$ the function

$$h_a = \tilde{g} + a \omega^a \left( \sum_{\alpha} \tau_\alpha \bar{t}_\alpha \right)$$

has a positive definite Levi form at each point of $K$.

Therefore there is a neighbourhood $U(K)$ of $K$ in $\mathcal{V}$ such that on any point of $U(K)$ the Levi form of $h_a$, for any $a > a_0(K)$, is positive definite.

Let $\sup g(x) = C$ and set $K = \overline{B}_{C+1}$, so that $A \subset K$, and take for $A$ the set $f(M \times A)$.

We can find $\varepsilon(K) > 0$ ($\varepsilon(K) < r_0$) such that

$$f(M_{\varepsilon(K)} \times K) \subset U(K).$$

We claim that the sets

$$\mathcal{B}_\nu = \{ x \in \mathcal{V} \mid h_\nu(x) < C + 1 \} \quad \nu = 1, 2, ... ,$$

form a decreasing system of neighbourhoods of $B_{C+1}$ in $\mathcal{V}$.

In fact, for any $\nu$, $\mathcal{B}_\nu \cap V_0 = B_{C+1}$. Moreover if $c = \inf g(x)$, one has

$$\mathcal{B}_{\nu} \subset f(M_{\frac{C+1}{\varepsilon}} \times B_{C+1}).$$

If $\frac{C}{\varepsilon} < \varepsilon(K)$, the sets $\mathcal{B}_\nu$ are relatively compact in $\mathcal{V}$, the function $h_\nu$ is strongly plurisubharmonic on $\mathcal{B}_\nu$ and the sets $\{ h_\nu(x) < \delta \}$ are relatively compact in $\mathcal{B}_\nu$ if $\delta < C + 1$. It follows that for these values of $\nu$ the sets $\mathcal{B}_\nu$ are 1-complete manifolds, i.e., holomorphically complete.

\(b\) Now let $\theta \in C^0 \omega (\mathcal{F})_0$; the class $\theta$ is defined by an element

$$\theta \in H^q (\omega^{-1}(M_\sigma), \mathcal{F})$$

where $\sigma > 0$ is sufficiently small.

Let $\nu$ be a positive integer, greater than $\frac{|C| + |\sigma|}{\varepsilon(K)}$, such that

$$\mathcal{B}_\nu \subset \omega^{-1}(M_\sigma).$$
We can find a positive number \( \varepsilon < \sigma \) such that
\[
\omega^{-1}(M_\varepsilon) \cap \mathcal{A} \subset \mathcal{B}_\sigma.
\]
The element
\[
r(\theta) \in \mathcal{R}^q|_{\mathcal{A}}(\mathcal{F}|_{\mathcal{A}})_0
\]
is defined by the image of \( \theta \) under the natural homomorphism
\[
H^q(\omega^{-1}(M_\varepsilon), \mathcal{F}) \to H^q(\omega^{-1}(M_\varepsilon), \mathcal{F}|_{\mathcal{A}}).
\]
On the other hand the triangle of restriction homomorphisms
\[
H^q(\omega^{-1}(M_\varepsilon), \mathcal{F}) \to H^q(\omega^{-1}(M_\varepsilon), \mathcal{F}|_{\mathcal{A}}) \to H^q(\mathcal{B}_\sigma, \mathcal{F}|_{\mathcal{B}_\sigma})
\]
is commutative.
Since \( \mathcal{B}_\sigma \) is holomorphically complete, \( H^q(\mathcal{B}_\sigma, \mathcal{F}|_{\mathcal{B}_\sigma}) = 0 \), for \( q \geq 1 \).
This shows that \( r(\theta) = 0 \).

b) We can now prove the following

**Theorem.** Every local deformation \((\mathcal{V}, \omega, M)\) of a holomorphically complete manifold \( V_0 \) over an open neighbourhood \( M \) of the origin in \( \mathbb{C}^n \) is a pseudo-trivial deformation.

**Proof.** By virtue of proposition 1 it is enough to show that for any relatively compact open subset \( A \subset V_0 \) we can find a neighbourhood \( \mathcal{A} \) of \( A \) in \( \mathcal{V} \), with \( \mathcal{A} \cap V_0 = A \), such that the homomorphism \( \tilde{\rho}_{\mathcal{A}} \) of Kodaira and Spencer for the family \((\mathcal{A}, \omega|_{\mathcal{A}}, \omega(\mathcal{A}))\) is the zero homomorphism.

If \( r \) is the restriction homomorphism
\[
\mathcal{R}^l \omega(\theta)_0 \to \mathcal{R}^l \omega|_{\mathcal{A}}(\theta|_{\mathcal{A}})_0,
\]
then we have the factorisation \( \tilde{\rho}_{\mathcal{A}} = r \circ \tilde{\rho} \).

Choosing \( \mathcal{A} \) as in proposition 2 we see that \( r = 0 \); hence \( \tilde{\rho}_{\mathcal{A}} = 0 \) as we wanted.

c) Application. Given a compact complex manifold \( V \) let us denote by \( d(V) \) the minimal number of Stein manifolds by which \( V \) can be co-
vered. If \((\mathcal{Y}, \omega, M)\) is a family of deformations of compact complex mani-
folds, \(\mathcal{Y} = \{V_t\}_{t \in M}\), then \(d(V_t)\) is an upper semicontinuous function of \(t\)
for \(t \in M\).

This fact can also be proved directly, using part of the argument
given in \(a\).

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