

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

M. K. VENKATESHA MURTHY

**A remark on the regularity at the boundary for solutions
of elliptic equations**

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^e série, tome 15,
n° 4 (1961), p. 355-370

http://www.numdam.org/item?id=ASNSP_1961_3_15_4_355_0

© Scuola Normale Superiore, Pisa, 1961, tous droits réservés.

L'accès aux archives de la revue « *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* » (<http://www.sns.it/it/edizioni/riviste/annaliscienze/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques*
<http://www.numdam.org/>

A REMARK ON THE REGULARITY AT THE BOUNDARY FOR SOLUTIONS OF ELLIPTIC EQUATIONS

by M. K. VENKATESHA MURTHY (Bombay).

§ 1. Introduction.

The object of this note is to prove the following result.

Let A be a linear elliptic operator (of order $2m$) with infinitely differentiable coefficients in a domain Ω , having a smooth boundary $\partial\Omega$, in a euclidean space and let B_j ($0 \leq j \leq 2m - 1$) be differential operators, with infinitely differentiable coefficients on $\partial\Omega$. If $(A, \{B_j\})$ is an admissible system (see § 2), and f and g_j are functions in certain classes of infinitely differentiable functions (referred to as Friedman classes in the sequel), then any function u infinitely differentiable in $\bar{\Omega}$ and satisfying $Au = f$ in Ω , $B_j u = g_j$ in $\partial\Omega$, is itself in a Friedman class in $\bar{\Omega}$ when the coefficients of A and of B_j and the functions which define the boundary $\partial\Omega$ locally, are in certain Friedman classes.

When f , g_j , and the coefficients of A and of B_j are real analytic functions of their arguments the real analyticity of the solution u (of the system) upto the boundary has been proved, using a method of Morrey and Nirenberg [4], by Magenes and Stampacchia [3] assuming that $(A, \{B_j\})$ is an admissible system and $\partial\Omega$ is analytic. Our result includes that of Magenes and Stampacchia. In the case of the Dirichlet problem this result was proved in the case of real analytic functions by Morrey and Nirenberg [4] and in the case of functions in Friedman classes by Friedman in [2].

The notation, necessary norms, and other preliminaries are introduced in § 2. In § 3 two lemmas, which lead to L_2 -estimates for u and its tangential derivatives of all orders and normal derivatives of order upto $2m$, are proved. In § 4 L_2 -estimates for derivatives of all orders of u are obtained and finally an application of Sobolev's lemma yields the result.

The author wishes to thank Mr. B. V. Singbal for his valuable suggestions and help.

§ 2. Notation and Preliminaries.

Let Ω denote a bounded domain in a ν -dimensional Euclidean space and let (x_1, \dots, x_ν) be a coordinate system in Ω . First we define certain classes of infinitely differentiable (C^∞) functions on Ω . Let $\{M_n\}$ be a sequence of positive numbers satisfying the following condition: there exists a positive constant C , independent of n , such that

$$(1) \quad \binom{n}{\lambda} M_\lambda M_{n-\lambda} \leq CM_n \quad (\text{for } \lambda = 1, 2, \dots, n; n = 1, 2, \dots)$$

Then if p is any non-negative integer, we denote by $C\{M_{n-p}; \Omega\}$ the class of C^∞ -functions f on Ω satisfying the following condition:

(QA) For every closed subdomain Ω_0 of Ω there exist two constants H_1 and H_2 , depending on f and on Ω_0 , such that, for any $x \in \Omega_0$, we have

$$\left| \frac{\partial^{|k|} f(x)}{\partial x_1^{k_1} \dots \partial x_\nu^{k_\nu}} \right| \leq H_1 H_2^{|k|} M_{|k|-p}$$

where $k = (k_1, \dots, k_\nu)$ and $|k| = k_1 + \dots + k_\nu$.

We call a class of the type $C\{M_{n-p}; \Omega\}$ a *Friedman class*.

Similarly we define the classes $C\{M_{n-p}; \bar{\Omega}\}$ when the condition (QA) is satisfied in $\bar{\Omega}$.

It is clear that (1) implies

$$(1') \quad (n+1) M_n \leq C_1 M_{n+1}$$

with a positive constant C_1 independent of n (in fact we can take $C_1 = \frac{C}{M_1}$).

Now we make some remarks on the Friedman classes $C\{M_n; \Omega\}$ which will be of use in the sequel.

(i) If f is a function in the class $C\{M_n; \Omega\}$ such that $f(x) \neq 0$ for $x \in \Omega$ then $\frac{1}{f}$ is itself in the class $C\{M_n; \Omega\}$. For, let (x_1, \dots, x_ν) be a coordinate system in Ω and let d_{i_μ} denote a generic partial differentiation operator of order one. Therefore, a generic partial differentiation operator of order k can be written in the form $d_{i_1} d_{i_2} \dots d_{i_k}$. Then

$$(d_n d_{n-1} \dots d_1) \left(\frac{1}{f} \right) = \frac{1}{f^{n+1}} H_n(f)$$

where $H_n(f)$ is a homogeneous polynomial of degree n in the set of arguments $(f, \dots, f^k, \dots, d_i f, \dots, d_{i_1} \dots d_{i_r} f, \dots, d_{i_1} \dots d_{i_n} f)$ and where the degree of $d_{i_1} \dots d_{i_r} f$ is taken to be r . Any monomial of degree k in these arguments is majorized on any closed subset Ω_0 of Ω by $(H_0 H_1)^k C^{k-1} M_k \leq H_0 (H_0 H_1 C)^k M_k$. Hence one can easily see that

$$|H_n(f)| \leq H_0 (c H_0 H_1 C)^n M_n$$

with a suitable positive constant c independent of n .

Let $\delta = \min_{x \in \Omega_0} |f(x)|$. Taking $K_0 = \frac{H_0}{\delta}$ and $K_1 = \frac{c H_0 H_1 C}{\delta}$ we see that

$$\left| d_n d_{n-1} \dots d_1 \left(\frac{1}{f} \right) \right| \leq K_0 K_1^n M_n$$

on Ω_0 which establishes (i).

(ii) If f, g are in $C\{M_n; \Omega\}$ then their product fg is itself in $C\{M_n; \Omega\}$. In fact, we have for an $\alpha = (\alpha_1, \dots, \alpha_r)$

$$D^\alpha (fg) = \sum_{\beta} \binom{\alpha}{\beta} D^\beta (f) \cdot D^{\alpha-\beta} (g)$$

by Leibniz formula, where $\beta = (\beta_1, \dots, \beta_r)$ and $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \dots \binom{\alpha_r}{\beta_r}$.

Then $|D^\alpha (fg)| \leq \sum \binom{\alpha}{\beta} H_1 H_2^\beta M_\beta H_1 H_2^{\alpha-\beta} M_{\alpha-\beta} \leq C' H_1^2 H_2^\alpha M_\alpha$, with a suitable constant $C' > 0$.

The remarks (i) and (ii) together imply the following:

(iii) If f, g belong to $C\{M_n; \Omega\}$ with g non-vanishing in Ω then f/g is itself in the class $C\{M_n; \Omega\}$.

Let s denote a fixed positive real number. Let H^s denote the space of all tempered distributions φ such that its Fourier transform $\widehat{\varphi}$ satisfies the condition that $(1 + |\xi|^2)^{s/2} \widehat{\varphi}$ is square integrable and we define the scalar product in H^s by

$$(\varphi, \psi)_s = \int \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)} (1 + |\xi|^2)^s d\xi \text{ for any two } \varphi, \psi \in H^s$$

and the corresponding norm

$$\|\varphi\|_s = \left[\int |\widehat{\varphi}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right]^{\frac{1}{2}} \text{ for any } \varphi \in H^s.$$

(see [3]).

In view of the local nature of the problem it is enough to consider the solution of the problem in a hemi-sphere with the boundary conditions defined on the plane part of the boundary.

Throughout, the function u is assumed to be infinitely differentiable in the hemi-sphere together with the plane part of the boundary. This is so, for example, in the following cases :

Let the coefficients of A and B_j , and f, g_j be infinitely differentiable functions of their arguments. Then any solution u of $Au = f, B_j u = g_j$ is infinitely differentiable either when $(A, \{B_j\})$ is an elliptic system in the sense defined by J. Peetre [5] or when the boundary operators B_j satisfy the complementing condition of Agmon, Douglis and Nirenberg [1] with respect to the elliptic operator A .

Next we introduce the differential operators. Let ω_r denote the hemi-sphere $\{x_1^2 + \dots + x_\nu^2 < r^2, x_\nu > 0\}$ and $\partial_1 \omega_r$ the plane part $\{x_\nu = 0\}$ of the boundary of ω_r . Let π_0 denote the $(\nu - 1)$ -dimensional subspace $\{(x_1, \dots, x_{\nu-1}, 0)\}$ and let x' denote either $(x_1, \dots, x_{\nu-1}, 0)$ or $(x_1, \dots, x_{\nu-1})$ inadvertently in the context. We adopt the following notation throughout :

$$\text{If } p = (p_1, \dots, p_\nu) \text{ then } a_p(x) \text{ denotes a function } a_{p_1 \dots p_\nu}(x) \text{ and } D^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \dots \partial x_\nu^{p_\nu}}.$$

Similar notation is used in π_0 also with x' in place of x and D_x^p in place of D^p .

All our functions are defined in ω_{R_0} together with the plane part $\partial_1 \omega_{R_0}$ of the boundary of ω_{R_0} where R_0 is a fixed positive number. Let

$$A = \sum_{|p| \leq 2m} a_p(x) D^p$$

be an elliptic linear partial differential operator of order $2m$ on ω_{R_0} with the coefficients $a_p(x) \in C^\infty$ in $\overline{\omega_{R_0}}$ and let

$$B_j = \sum_{|p| \leq 2m-1-j} b_p^j(x') D^p \quad (0 \leq j \leq 2m - 1)$$

be differential operators (boundary operators) where $b_p^j(x')$ are C^∞ functions on $\partial_1 \omega_{R_0}$.

Let $\varrho(t)$ be a real valued C^∞ function of the variable t ($-\infty < t < \infty$) such that

$$\begin{aligned} \varrho(t) &= 1 \text{ for } t \leq 0, \\ &= 0 \text{ for } t \geq 1, \end{aligned}$$

then for any pair of positive numbers r and h with $0 < r < r + h < R_0$ define

$$\varphi_{r,h}(x) = \varrho\left(\frac{|x| - r}{h}\right).$$

Then clearly we have

$$\begin{aligned} \varphi_{r,h}(x) &= 1 \text{ for } |x| \leq r \\ &= 0 \text{ for } |x| \geq r + h \end{aligned}$$

and further for any $p = (p_1, \dots, p_r)$ we have

$$|D^p \varphi_{r,h}(x)| \leq C_2 h^{-|p|} \text{ (when } h < r),$$

where C_2 is a positive constant depending on ν, p and the bounds for the derivatives of ϱ .

DEFINITION. The system $(A, \{B_j\})$ is said to be an admissible system if A and the boundary operators $\{B_j\}$ satisfy the following condition: there exists a constant C_3 such that for any C^∞ function u and for any r with $0 < r < r + h < R_0$, we have

$$(2) \quad \sum_{q=2m} \|D^q \varphi_{r,h} u\|_{0, \omega_{R_0}}^2 \leq C_3 \left\{ \|A(\varphi_{r,h} u)\|_{0, \omega_{R_0}}^2 + \sum_{j=0}^{2m-1} \|B_j(\varphi_{r,h} u)\|_{j+\frac{1}{2}, \pi_0}^2 \right\}$$

where if $B_j(\varphi_{r,h} u)$ is considered as having its support contained in $\partial_1 \omega_{r+h}$ then $B_j(\varphi_{r,h} u)$ is extended to the whole of π_0 by taking it to be equal to zero in $\pi_0 - \partial_1 \omega_{r+h}$. Here $\|f\|_{0, \Omega}$ is defined by $\|f\|_{0, \Omega}^2 = \int_{\Omega} |f(x)|^2 dx$.

REMARK. (a) When $(A, \{B_j\})$ is an admissible system the analyticity of a solution u of $Au = f, B_j u = g_j$ upto the boundary (the coefficients of A and of B_j , and f, g_j being real analytic functions of their arguments) was proved by Magenes and Stampacchia [3].

(b) The inequality (2) has been obtained by J. Peetre when $(A, \{B_j\})$ is an elliptic system in the sense defined in [5]. When A is elliptic and B_j satisfy the complementing condition with respect to A an analogous inequality has been proved by Agmon, Douglis and Nirenberg [1].

The following is the precise statement of our theorem.

THEOREM. Let $(A, \{B_j\})$ be an admissible system, with A elliptic, such that the following conditions are satisfied:

- (i) the coefficients $a_p(x)$ of A are in $C\{M_n; \omega_{R_0}\}$;
- and (ii) the coefficients $b_p^j(x')$ of B_j are in $C\{M_n; \partial_1 \omega_{R_0}\}$.

Then any function u , C^∞ in $\overline{\omega_{R_0}}$ and satisfying the system

$$Au = f \text{ in } \omega_{R_0},$$

$$B_j u = g_j \text{ in } \partial_1 \omega_{R_0} \quad (0 \leq j \leq 2m - 1),$$

where f is in $C\{M_n; \omega_{R_0}\}$ and g_j are in $C\{M_{n-j-1}; \partial_1 \omega_{R_0}\}$ respectively, is a function in $C\{M_{n-2m+[r/2]+1}; \omega_{R_0} \cup \partial_1 \omega_{R_0}\}$.

In the course of the proof of the theorem we need the following norms (introduced in [3]):

$$e_{k,r}(f) = \left(\sum_{|q|=k} \|\varphi_{r,h} D_x^q f\|_{0,\omega_{r+h}}^2 \right)^{\frac{1}{2}} \text{ with } h = \frac{R-r}{k+1}, \quad k = 0, 1, 2, \dots$$

$$e_{j,k,r}(g) = \left(\sum_{|q|=k} \|\varphi_{r,h} D_x^q g\|_{j+\frac{1}{2},\pi_0}^2 \right)^{\frac{1}{2}} \text{ with } h = \frac{R-r}{k+1}, \quad k = 0, 1, 2, \dots$$

(0 ≤ j ≤ 2m - 1)

and
$$d_{k,r}(u) = \left(\sum_{|q|=2m} \sum_{|p|=k} \|D_x^q D_x^p u\|_{0,\omega_r}^2 \right)^{\frac{1}{2}} \text{ for } k = 0, 1, 2, \dots$$

$$= \left(\sum_{|q|=2m+k} \|D_x^q u\|_{0,\omega_r}^2 \right)^{\frac{1}{2}} \text{ for } k = -2m, \dots, 0.$$

We make the convention that

$$[M_k] = M_k \quad \text{if } k \geq 0 \text{ and}$$

$$= 1 \quad \text{if } k < 0$$

and introduce the following notation (in analogy with that introduced in [4])

$$M_{R,k}(f) = \frac{1}{M_k} \sup_{R/2 \leq r < R} (R - (r + h))^{2m+k} e_{k,r}(f) \quad \text{for } k = 0, 1, 2, \dots$$

$$M_{j,R,k}(g) := \frac{1}{M_k} \sup_{R/2 \leq r < R} (R - (r + h))^{2m+k} e_{j,k,r}(g)$$

for $k = 0, 1, 2, \dots; 0 \leq j \leq 2m - 1$

and

$$N_{R,k}(u) = \frac{1}{[M_k]} \sup_{R/2 \leq r < R} (R - r)^{2m+k} d_{k,r}(u)$$

for $k = -2m, -2m + 1, \dots, 0, 1, 2, \dots$

§ 3. In this paragraph we present two lemmas leading to the proof of the main theorem stated in the previous paragraph. In principle we obtain an L_2 -estimate for the derivatives, upto order $2m$ in the transverse direction and of all orders in the tangential direction, for a function satisfying the system. To begin with we have the following result due to Magenes and Stampacchia (see [3] p. 331).

If u is any C^∞ function and if $(A, \{B_j\})$ is an admissible system then there exists a constant C_4 , independent of u , r and h such that for $0 < r < r + h < R_0$, $r > h$ we have

$$(3) \quad \sum_{|q|=2m} \|D^q u\|_{0,\omega_r}^2 \leq C_4 \left\{ \|\varphi_{r,h} Au\|_{0,\omega_{r+h}}^2 + \sum_{j=0}^{2m-1} \|\varphi_{r,h} B_j u\|_{j+\frac{1}{2},\pi_0}^2 + \sum_{\lambda=0}^{2m-1} \|u\|_{\lambda,\omega_{r+h}}^2 h^{2\lambda-4m} \right\}.$$

Now we observe that, for any positive integers λ, l, k , we have

$$(4) \quad \sum_{|q|=\lambda+l} \sum_{|p|=k} |D^q D_x^p u|^2 \leq \begin{cases} \sum_{|q|=\lambda} \sum_{|p|=k-l} |D^q D_x^p u|^2 & \text{for } l \leq k \\ \sum_{|q|=\lambda+k-l} |D^q u|^2 & \text{always.} \end{cases}$$

It follows from this that, for $k \geq 0$, $0 \leq \lambda \leq 2m$ we have

$$(5) \quad \sum_{|p|=k} \bar{a}_{-\lambda,r}^2(D_x^p u) \leq \bar{a}_{k-\lambda,r}^2(u).$$

On the otherhand we also have

$$(6) \quad \sum_{|p|=k} e_{0,r}^2(D_x^p f) = e_{k,r}^2(f)$$

$$\sum_{|p|=k} e_{j,0,r}^2(D_x^p g) = e_{j,k,r}^2(g).$$

Taking $h = \frac{R-r}{k+1}$ ($R/2 \leq r < R$), (3) can now be written in the form

$$(7) \quad \bar{a}_{0,r}^2(u) \leq C_4 \left\{ e_{0,r}^2(Au) + \sum_{j=0}^{2m-1} e_{j,0,r}^2(B_j u) + \sum_{\lambda=1}^{2m} \bar{a}_{-\lambda,r+h}^2(u) \cdot h^{-2\lambda} \right\}.$$

LEMMA 3.1. If u is any C^∞ function and if $(A, \{B_j\})$ is an admissible system then there exists a positive constant C_5 independent of u, R and of

k , such that for any $R < R_1, k > 0$ the following inequality holds :

$$(8) \quad N_{R,k}(u) \leq C_5 \left\{ M_{R,k}(Au) + \sum_{j=0}^{2m-1} M_{j,R,k}(B_j u) + \sum_{\lambda=1}^{2m} N_{R,k-\lambda}(u) + \sum_{\tau=1}^{2m+k} (H_2 R)^\tau N_{R,k-\tau}(u) \right\}.$$

PROOF. Consider any one of the tangential derivatives $D_{x'}^q u$ of u with $|q| = k$ and apply (3) in the form (7) taking $R/2 \leq r < R$ and $h = \frac{R-k}{k+1}$.

We obtain

$$(9) \quad d_{0,r}^2(D_{x'}^q u) \leq C_4 \left\{ e_{0,r}^2(A(D_{x'}^q u)) + \sum_{j=0}^{2m-1} e_{j,0,r}^2(B_j(D_{x'}^q u)) + \sum_{\lambda=0}^{2m} d_{-\lambda,r+h}^2(D_{x'}^q u) \cdot h^{-2\lambda} \right\}.$$

Using Leibniz formula for the derivation of a product of two functions and the fact that

$$\binom{q_1}{s_1} \binom{q_2}{s_2} \dots \binom{q_\nu}{s_\nu} \leq \binom{k}{\mu}$$

where q_i and s_i are non-negative integers such that $q_1 + \dots + q_\nu = k$ and $s_1 + \dots + s_\nu = \mu$ we have the inequalities

$$\sum_{|q|=k} |A(D_{x'}^q u)| \leq \sum_{|q|=k} |D_{x'}^q(Au)| + \sum_{|p| \leq 2m} \sum_{\mu=1}^k \binom{k}{\mu} \sum_{|s|=\mu} |D_{x'}^s a_p| \sum_{|t|=k-\mu} |D_{x'}^t D^p u|$$

and

$$\sum_{|q|=k} |B_j(D_{x'}^q u)| \leq \sum_{|q|=k} |D_{x'}^q(B_j u)| + \sum_{|p| \leq 2m-1-j} \sum_{\mu=1}^k \binom{k}{\mu} \sum_{|s|=\mu} |D_{x'}^s b_p^j| \sum_{|t|=k-\mu} |D_{x'}^t D^p u|.$$

Summing over all q with $|q| = k$ in (9), using the following majorizations

$$\left(\sum_{|q|=\lambda} |D^q a_p(x)|^2 \right)^{\frac{1}{2}}, \quad \left(\sum_{|q|=\lambda} |D_{x'}^q b_p^j(x')|^2 \right)^{\frac{1}{2}} \leq H_1 H_2^\lambda M_\lambda$$

(with the constants H_1, H_2 suitably changed) and making use of (5), (6) we obtain

$$\begin{aligned} d_{k,r}(u) \leq C_6 & \left\{ e_{k,r}(Au) + \sum_{j=0}^{2m-1} e_{j,k,r}(B_j u) + \sum_{\lambda=1}^{2m} d_{k-\lambda,r+h}(u) h^{-\lambda} + \right. \\ & + \sum_{\mu=1}^k \binom{k}{\mu} H_1 H_2^\mu M_\mu \left(\sum_{|p| \leq 2m} \sum_{|t| = k-\mu} \| D_{x'}^t D^p u \|_{0, \omega_{r+h}}^2 \right)^{\frac{1}{2}} + \\ & \left. + \sum_{j=0}^{2m-1} \sum_{\mu=1}^k \binom{k}{\mu} H_1 H_2^\mu M_\mu \left(\sum_{|p| \leq 2m-1-j} \sum_{|t| = k-\mu} \| \varphi_{r,h} D_{x'}^t D^p u \|_{j+\frac{1}{2}, \pi_0}^2 \right)^{\frac{1}{2}} \right\} \end{aligned}$$

where C_6 is a positive constant independent of u, r, h, k, R . Moreover we have $\| \varphi_{r,h} v \|_{j+\frac{1}{2}, \pi_0} \leq C_7 \| v \|_{j+1, \omega_{r+h}}$ (see [3]) with C_7 independent of v . From this remark it is clear that the last term of the second member of the above inequality can be majorized by the last but one term. Hence

$$\begin{aligned} (10) \quad d_{k,r}(u) \leq C_6 & \left\{ e_{k,r}(Au) + \sum_{j=0}^{2m-1} e_{j,k,r}(B_j u) + \sum_{\lambda=1}^{2m-1} d_{k-\lambda,r+h}(u) \cdot h^{-\lambda} + \right. \\ & \left. + C_8 \sum_{\mu=1}^k \binom{k}{\mu} H_2^\mu M_\mu \left(\sum_{|p| \leq 2m} \sum_{|t| = k-\mu} \| D_{x'}^t D^p u \|_{0, \omega_{r+h}}^2 \right)^{\frac{1}{2}} \right\} \end{aligned}$$

where C_8 is a positive constant independent of u, r, h and k . Applying the inequality (4) to the last term of the second member of (10) we obtain

$$\begin{aligned} (11) \quad d_{k,r}(u) \leq C_8 & \left\{ e_{k,r}(Au) + \sum_{j=0}^{2m-1} e_{j,k,r}(B_j u) + \sum_{\lambda=1}^{2m} d_{k-\lambda,r+h}(u) \cdot h^{-\lambda} + \right. \\ & \left. + C_8 \sum_{\mu=1}^k \binom{k}{\mu} H_2^\mu M_\mu \sum_{l=0}^{2m} d_{k-\mu-l,r+\mu}(u) \right\}. \end{aligned}$$

Multiplying both sides of (11) by $\frac{1}{M_k} (R-r)^{2m+k}$ we have the following estimates :

$$\begin{aligned} \frac{1}{M_k} (R-r)^{2m+k} e_{k,r}(Au) &= \frac{1}{M_k} \left[\frac{R-r}{R-(r+h)} \right]^{2m+k} (R-(r+h))^{2m+k} e_{k,r}(Au) \leq \\ &\leq \left(1 + \frac{1}{k} \right)^{2m+k} M_{R,k}(Au) \\ \frac{1}{M_k} (R-r)^{2m+k} e_{j,k,r}(B_j u) &\leq \left(1 + \frac{1}{k} \right)^{2m+k} M_{j,R,k}(B_j u) \quad \text{for } 0 \leq j \leq 2m-1. \end{aligned}$$

Further since $h = \frac{R-r}{k+1}$

$$\begin{aligned} \frac{1}{M_k} (R-r)^{2m+k} d_{k-\lambda, r+h}(u) h^{-\lambda} &\leq \frac{1}{M_k} (k+1)^{\lambda} \left(1 + \frac{1}{k}\right)^{2m+k-\lambda} [M_{k-\lambda}] N_{R, k-\lambda}(u) \leq \\ &\leq \frac{[(k-\lambda)!]}{k!} C_1^{\lambda} (k+1)^{\lambda} \left(1 + \frac{1}{k}\right)^{2m+k-\lambda} N_{R, k-\lambda}(u) \end{aligned}$$

because $[M_{k-\lambda}] \leq \frac{(k-\lambda)!}{k!} C_1^{\lambda} M_k$. Similarly we have

$$\begin{aligned} \binom{k}{\mu} \frac{M_{\mu}}{M_k} (R-r)^{2m+k} d_{k-\mu-l, r+h}(u) &\leq \binom{k}{\mu} \frac{M_{\mu} [M_{k-\mu-l}]}{M_k} \left(1 + \frac{1}{k}\right)^{2m+k-\mu-l} \\ &\cdot R^{\mu+l} N_{R, k-\mu-l}(u). \end{aligned}$$

But by (1) $M_{\mu} [M_{k-\mu-l}] \leq \left[\binom{k-l}{\mu}\right]^{-1} C M_{k-l}$ and by (1') it follows that $M_{k-1} \leq C_1^l \frac{[(k-l)!]}{k!} M_k$. Hence we have:

$$\begin{aligned} \binom{k}{\mu} \frac{M_{\mu}}{M_k} (R-r)^{2m+k} d_{k-\mu-l, r+h}(u) &\leq C C_1^l \frac{[(k-\mu-l)!]}{(k-\mu)!} \left(1 + \frac{1}{k}\right)^{2m+k-\mu-l} \\ &\cdot R^{\mu+l} N_{R, k-\mu-l}(u). \end{aligned}$$

Then the inequality (11) becomes

$$\begin{aligned} \frac{1}{M_k} (R-r)^{2m+k} d_{k,r}(u) &\leq C_8 \left(1 + \frac{1}{k}\right)^{2m+k} \left\{ M_{R,k}(Au) + \sum_{j=0}^{2m-1} M_{j,R,k}(B_j u) + \right. \\ &\quad \left. + \sum_{\lambda=1}^{2m} \frac{[(k-\lambda)!]}{k!} (k+1)^{\lambda} C_1^{\lambda} \left(1 + \frac{1}{k}\right)^{-\lambda} N_{R, k-\lambda}(u) + \right. \\ &\quad \left. + C_8 \sum_{\mu=1}^k \sum_{l=0}^{2m} H_2^{\mu} R^{\mu+l} C C_1^l \frac{[(k-\mu-l)!]}{(k-\mu)!} \left(1 + \frac{1}{k}\right)^{-(\mu+l)} \cdot N_{R, k-\mu-l}(u) \right\}. \end{aligned}$$

Since $k^{\lambda} \frac{[(k-\lambda)!]}{k!} \leq \lambda^{\lambda} \leq 2m^{2m}$ and $\frac{[(k-\mu-l)!]}{(k-\mu)!} \leq 1$ it follows that there

exists a constant C_5 such that

$$\frac{1}{M_k} (R - r)^{2m+k} d_{k,r}(u) \leq C_5 \left\{ M_{R,k}(Au) + \sum_{j=0}^{2m-1} M_{j,R,k}(B_j u) + \sum_{\lambda=1}^{2m} N_{R,k-\lambda}(u) + \sum_{\mu=1}^k \sum_{l=0}^{2m} (H_2 R)^{\mu+l} N_{R,k-\mu-l}(u) \right\}.$$

Taking $\mu + l = \tau$ in the last term of the second member and the supremum for $R/2 \leq r < R$ of the first member we obtain (8) and this completes the proof of the lemma.

LEMMA 3.2. Let $(A, \{B_j\})$ be an admissible system and u be any C^∞ function satisfying the system

$$\begin{aligned} Au &= f \quad \text{in } \omega_{R_0}, \\ B_j u &= g_j \quad \text{in } \partial_1 \omega_{R_0} \quad (0 \leq j \leq 2m - 1), \end{aligned}$$

with f and g_j respectively in the classes $C\{M_n; \omega_{R_0}\}$ and $C\{M_{n-j-1}; \partial_1 \omega_{R_0}\}$. Then there exist two positive constants M and λ such that

$$(12) \quad N_{R,k}(u) \leq M \lambda^k \quad \text{for } k = -2m, -2m + 1, \dots$$

PROOF. We can suppose, if necessary after some modification that the constants H_1, H_2 and R_1 are the same as before and are such that

$$\left(\sum_{|q|=k} |D^q f(x)|^2 \right)^{\frac{1}{2}} \leq H_1 H_2^k M_k \quad \text{for } x \in \bar{\omega}_{R_1}, \quad k = 0, 1, 2, \dots;$$

and

$$\left(\sum_{|q|=k} |D_{x'}^q g_j(x')|^2 \right)^{\frac{1}{2}} \leq H_1 H_2^k M_k \quad \text{for } x' \in \partial_1 \omega_{R_1}, \quad k = 0, 1, 2, \dots; \quad 0 \leq j \leq 2m - 1.$$

Let β_ν^2 denote the volume of the unit ball in the ν -dimensional Euclidean space. Then for $R < R_1$ we have

$$M_{R,k}(f) \leq \frac{1}{M_k} R^{2m+k} \left(\sum_{|q|=k} \int_{\omega_R} |D_{x'}^q f|^2 dx \right)^{\frac{1}{2}} \leq H_1 R^{2m+k} H_2^k R^{\nu/2} \beta_\nu.$$

Similarly using $\|\varphi_{r,h} W\|_{j+\frac{1}{2}, \pi_0} \leq \tilde{C} \|W\|_{j+1, \omega_{r+h}}$, with a positive constant \tilde{C}

independent of W , we obtain

$$\begin{aligned} M_{j,R,k}(g_j) &\leq \tilde{C} \frac{1}{M_k} R^{2m+k} \left(\sum_{l=0}^{j+1} \sum_{|q|=-k+l} \int_{\partial_1 \omega_R} |D_{x'}^q g_j|^2 dx' \right)^{\frac{1}{2}} \leq \\ &\leq C_9 R^{2m+k} H_1 H_2^k R^{\frac{\nu-1}{2}} \beta_{\nu-1} \left(\sum_{l=0}^{j+1} \left(\frac{M_{k+l-j-1}}{M_k} \right)^2 \right)^{\frac{1}{2}} \leq \\ &\leq C_{10} R^{2m+k} H_1 H_2^k R^{\frac{\nu-1}{2}} \beta_{\nu-1} \end{aligned}$$

using (1'), where C_9, C_{10} are positive constants independent of R and k . Then the inequality (8) becomes, for any k and $R < R_1$,

$$N_{R,k}(u) \leq C_5 \left\{ C_{11} (H_2 R)^k + \sum_{\lambda=1}^{2m} N_{R,k-\lambda}(u) + \sum_{r=1}^{2m+k} (H_2 R)^r N_{R,k-r}(u) \right\}.$$

Now proceeding, as in the proof of Magenes and Stampacchia, with the constants $M \geq 3C_5 C_{11}$ and $\lambda = (3C_5 + 1)(H_2 R_1 + 1)$ we obtain

$$N_{R,k}(u) \leq M \lambda^k \quad \text{for } k = -2m, -2m + 1, \dots$$

after using an induction argument on k . This completes the proof of lemma 3.2.

§ 4. We complete the proof of the main theorem (see § 2) in this paragraph. For this purpose it is necessary to obtain estimates of the type (12) for all derivatives, tangential as well as transversal, of u . To obtain such estimates we follow a procedure used by Morrey and Nirenberg in [4]. We introduce the following norms analogous to those in § 2.

For $p \geq 0, q \geq -2m$ define

$$(13) \quad N_{R,p,q}(u) = \frac{1}{[M_{p+q}]} \sup_{R/2 \leq r < R} (R-r)^{2m+p+q} \left(\sum_{|\lambda|=p} \int_{\omega_r} |D_y^{2m+q} D_{x'}^\lambda u|^2 dx \right)^{\frac{1}{2}}.$$

Analogous to (4) we have

$$\begin{aligned} \sum_{|\lambda|=p} |D_y^{2m+q} D_{x'}^\lambda u| &\leq \sum_{|\mu|=q} \sum_{|\lambda|=p} |D^{2m+\mu} D_{x'}^\lambda u| \leq \\ &\leq \begin{cases} \sum_{|\mu|=2m} \sum_{|\lambda|=p+q} |D^{2m} D_{x'}^{\mu+\lambda} u| & \text{if } p \geq 0, q \leq 0 \\ \sum_{|\mu|=p+q+2m} |D^\mu u| & \text{in all cases.} \end{cases} \end{aligned}$$

This implies that

$$(14) \quad N_{R,p,q}(u) \leq N_{R,p+q}(u) \text{ if } p \geq 0, q \leq 0.$$

We now prove the following extension of the estimation (12): if R is smaller than or equal to a fixed number depending only on the given differential equation, then

$$(15) \quad N_{R,p,q}(u) \leq \bar{M} \bar{\lambda}^{p+q} \theta^p, (p \geq 0, q \geq -2m)$$

with $\bar{M}, \bar{\lambda} \geq 1$ and $\theta \leq \frac{1}{2}$ fixed constants, $\bar{\lambda}$ and θ depending only on the equation.

The following is a sketch of the derivation of the estimate (15). Let us denote x , by y for convenience. By assumption $y = 0$ is not a characteristic surface for the given equation $Au = f$. Hence one can solve for the normal derivative $D_y^{2m} u$ of u in terms of the derivatives involving normal derivatives of u of orders less than $2m$:

$$(16) \quad D_y^{2m} u = g + \sum_{t=1}^{2m} b_t D_y^{2m-t} D_{x'}^t u$$

where in view of the remarks on the classes $C\{M_n; \Omega\}$, made in § 2, g and b_t are functions belonging to the class $C\{M_n; \omega_{R_0}\}$. This implies that both

$$(17) \quad \sum_{|\lambda|-p} |D_y^q D_{x'}^\lambda g(x)|, \sum_{|\lambda|-p} |D_y^q D_{x'}^\lambda b_t(x)| \leq H_1 H_2^{p+q} M_{p+q}$$

for suitable constants H_1, H_2 and $R_0 \leq 1$. We can assume these constants to be the same as before by suitable choice. Then we have from (16)

$$\sum_{|\lambda|-p} |D_y^{2m+q} D_{x'}^\lambda u| \leq \sum_{|\lambda|-p} |D_y^q D_{x'}^\lambda g| + \sum_{|\lambda|-p} \sum_{t=1}^{2m} |D_y^q D_{x'}^\lambda (b_t D_y^{2m-t} D_{x'}^t u)|.$$

Hence

$$(18) \quad \sum_{|\lambda|-p} |D_y^{2m+q} D_{x'}^\lambda u| \leq H_1 H_2^{p+q} M_{p+q} + \sum_{t=1}^{2m} \sum_{|\lambda|-p} \sum_{\alpha} \sum_{\beta=0}^q \binom{\lambda}{\alpha} \binom{q}{\beta} H_1 H_2^{|\alpha|+\beta} M_{|\alpha|+\beta} |D_y^{2m+q-\beta-t} D_{x'}^{\lambda-\alpha+t} u|.$$

It is clear from (12) and (14) that (15) follows for $-2m \leq q \leq 0$ and all $p \geq 0$ provided that $R < R_1 < R_0$ (R_1 chosen suitably) and

$$(19) \quad \bar{M} \bar{\lambda}^{p+q} \theta^p \geq M \lambda^{p+q} \text{ for } -2m \leq q \leq 0, p \geq 0.$$

We prove (15) for $q \geq 0, p \geq 0$ by induction on q . Let us assume that (15) holds for all values of q less than a certain positive integer which we again denote by q . Squaring both sides of (18) and integrating over ω_r we obtain

$$\begin{aligned} & \left(\sum_{|\lambda|=p} \int_{\omega_r} |D_y^{2m+q} D_x^\lambda u|^2 dx \right)^{\frac{1}{2}} \\ & \leq H_1 H_2^{p+q} M_{p+q} \beta_r r^{p/2} + \sum_{t=1}^{2m} \sum_{|\alpha|=0}^p \sum_{\beta=0}^q \binom{p}{|\alpha|} \binom{q}{\beta} H_1 H_2^{|\alpha|+\beta} M_{|\alpha|+\beta} \cdot \\ & \qquad \qquad \qquad \cdot \left(\sum_{|\lambda|=p} \int_{\omega_r} |D_y^{2m+q-\beta-t} D_x^{\lambda-\alpha+t} u|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

Multiplying both sides of this inequality by

$$\frac{(R-r)^{2m+p+q}}{[M_{p+q}]} \bar{M}^{-1} \bar{\lambda}^{-(p+q)} \theta^{-p}$$

for $R < R_1$, taking the supremum over all r with $R/2 \leq r < R$ and using the induction assumption we obtain

$$\begin{aligned} (20) \quad & \bar{M}^{-1} \bar{\lambda}^{-(p+q)} \theta^{-p} N_{R,p,q} \leq \frac{KH_1}{\bar{M}} \left(\frac{H_2 R}{\bar{\lambda}} \right)^{p+q} \theta^{-p} + \\ & + \sum_{t=1}^{2m} \theta^t \sum_{|\alpha|=0}^p \sum_{\beta=0}^q \binom{p}{|\alpha|} \binom{q}{\beta} M_{|\alpha|+\beta} \left(\frac{H_2 R}{\theta \bar{\lambda}} \right)^{|\alpha|} \left(\frac{H_2 R}{\bar{\lambda}} \right)^\beta \frac{[M_{p+q-|\alpha|-\beta}]}{[M_{p+q}]} \end{aligned}$$

where K is a suitable constant.

But by (1) we have the inequality

$$[M_{p+q-|\alpha|-\beta}] M_{|\alpha|+\beta} \leq C \left[\binom{p+q}{|\alpha|+\beta} \right]^{-1} [M_{p+q}].$$

Then the inequality (20) becomes

$$\begin{aligned} & \bar{M}^{-1} \bar{\lambda}^{-(p+q)} \theta^{-p} N_{R,p,q}(u) \leq \frac{KH_1}{\bar{M}} \left(\frac{H_2 R}{\bar{\lambda}} \right)^{p+q} \theta^{-p} + \\ & + H_1 C \sum_{t=0}^{2m} \theta^t \sum_{|\alpha|=0}^p \sum_{\beta=0}^q \binom{p}{|\alpha|} \binom{q}{\beta} \left[\binom{p+q}{|\alpha|+\beta} \right]^{-1} \left(\frac{H_2 R}{\theta \bar{\lambda}} \right)^{|\alpha|} \left(\frac{H_2 R}{\bar{\lambda}} \right)^\beta \end{aligned}$$

Here all the terms in the summation over α, β are less than unity. Taking $\frac{H_2 R}{\theta \bar{\lambda}} \leq \frac{1}{2}$ and $\theta \leq \frac{1}{2}$ the second member does not exceed $\frac{KH_1}{\bar{M}} + 8 CH_1 \theta$ which is again less than unity if $\bar{M} \geq 2KH_1$, and $\theta \leq \frac{1}{16 CH_1}$. Thus we have proved that

$$N_{R,p,q}(u) \leq \bar{M} \bar{\lambda}^{p+q} \theta^p$$

holds for all q , with $q \geq -2m$ and for $R < R_1$ if we take $\theta = \frac{1}{16 CH_1}$, $\bar{\lambda} = \max\left(\frac{2H_2 R_1}{\theta}, \frac{\lambda}{\theta}\right)$ and $\bar{M} \geq 2KH_1$.

As we have already said in the introduction the result is deduced by applying Sobolev's lemma to the L_2 -norms of the derivatives of u . For this we need estimates for the square integrals of the type

$$\tilde{d}_p^2(u, \omega_r) = \sum_{|q|=p} \int_{\omega_r} |D^q u|^2 dx.$$

These are easily obtained from (15) as follows :

$$\begin{aligned} \tilde{d}_p^2(u, \omega_r) &= \sum_{t=0}^p \sum_{|q|=p-t} \int_{\omega_r} |D_y^t D_x^q u|^2 dx \\ &\leq \sum_{t=0}^p \left[\frac{\bar{M} \bar{\lambda}^{p-2m} \theta^{p-t} M_{p-2m}}{(R-r)^p} \right]^2. \end{aligned}$$

Hence

$$\tilde{d}_p(u, \omega_r) \leq \frac{\bar{M} \bar{\lambda}^p}{(R-r)^p} M_{p-2m} \left(\sum_{t=0}^p \theta^{2t} \right)^{\frac{1}{2}}$$

Thus we obtain

$$(21) \quad \tilde{d}_p(u, \omega_r) \leq \frac{2 \bar{M} (\bar{\lambda})^p M_{p-2m}}{(R-r)^p}.$$

Now we apply Sobolev's lemma in the form used in [4], namely, for $x \in \omega_r \cup \partial_1 \omega_r$

$$\begin{aligned} |D^p u(x)| &\leq C' \left[\sum_{l=0}^{[v/2]+1} r^{2l-v} \tilde{d}_l^2(D^p u, \omega_r) \right]^{\frac{1}{2}} \\ &\leq C' \left[\sum_{l=0}^{[v/2]+1} r^{2l-v} \left\{ \frac{2 \bar{M} (\bar{\lambda})^{p+l}}{(R-r)^{p+l}} M_{p+l-2m} \right\}^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Since $R - r \leq r$ and $R \leq 1$ we obtain the following inequality

$$|D^p u(x)| \leq \frac{K' \bar{\lambda}^{[v/2]+p+1}}{(R-r)^{[v/2]+p+1}} M_{p+[v/2]+1-2m}$$

after using (1') (K' being a positive constant independent of p). This proves the fact that $u \in C\{M_{n-2m+[v/2]+1}; \omega_{R_0} \cup \partial_1 \omega_{R_0}\}$ thus completing the proof of the theorem.

B I B L I O G R A P H Y

- [1] S. AGMON, A. DOUGLIS and L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Comm. Pure. Appl. Math., 12 (1959), pp. 623-727.
- [2] A. FRIEDMAN, *On the regularity of the solutions of non-linear elliptic and parabolic systems of partial differential equations*, Journal of Mathematics and Mechanics, 7 (1958), pp. 43-59.
- [3] E. MAGENES and G. STAMPACCHIA, *I problemi al contorno per le equazioni differenziali di tipo ellittico*, Scuola Normale Superiore, Pisa, 12 (1958), pp. 247-257.
- [4] C. B. MORREY and L. NIRENBERG, *On the analyticity of the solutions of linear elliptic systems of partial differential equations*, Comm. Pure. Appl. Math., 10 (1957), pp. 271-290.
- [5] J. PEETRE, *Théorèmes de régularité pour quelques classes d'opérateurs différentiels*, Communications du Séminaire Mathématique de l'université de Lund, Tome 16 (1959).