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A GENERALIZATION OF THE PROBLEM OF TRANSMISSION (*)

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1. Introduction.

In the past few years there has been increasing interest in so called « transmission » problems for elliptic equations and systems (cf., eg., Picone [12], Lions [9, 10, 11], Stampacchia [17], Campanato [18, 19]). These problems may be described as follows. There are given two domains in Euclidean n — space which have a portion Σ_0 of their boundaries in common. A boundary value problem is than posed for each domain with the boundary conditions on Σ_0 being double the usual number and involving the solutions of both problems.

To give a simple example, let $G^{(1)}$ and $G^{(2)}$ be the domains in question with boundaries $\partial G^{(1)}$ and $\partial G^{(2)}$, respectively. One might ask to find functions $u^{(1)}$ and $u^{(2)}$ harmonic in $G^{(1)}$ and $G^{(2)}$, respectively, for which $a_1 u^{(1)} + a_2 u^{(2)}$ and $b_1 \dfrac{\partial u^{(1)}}{\partial n} + b_2 \dfrac{\partial u^{(2)}}{\partial n}$ are prescribed on Σ_0, while $u^{(1)}$ and $\dfrac{\partial u^{(2)}}{\partial n}$ are given on $\Sigma_1 = \partial G^{(1)} - \Sigma_0$ and $\Sigma_2 = \partial G^{(2)} - \Sigma_0$, respectively (here a_1, a_2, b_1, b_2 are given functions and $\dfrac{\partial}{\partial n}$ denotes a normal derivative).

Problems of this type for general second order elliptic equations have been treated as well as problems for some systems (cf. the references mentioned above).

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In this paper we consider a transmission problem for general $m - th$ order elliptic equations and general boundary conditions. More precisely, let A_1 and A_2 be two elliptic operators of order $m = 2r$, $r \geq 1$, with smooth complex coefficients. On Σ_1 we consider a set $B_{11}, B_{12}, \ldots, B_{1r}$, of partial differential operators with smooth coefficients which need only be defined on Σ_1. A similar set $B_{21}, B_{22}, \ldots, B_{2r}$ is to be defined on Σ_2. On Σ_0 we prescribe $4r$ operators $C_{11}, C_{12}, \ldots, C_{1,2r}, C_{21}, C_{22}, \ldots, C_{2,2r}$ of the same type. The problem we consider is the following: Given two functions $f^{(1)}$, $f^{(2)}$ defined in $G^{(1)}$ and $G^{(2)}$ respectively, to find functions $u^{(1)}$ and $u^{(2)}$ such that

$$A_1 u^{(1)} = f^{(1)} \quad \text{in} \quad G^{(1)}, \quad A_2 u^{(2)} = f^{(2)} \quad \text{in} \quad G^{(2)}$$

$$B_{1j} u^{(1)} = 0 \quad \text{on} \quad \Sigma_1, \quad B_{2j} u^{(2)} = 0 \quad \text{on} \quad \Sigma_2, \quad j = 1, 2, \ldots, r,$$

$$C_{1j} u^{(1)} = C_{2j} u^{(2)} \quad \text{on} \quad \Sigma_0, \quad j = 1, 2, .. , 2r.$$

Our assumptions on the B_{ij} and C_{ij} are mild, being no more than those usually imposed in regular elliptic problems (cf. [2, 7]) plus a compatibility condition at points where Σ_0 and Σ_1 (or Σ_2) meet (cf. Section 2). We define an « adjoint » problem and show, among other things, that existence of solution of the original problem is guaranteed by the uniqueness of solution of the adjoint problem (cf. Section 2 for a more general statement). Moreover we show the existence of solutions which are smooth up to the boundary except possibly at points where Σ_0 and Σ_1 join and which do not behave badly at such points. Our theorem, when specialized to second order equations gives more general results than those previously obtained.

For simplicity we consider the case when both $G^{(1)}$ and $G^{(2)}$ are bounded and smooth. Under such circumstances we can map the closure of $G^{(2)}$ onto the closure of $G^{(1)}$ and then consider both boundary problems appertaining only to $G^{(1)}$. Our transmission problem then becomes a « mixed » boundary value problem for a *system* of equations (cf. [16]). This is essentially the way we treated the problem, although we retained the original notation and terminology.

Our method employs a coerciveness inequality specially adapted to the problem. Near points of Σ_1 and Σ_2, no new inequalities were needed, the proper estimated being already available in the literature (cf. [1, 2, 14]). For Σ_0 we derive new inequalities which are essentially those for systems (cf. Section 7). More general estimates of this nature will be given in the second part of [2]. Finally, for points where Σ_0 and Σ_1 intersect we obtain special inequalities peculiar to this particular problem (cf. Section 8).

In Section 2 we state our hypotheses and main theorem (Theorem 2.1). The coerciveness inequality is described in Section 3, and its proof is given in Sections 5, 7, and 8. Our existence and regularity proof is given in Section 4. In Section 6 some algebraic theorems which are needed for our estimates are discussed.

2. Assumptions and Results.

We consider two bounded domains, $G^{(1)}$ and $G^{(2)}$ in Euclidean n space E^n with boundaries $\partial G^{(1)}$ and $\partial G^{(2)}$ which are each of class C^∞. We assume that $G^{(1)} \cap G^{(2)} = 0$, but also that $\partial G^{(1)} \cap \partial G^{(2)} = \bar{\Sigma}_0$, the closure of a non-empty set Σ_0 open in the topology of $\partial G^{(1)}$ or $\partial G^{(2)}$. We set $\Sigma_i = \partial G^{(i)} - \bar{\Sigma}_0$.

For any appropriate subset $\mathcal{S}$ of E^n, we let $C^\infty(\mathcal{S})$ denote the set of complex valued functions which are infinitely differentiable in $\mathcal{S}$. We shall deal with vector functions $v = (v^{(1)}, v^{(2)})$ where $v^{(i)} \varepsilon C^\infty(\overline{G^{(i)}})$ and the following norms and inner products:

$$(u, v)_s = \sum_{i=1}^{2} \sum_{|\mu| \leq s} \int_{G^{(i)}} D^\mu u^{(i)} \overline{D^\mu v^{(i)}} \, dx$$

$$\| v \|_s^2 = (v, v)_s, \quad (u, v) = (u, v)_0,$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is a multi-index with non negative components, $|\mu| = \mu_1 + \mu_2 + \ldots + \mu_n$, and

$$D^\mu = \left(\frac{\partial}{\partial x_1}\right)^{\mu_1} \left(\frac{\partial}{\partial x_2}\right)^{\mu_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\mu_n}.$$

By a *boundary triple* (x^0, τ, ν) we shall mean a point $x^0 \varepsilon \partial G^{(1)} \cap \partial G^{(2)}$, a real vector $\tau \neq 0$ tangent to $\partial G^{(1)}$ (or $\partial G^{(2)}$) at x^0 and a real vector $\nu \neq 0$ normal to $G^{(1)}$ (or $G^{(2)}$) at x^0. We shall make the following assumptions (referred to as Hypotheses 1 — 9).

1. In each $\overline{G^{(i)}}$ there is defined a partial differential operator

$$A_i = \sum_{|\mu| \leq 2r} a_{i\mu}(x) D^\mu, \tag{2.1}$$

with complex coefficients in $C^\infty(\overline{G^{(i)}})$.

2. Each A_i is elliptic in $\overline{G^{(i)}}$, i.e., the characteristic polynomial

$$P_i(x, \xi) = \sum_{|\mu| = 2r} a_{i\mu}(x) \xi^\mu, \quad \xi^\mu = \xi_1^{\mu_1} \xi_2^{\mu_2} \ldots \xi_n^{\mu_n}, \tag{2.2}$$

of A_i does not vanish at any point $x \varepsilon \overline{G^{(i)}}$ when $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ is real and $\neq 0$.

3. Each A_i is *properly* elliptic in $\overline{G^{(i)}}$. By this we meam that A_i is elliptic in $\overline{G^{(i)}}$ and that for every boundary triple (x^0, τ, ν) with $x^0 \varepsilon \partial G^{(i)}$, the polynomial in z

$$P_i(z) = P_i(x^0, \tau + z\nu) \tag{2.3}$$

has exactly r roots $\lambda_1^{(i)}(x^0, \tau, \nu), \lambda_2^{(i)}(x^0, \tau, \nu), \dots, \lambda_r^{(i)}(x^0, \tau, \nu)$ with positive imaginary parts. When $n > 2$, every elliptic operator is properly elliptic.

4. On each $\Sigma_i = \partial G^{(i)} - \overline{\Sigma}_0$ there are defined r partial differential operators

$$B_{ij} = \sum_{|\mu| \le m_{ij}} b_{ij\mu}(x) D^\mu, \qquad j = 1, 2, \dots, r. \tag{2.4}$$

where each $m_{ij} < 2r$ and the complex coefficients are in $C^\infty(\overline{\Sigma}_i)$.

5. The set $\{B_{ij}\}_{j=1}^r$ *covers* A_i. This means that if

$$Q_{ij}(x, \xi) = \sum_{|\mu| = m_{ij}} b_{ij\mu}(x)\, \xi^\mu, \qquad j = 1, 2, \dots, r, \tag{2.5}$$

is the characteristic polynomial of B_{ij}, then for every boundary triple (x^0, τ, ν) with $x^0 \varepsilon \overline{\Sigma}_i$ the polynomials in z

$$Q_{ij}(z) \equiv Q_{ij}(x^0, \tau + z\nu), \qquad j = 1, 2, \dots, r, \tag{2.6}$$

are linearly independent modulo

$$P_i^+(z) = \prod_{k=1}^{r} (z - \lambda_k^{(i)}(x^0, \tau, \nu)), \tag{2.7}$$

where the $\lambda_k^{(i)}(x^0, \tau, \nu)$ are the roots of (2.3) with positive imaginary parts (cf. Hypothesis 3).

6. The set $\{B_{ij}\}_{j=1}^r$ is *normal,* i.e., $m_{ij} \neq m_{ik}$ for $j \neq k$ and no $Q_{ij}(x^0, \nu)$ vanishes for any $x^0 \varepsilon \overline{\Sigma}_i$ and $\nu \neq 0$ normal to $\partial G^{(i)}$ at x^0.

7. There are $4r$ boundary operators defined on $\overline{\Sigma}_0$

$$C_{ij} = \sum_{|\mu| \le m_j} c_{ij\mu}(x) D^\mu, \quad i = 1, 2, \quad j = 1, 2, \dots, 2r. \tag{2.8}$$

Each set $\{C_{1j}\}_{j=1}^{2r}$, $\{C_{2j}\}_{j=1}^{2r}$ is assumed normal.

8. Let

$$R_{ij}(x, \xi) = \sum_{|\mu|=m_j} c_{ij\mu}(x)\, \xi^{\mu}, \quad i = 1, 2\,; \quad j = 1, 2, \dots, 2r, \tag{2.9}$$

be the characteristic polynomial of C_{ij}. Then for any boundary triple (x^0, τ, ν) with $x^0 \varepsilon \bar{\Sigma}_0$, the relations

$$\sum_{j=1}^{2r} \lambda_j R_{1j}(z) = U_1(z) P_1^+(z) \tag{2.10}$$

$$\sum_{j=1}^{2r} \lambda_j R_{2j}(z) = U_2(z) P_2^+(z) \tag{2.11}$$

imply that $U_1(z)$, $U_2(z)$ and the λ_j all vanish, where the λ_j are complex constants, the $U_1(z)$ are polynomials, the $P_i^+(z)$ are defined by (2.7), and

$$R_{ij}(z) \equiv R_{ij}(x^0, \tau + z\nu), \quad i = 1, 2\,; \quad j = 1, 2, \dots, r. \tag{2.12}$$

Before stating the last stipulation, we mention several consequences of Hypotheses 1 — 8.

REMARK 2.1. It follows from Hypotheses 6 that to $\{B_{ij}\}_{j=1}^{r}$ there corresponds a normal set $\{B'_{ij}\}_{j=1}^{r}$, called adjoint to it relative to A, such that

$$(A_i u^{(i)}, v^{(i)}) = (u^{(i)}, A_i^* v^{(i)}) \tag{2.13}$$

hold whenever $u^{(i)}$ and $v^{(i)}$ vanish near $\bar{\Sigma}_0$, and

$$B_{ij} u^{(i)} = 0 \quad \text{on} \quad \Sigma_i, \quad j = 1, 2, \dots, r, \tag{2.14}$$

$$B'_{ij} v^{(i)} = 0 \quad \text{on} \quad \Sigma_i, \quad j = 1, 2, \dots, r. \tag{2.15}$$

(Here A_i^* denotes the formal adjoint of A_i). In addition, if $u^{(i)}$ satisfies (2.13) for all $v^{(i)}$ which vanish near $\bar{\Sigma}_0$ and satisfy (2.15), then (2.14) holds. Conversely, if $v^{(i)}$ satisfies (2.13) for all $u^{(i)}$ which vanish near $\bar{\Sigma}_0$ and satisfy (2.14), then (2.15) holds (See [3, 15]).

REMARK 2.2. Similarly, Hypothesis 7 implies that there are normal sets $\{C'_{ij}\}_{j=1}^{2r}$ such that

$$(A_i u^{(i)}, v^{(i)}) - (u^{(i)}, A_i^* v^{(i)}) = \sum_{j=1}^{2r} \int_{\Sigma_0} C_{ij} u^{(i)}\, \overline{C'_{ij} v^{(i)}}\, ds \tag{2.16}$$

for all $u^{(i)}$ and $v^{(i)}$ which vanish near $\bar{\Sigma}_i$, where ds denotes the element of surface on $\partial G^{(i)}$.

REMARK 2.3. From Hypothesis 8 it follows that for avery boundary triple (x_0, τ, ν) with $x_0 \varepsilon \Sigma_0$, the relations

$$\sum_{j=1}^{2r} \lambda_j R_{1j}(z) = U_1(z) P_1^-(z) \tag{2.17}$$

$$\sum_{j=1}^{2r} \lambda_j R_{2j}(z) = U_2(z) P_2^-(z) \tag{2.18}$$

imply that $U_1(z)$, $U_2(z)$, and the λ_j all vanish, where $P_i^-(z) = P_i(z)/P_1^+(z)$ $P_i(x^0, \nu)$. This follows from the fact that $R_{ij}(-z)$ and $P_i^+(-z)$ for (x^0, τ, ν) are equal, respectively, to $R_{ij}(z)$ and $P_i^-(z)$ for $(x^0, \tau, -\nu)$. Hence relations (2.17) and (2.18) at a boundary triple (x^0, τ, ν) imply relations similar to (2.10) and (2.11) for the boundary triple $(x_0, \tau, -\nu)$. Thus the λ_j must vanish.

The following notation will be useful in fomulating our last assumption. If $H(z) = \sum_{k=0}^{m} \alpha_k z^k$ is any polynomial of degree $\leq m$ and $\omega = (\omega_0, \omega_1,$ $, \dots \omega_m)$ is any $(m+1)$ dimensional complex vector, we write $H(\omega) = \sum_{k=0}^{m} \alpha_k \omega_k$. We also write $\bar{H}(z) = \sum_{k=0}^{m} \bar{\alpha}_k z^k$, where the bar denotes complex conjugation.

For every boundary triple (x^0, τ, ν) we can define

$$H'_{ij}(z) \equiv z^{j-1} \bar{P}_i^-(z), \; i = 1, 2, \; j = 1, 2, \dots, r, \tag{2.19}$$

(cf. (2.7)). Hypothesis 9 can now be stated as follows.

9. For every point $x^0 \varepsilon S = \bar{\Sigma}_0 \cap \bar{\Sigma}_1$ and every $\nu \neq 0$ normal to $\partial G^{(1)}$ (or $\partial G^{(2)}$) at x^0, there are polynomials in the components of τ

$$E_{sjk}(x^0, \tau, \nu) = \sum_{|\mu| = 4r - m'_j - m'_{sk} - 1} e_{sjk\mu}(x^0, \nu) \tau^\mu \tag{2.20}$$

such that

$$\operatorname{Re} \sum_{s=1}^{2} \sum_{j=1}^{2r} \sum_{k=1}^{r} E_{sjk} \{R'_{1j}(\omega^{(1)}) + R'_{2j}(\omega^{(2)})\} \overline{Q'_{sk}(\omega^{(s)})}$$

is positive for all complex vectors $\omega^{(1)}$ and $\omega^{(2)}$ satisfyng

$$H'_{ij}(\omega^{(i)}) = 0, \qquad i = 1, 2; \; j = 1, 2, \dots, r,$$

unless $\omega^{(1)} = \omega^{(2)} = 0$. Here R'_{ij} and Q'_{ij} denote the characteristic polynomials of the C'_{ij} and B'_{ij}, respectively, which are assumed of orders m'_j and m'_{ij}, respectively.

By a solution of problem $\Pi(A, f, B_j, C_j)$ we shall mean a vector function $u = (u^{(1)}, u^{(2)})$ such that $u^{(i)} \varepsilon L^2(G^{(i)}) \cap C^\infty(K^{(i)})$ for every compact subset $K^{(i)}$ of $\overline{G^{(i)}}$ which is bounded away from S, and such that

$$A_i u^{(i)} = f^{(i)} \quad \text{in } G^{(i)},\ i = 1, 2, \tag{2.21}$$

$$B_{ij} u^{(j)} = 0 \quad \text{on } \Sigma_{ij}\ i = 1, 2\,;\ j = 1, 2, \dots, r, \tag{2.22}$$

$$C_{1j} u^{(i)} = C_{2j} u^{(2)} \text{ on } \Sigma_0,\ j = 1, 2, \dots, 2r. \tag{2.23}$$

THEOREM 2.1. *Assume that Hypotheses* 1-9 *hold. Then a sufficient condition for problem* $\Pi(A, f, B_j, C_j)$ *to have a solution is that* $(f, v) = 0$ *for every solution* v *of* $\Pi(A^*, 0, B_j', C_j')$.

As an illustration we consider the following generalization of the problem mentioned in the introduction.

$$A_i = \sum_{j,k} a_{jk}^{(i)} \frac{\partial^2}{\partial x_j\, \partial x_k} + \sum_k a_k^{(i)} \frac{\partial}{\partial x_k} + a^{(i)}, \qquad i = 1, 2,$$

$$B_{11} = 1\,;\ B_{21} = \Sigma\, a_{jk}^{(2)} \nu_j \frac{\partial}{\partial x_k} + b^{(0)}.$$

$$C_{i1} = \beta_{i1}\,;\ C_{i2} = \beta_{i2}\, \Sigma\, a_{jk}^{(i)} \nu_j \frac{\partial}{\partial x_k} + b^{(i)}, \qquad i = 1, 2,$$

where $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ is a unit normal vector to the surface in question and the β_{ij} and $b^{(i)}$ are complex functions. We assume that $a_{jk} = a_{kj}$ and that the A_i are properly elliptic (cf. below). We shall show that Theorem 2.1 applies when a certain expression Z defined below is real and negative on $\bar{\Sigma}_0$.

First let us consider an arbitrary boundary triple (x^0, τ, ν). Then

$$P_i(z) = \Sigma\, a_{jk}^{(i)} (\tau_j + z\,\nu_j)(\tau_k + z\nu_k)$$

$$= \alpha^{(i)} (z - \lambda_1^{(i)})(z - \lambda_2^{(i)})$$

where

$$\alpha^{(i)} = \Sigma\, a_{jk}^{(i)} \nu_j \nu_k,\ \lambda_1^{(i)} \lambda_2^{(i)} \alpha^{(i)} = \Sigma\, a_{jk}^{(i)} \tau_j \tau_k$$

$$(\lambda_1^{(i)} + \lambda_2^{(i)})\, \alpha^{(i)} = -2\, \Sigma\, a_{jk}^{(i)} \nu_j \tau_k.$$

We assume $\mathcal{I}\lambda_1^{(i)} > 0$, $\mathcal{I}\lambda_2^{(i)} < 0$, so that Hypothesis 3 is satisfied. Moreover, $P_i^+(z) = z - \lambda_1^{(i)}$. If the boundary triple is on Σ_2, $Q_{21}(z) =$

$= \Sigma\, a_{jk}^{(2)}\, \nu_j\, (\tau_k + z\nu_k) = \alpha^{(2)}\, (z - \mu^{(2)})$, where $2\mu_1^{(i)} = \lambda_1^{(i)} + \lambda_2^{(i)}$. Hence $Q_{21}\,(z)$ can never vanish for $z = \lambda_1^{(2)}$ and Hypothesis 5 is satisfied.

Next, assume that the boundary triple is on Σ_0. We must show that the identities

$$\gamma_1\, \beta_{i1} + \gamma_2\, \beta_{i2}\, \alpha^{(i)}\, (z - \mu^{(i)}) \equiv C^{(i)}\, (z - \lambda_1^{(i)}), \qquad i = 1, 2, \tag{2.24}$$

imply that $\gamma_1 = \gamma_2 = 0$. Here the $C^{(i)}$ are complex constants and we have made use of the fact that

$$R_{i2} = \beta_{i2}\, \alpha^{(i)}\, (z - \mu^{(i)}), \qquad i = 1,\ 2.$$

Now (2.24) implies that

$$C^{(i)} = \gamma_2\, \beta_{i2}\, \alpha^{(i)},\ \gamma_1\, \beta_{i1} + \gamma_2\, \beta_{i2}\, \sigma^{(i)} = 0, \qquad i = 1, 2,$$

where $\sigma^{(i)} = \alpha^{(i)}\, (\lambda_1^{(i)} - \mu^{(i)}) = \dfrac{1}{2}\, \alpha^{(i)} (\lambda_1^{(i)} - \lambda_2^{(i)})$

$= \pm \sqrt{(\Sigma\, a_{jk}^{(i)}\, \nu_j\, \tau_k)^2 - \alpha\, \Sigma\, a_{jk}^{(i)}\, \tau_j\, \tau_k}$. Hence the γ_i must equal zero if the determinant

$$\begin{vmatrix} \beta_{11} & \sigma^{(1)} & \beta_{12} \\ \beta_{21} & \sigma^{(2)} & \beta_{22} \end{vmatrix} \tag{2.25}$$

does not vanish on Σ_0.

Finally, we assume that the boundary triple is on S. We first note that

$$B'_{11} = 1\ ;\ B'_{21} = \Sigma\, \overline{a_{jk}^{(2)}}\, \nu_j\, \frac{\partial}{\partial x_k} + b^{(0)},$$

$$C'_{i1} = \overline{\beta}_{i1}\ ;\ C'_{i2} = \beta_{i2}\, \Sigma\, \overline{a_{jk}^{(i)}}\, \nu_j\, \frac{\partial}{\partial x_k} + b^{(i)'},\ i = 1, 2$$

where the $b^{(i)'}$ depend on the $a_{jk}^{(i)}$ and the $b^{(i)}$. Thus

$$Q'_{11}\,(z) = 1\ ;\ Q'_{21} = \overline{\alpha^{(2)}}\, (z - \overline{\mu^{(2)}})$$

$$R'_{i1} = \overline{\beta}_{i1}\ ;\ R'_{i2} = \overline{\beta}_{i2}\, \alpha^{(i)}\, (z - \overline{\mu^{(i)}}),\ i = 1, 2$$

$$H'_{i1}\,(z) = z - \overline{\lambda_2^{(i)}},\ i = 1, 2.$$

Now let $\omega^{(i)} = (\omega_0^{(i)}, \omega_1^{(i)})$, $i = 1, 2$, be two complex vectors. Thus

$$H'_{i1}\,(\omega^{(i)}) = \omega_1^{(i)} - \overline{\lambda_2^{(i)}}\, \omega_0^{(i)}, i = 1, 2.$$

Hence, when $H'_{i1}(\omega^{(i)}) = 0$,

$$R_{i2}(\omega^{(i)}) = \overline{\beta}_{i2}\, \overline{\alpha^{(i)}}\, (\overline{\lambda_2^{(i)}} - \overline{\mu^{(i)}})\, \omega_0^{(i)}$$

$$= -\overline{\beta}_{i2}\, \overline{\sigma^{(i)}}\, \omega_0 .$$

In order to satisfy Hypothesis 9 we must therefore show that there are polynomials e_s in τ such that the real part of

$$(\overline{\beta}_{11}\, \omega_0^{(1)} + \overline{\beta}_{21}\, \omega_0^{(2)})\, (e_1\, \overline{\omega_1^{(1)}} + e_2\, \sigma^{(2)}\, \overline{\omega_0^{(2)}})$$

$$+ (\overline{\beta}_{21}\, \overline{\sigma^{(1)}}\, \omega_0^{(1)} + \overline{\beta}_{22}\, \overline{\sigma^{(2)}}\, \omega_0^{(2)})\, (e_3\, \overline{\omega_0^{(1)}} + e_4\, \sigma^{(2)}\, \overline{\omega_0^{(2)}})$$

is positive definite in the $\omega_0^{(i)}$. Moreover the polynomials $e_1, \dots, e_4$ must be homogeneous in the components of τ of orders 3, 2, 2, 1, respectively. It is therefore clear that e_1 and e_4 cannot help in making the expression positive definite. We take them to be zero. We are then left with

$$\mathrm{Re}\, e_3\, \overline{\beta}_{12}\, \overline{\sigma^{(1)}}\, |\, \omega_0^{(1)}\, |^2 + \mathrm{Re}\, e_2\, \overline{\beta}_{21}\, \sigma^{(2)}\, |\, \omega_0^{(2)}\, |^2$$

$$+ \mathrm{Re}\, (e_2\, \overline{\beta}_{11} + \overline{e}_3\, \beta_{22})\, \sigma^{(2)}\, \omega_0^{(1)}\, \overline{\omega_0^{(2)}} .$$

Now by definition, $\mathcal{I}\, \overline{\alpha^{(i)}}\, \sigma^{(i)} > 0,\ i = 1, 2$. From this we can easily show that the expression can be made positive in the $\omega_0^{(i)}$ if $Z \equiv \beta_{11}\, \beta_{22}\, \overline{\beta}_{21}\, \overline{\beta}_{12}\, \overline{\alpha^{(1)}}\, \alpha^{(2)}$ is real and negative on S. For then we may take

$$e_2 = -i\, |\, \beta_{22}\, |^2\, \beta_{21}\, \overline{\alpha^{(2)}}\, |\, \tau\, |^2;\ e_3 = -i\, \beta_{11}\, \beta_{22}\, \overline{\beta_{21}}\, \alpha^{(2)}\, |\, \tau\, |^2$$

For then

$$e_3\, \overline{\beta}_{12}\, \overline{\sigma^{(i)}} = -\beta_{11}\, \beta_{22}\, \overline{\beta}_{21}\, \beta_{12}\, \alpha^{(2)}\, (i\, \overline{\sigma^{(i)}})\, |\, \tau\, |^2$$

$$= -Z\, |\, \alpha^{(1)}\, |^{-2}\, (i\, \alpha^{(i)}\, \overline{\sigma^{(i)}})\, |\, \tau\, |^2,$$

$$e_2\, \overline{\beta}_{21}\, \sigma^{(2)} = |\, \beta_{22}\, \beta_{21}\, |^2\, (-\, i\, \overline{\alpha^{(2)}}\, \sigma^{(2)})\, |\, \tau\, |^2,$$

both of which have positive real parts, and

$$e_2\, \overline{\beta}_{11} + \overline{e}_3\, \beta_{22} = -\overline{\beta}_{11}\, |\, \beta_{22}\, |^2\, \beta_{21}\, \overline{\alpha^{(2)}}\, |\, \tau\, |^2$$

$$+ i\, \overline{\beta}_{11}\, |\, \beta_{22}\, |^2\, \beta_{21}\, \overline{\alpha^{(2)}}\, |\, \tau\, |^2 = 0.$$

Mareover, if $Z < 0$ on the whole of $\bar{\Sigma}_0$, then (2.25) cannot vanish there. For otherwise we vould have

$$\beta_{11}\,\beta_{22}\,\bar{\beta}_{12}\,\bar{\beta}_{21}\,\overline{\alpha^{(1)}}\,\alpha^{(2)}\,\overline{\alpha^{(2)}}\,\sigma^{(2)}$$
$$= |\,\beta_{12}\,\beta_{21}\,\alpha^{(2)}\,|^2\,\overline{\alpha^{(1)}}\,\sigma^{(1)}.$$

which is impossible. For the imaginary part of the right hand side is positive, while of the left hand side is negative. Thus Theorem 2.1 applies when $Z < 0$ on Σ_0. Notice, that when $A_1 = A_2$ on $\bar{\Sigma}_0$, this criterion reduces to $\beta_{11}\,\beta_{22}\,\bar{\beta}_{12}\,\bar{\beta}_{21} < 0$.

A proof of Theorem 2.1 will be given in Section 5 after we discuss a basic inequality in the next section.

3. The Inequality.

An important tool in establishing our existence theorem will be a coerciveness inequality adapted to this particular problem. In it we employ a boundary norm which, while not needed in obtaining a weak solution, enables us to prove smoothness up to the boundary. We shall follow the methods of [15, 16] very closely.

Let x^0 be any point of $\partial G^{(1)}$ (x^0 may be either in $\bar{\Sigma}_0$ or $\bar{\Sigma}_1$). Since $\partial G^{(1)}$ is of class C^∞, there is a neighborhood $\mathcal{N}(x^0)$ of x^0 such that $\partial G^{(1)} \cap \mathcal{N}(x^0)$ can be mapped in a one-to-one C^∞ way onto the hyperplane $\mathcal{T}(x^0)$ tangent to $\partial G^{(1)}$ at x^0. Let φ be a smooth complex valued function defined on and having compact support in $\partial G^{(1)} \cap \mathcal{N}(x^0)$. By the mapping we may consider φ defined on part of $\mathcal{T}(x^0)$. Defining it to be zero on the rest of $\mathcal{T}(x^0)$, we set

$$F_l[\varphi] = \widehat{\varphi}_l(\xi') = \int_{\mathcal{V}_l(x^0)} e^{-i\xi' x'}\,\varphi(x')\,dx',\quad l = 0, 1, \tag{3.1}$$

$$F[\varphi] = F_0[\varphi] + F_1[\varphi], \tag{3.2}$$

where $x' = (x'_1, x'_2, \ldots x'_{n-1})$ is a coordinate system on $\mathcal{T}(x^0)$, $\xi' = (\xi'_1, \xi'_2, \ldots, \xi'_{n-1})$, $\xi' x' = \xi'_1 x'_1 + \xi'_2 x'_2 + \ldots + \xi'_{n-1} x'_{n-1}$, and $\mathcal{V}_l(x^0)$ is the image of $\Sigma_l \cap \mathcal{N}(x^0)$ under the mapping. We then set

$$\begin{aligned} \langle \varphi, \psi \rangle_{x^0,s,l} &= \int |\,\xi'\,|^{2s-1}\,F_l[\varphi]\,\bar{F}_l[\psi]\,d\xi',\quad l = 0, 1;\ s = 0, 1, 2, \ldots, \\ \langle \varphi \rangle^2_{x^0,s,l} &= \langle \varphi, \varphi \rangle_{x^0,s,l} \end{aligned} \tag{3.3}$$

for any two functions φ, ψ, where $\bar{F}_l[\psi]$ denotes the complex conjugate of $F_l[\psi]$. Similarly we set

$$(3.4)\qquad \begin{aligned} \langle \varphi, \psi \rangle_{x^0,s} &= \int |\xi'|^{2s-1} F[\varphi]\, \bar{F}[\psi]\, d\xi' \\ \langle \varphi \rangle^2_{x^0,s} &= \langle \varphi, \varphi \rangle_{x^0,s}. \end{aligned}$$

Analogous definitions are to be made for points x^0 on $\partial G^{(2)}$.

Next assume in addition that φ and ψ vanish near $S = \bar{\Sigma}_0 \cap \bar{\Sigma}_1$ (this is automatically the case when $\mathcal{N}(x^0) \cap S = 0$). If $E(\xi')$ is any polynomial in the components of ξ', we claim that

$$(3.5)\qquad \int E(\xi')\, F_0[\varphi]\, \bar{F}_1[\psi]\, d\xi' = 0.$$

In fact, if we apply Parseval's identity to (3.5), we obtain

$$(3.6)\qquad \int E(D')\, \varphi_0\, \bar{\psi}_1\, dx',$$

where $E(D')$ is a tangential differential operator, φ_0 is the function which equals φ on Σ_0 and equals zero on Σ_1, while ψ_1 has the opposite relationship to ψ. Thus $E(D')\varphi_0 \bar{\psi}_1$ vanishes throughout $\mathcal{T}(x^0)$ making (3.6) equal zero.

Now it follows from (3.5) that

$$\begin{aligned} &\int E(\xi')\, F[\varphi]\, \bar{F}[\psi]\, d\xi' \\ &= \int E(\xi') \{F_0[\varphi]\, \bar{F}[\psi] + F[\varphi]\, \bar{F}_1[\psi]\}\, d\xi', \end{aligned}$$

and hence, if $E(\xi')$ is homogeneous in ξ' of degree s, we have by the Schwarz inequality

$$(3.7)\qquad \begin{aligned} &\left| \int E(\xi')\, F[\varphi]\, \bar{F}[\psi]\, d\xi' \right| \\ &\leq K(\langle \varphi \rangle_{x^0,s,0} \langle \psi \rangle_{x^0,s} + \langle \varphi \rangle_{x^0,s} \langle \psi \rangle_{x^2,s,1}). \end{aligned}$$

It should be borne in mind that when corresponding definitions are made for points of $\partial G^{(2)}$, all of the above relationships hold with the subscript 1 replaced by 2.

For any subset $\mathcal{S}$ of E^n, let $C_0^\infty(\mathcal{S})$ be the collection of all $w \varepsilon C^\infty(\mathcal{S})$ which vanish near the boundary of $\mathcal{S}$. It follows from considerations in [1, 14] that there is a constant K depending only on $\mathcal{N}(x^0)$ and s such that

$$\langle v \rangle_{x^0,s} \leq K \| v \|_s \tag{3.8}$$

for all $v \varepsilon C_0^\infty(\mathcal{N}(x^0))$. From this it follows that

$$\langle v \rangle_{x_0,s,t} \leq K \| v \|_s \qquad t = 0, 1, 2 \tag{3.9}$$

whenever $v^{(i)} \varepsilon C^\infty(G^{(i)}) \cap C_0^\infty(\mathcal{N}(x^0))$ and $\mathcal{N}(x^0)$ is bounded away from S.

Let $\widetilde{C}^\infty(\overline{G})$ be the set of all vector functions $v = (v^{(1)}, v^{(2)})$ such that each $v^{(i)} \varepsilon C^\infty\overline{(G^{(i)})}$ and vanishes near S. Assume that all of the hypotheses of Theorem 2.1 are satisfied. We are going to show for every point $x^0 \varepsilon \overline{G^{(1)}} \cup \overline{G^{(2)}}$ there is a neighborhood $\mathcal{N}(x^0)$ such that for every $\varepsilon > 0$ and every $\zeta \varepsilon C_0^\infty(\mathcal{N}(x^0))$ there is a constant K such that

$$\begin{aligned}\| \zeta u \|_{2r}^2 - \varepsilon \| u \|_{2r}^2 \leq K (\| A^* u \|_0^2 &+ \sum_{i=1}^{2} \sum_{j=1}^{r} \langle \zeta B'_{ij} u^{(i)} \rangle^2_{x^0, 2r - m'_{ij}, i} \\ &+ \sum_{j=1}^{2r} \langle \zeta (C'_{ij} u^{(1)} + C'_{2j} u^{(2)}) \rangle^2_{x^0, 2r - m'_j, 0} + \| u \|_0^2)\end{aligned} \tag{3.10}$$

for all $u \varepsilon \widetilde{C}^\infty(\overline{G})$. As before m'_{ij} is the order of B'_{ij} and m'_j is the order of both C'_{ij} and C'_{2j}. By the compactness of the $\overline{G^{(i)}}$ we know that there is a finite set $\{x^{(k)}\}_{k=1}^p$ of points such that $\overline{G^{(1)}} \cup \overline{G^{(2)}} \subset \bigcup_{k=1}^{p} \mathcal{N}(x^{(k)})$. Let $1 = \Sigma \zeta_k$ be a partition of unity subordinate to this covering. We may assume that the ζ_k are infinitely differentiable. Taking $\varepsilon < \frac{1}{2p}$ in (3.10) we have

$$\begin{aligned}\| u \|_{2r}^2 = \| \Sigma \zeta_k u \|_{2r}^2 &\leq K_1 \Sigma \| \zeta_k u \|_{2r}^2 \\ &\leq K_2 (\| A^* u \|_0^2 + \sum_{i=1}^{2} \sum_{j=1}^{r} \langle B'_{ij} u^{(i)} \rangle^2_{2r - m'_{ij}, i} \\ &+ \sum_{j=1}^{2r} \langle C'_{1j} u^{(1)} + C'_{2j} u^{(2)} \rangle^2_{2r - m'_j, 0} + \| u \|_0^2) + \frac{1}{2} \| u \|_{2r}^2,\end{aligned}$$

where

$$\langle v \rangle^2_{s,t} = \sum_{k=1}^{p} \langle \zeta_k v \rangle_{x^k, s, t} \qquad t = 0, 1, 2. \tag{3.11}$$

Thus

$$\| u \|_{2r}^2 \leq K \Big(\| A^* u \|_0^2 + \sum_{i=1}^{2} \sum_{j=1}^{r} \langle B'_{ij} u^{(i)} \rangle^2_{2r-m'_{ij},i} \tag{3.12}$$

$$+ \sum_{j=1}^{2r} \langle C'_{1j} u^{(1)} + C'_{2j} u^{(2)} \rangle^2_{2r-m'_j,0} + \| u \|_0^2 \Big).$$

This is the appropriate coerciveness inequality which will be used in the next section in our existence and regularity proof. As we have just seen, it can be proved by establishing inequality (3.10) at each point x^0 of $\overline{G^{(1)}} \cup \overline{G^{(2)}}$. This program will be carried out in Sections 5, 7, and 8.

4. Existence and Regularity of the Solution.

In this section we give a proof of Theorem 2.1. The main tool in our approach in the coerciveness inequality (3.12).

Let H^* be the completion of $\widetilde{C}^\infty(\overline{G})$ with respect to the norm

$$\| v \|^{*2} = \| v \|_{2r}^2 + \sum_{i=1}^{2} \sum_{j=1}^{r} \langle B'_{ij} v^{(i)} \rangle^2_{2r-m'_{ij},i} + \sum_{j=1}^{2r} \langle C'_{1j} v^{(1)} + C'_{2j} v^{(2)} \rangle^2_{2r-m'_j,0}\,. \tag{4.1}$$

Clearly, H^* is a Hilbert space which may be identified with a subset of $H^{2r}(G)$, the completion of $C^\infty(\overline{G^{(1)}}) \times C^\infty(\overline{G^{(2)}})$ with respect tot he norm $\| \ \|_{2r}$. We also set

$$[u, v]^* = (A^* u, A^* v) + \sum_{i=1}^{2} \sum_{j=1}^{r} \langle B'_{ij} u^{(i)}, B'_{ij} v^{(i)} \rangle_{2-rm'_{ij},i}$$

$$+ \sum_{j=1}^{2r} \langle C'_{1j} u^{(1)} + C'_{2j} u^{(2)}, C'_{1j} v^{(1)} + C'_{2j} v^{(2)} \rangle_{2r-m'_j,0}.$$

It is easily seen from (3.12) that $[u, v]^*$ is defined for $u, v \,\varepsilon\, H^*$ and that there is a constant $c > 0$ such that

$$c^{-1} \| v \|^{*2} \leq [v, v]^* + \| v \|_0^2 \leq c \| v \|^{*2} \tag{4.3}$$

for all $v \,\varepsilon\, H^*$. Let N^* be the set of all $v \,\varepsilon\, H^*$ such that $[v, v]^* = 0$. It follows from (4.3) and Rellich's lemma that N^* is finite dimensional. Hence N^* is closed in both H^* and $L^2(G^{(1)}) \times L^2(G^{(2)})$. Now let M^* be the set of all $v \,\varepsilon\, H^*$ such that $(v, N^*) = 0$ (i.e., $(v, w) = 0$ for all $w \,\varepsilon\, N^*$). It follows again from (4.3) and Rellich's lemma that there is a constant $c_1 > 0$ such that

$$c_1^{-1} \| v \|^{*2} \leq [v, v]^* \leq c_1 \| v \|^{*2} \tag{4.4}$$

for all $v \varepsilon M^*$ (cfr. [15]).

Now assume for the moment that f is given and that

$$(f, N^*) = 0. \tag{4.5}$$

In view of (4.4) we may substitute $[u, v]^*$ for the inner product of the Hilbert space M^*. Since (v, f) is a bounded linear functional in M^*, there is a $g \varepsilon M^*$ such that

$$[g, v]^* = (f, v) \tag{4.6}$$

for all $v \varepsilon M^*$. Moreover, we claim that (4.5) implies that (4.6) holds for all $v \varepsilon H^*$. To see this, we first note that by (4.3) the norm $([v, v]^* + \| v \|_0^2)^{1/2}$ is equivalent to $\| v \|^*$ in H^*. Furthermore, since N^* is closed in H^*, every element $v \varepsilon H^*$ can be decomposed into the form $v = v' + v''$, where $v'' \varepsilon N^*$ and

$$[v', N^*]^* + (v', N^*) = 0.$$

But since $[N^*, N^*]^* = 0$, the first term vanishes showing that $v' \varepsilon M^*$. Now (4.5) and (4.6) imply that

$$[g, v]^* = [g, v']^* = (f, v') = (f, v) \tag{4.7}$$

for all $v \varepsilon H^*$, as was asserted.

Taking the special cases when $v = (v^{(1)}, 0)$ or $v = (0, v^{(2)})$ with $v^{(i)} \varepsilon C_0^\infty(G^{(i)})$, we see that

$$(A_i^* g^{(i)}, A_i^* v^{(i)}) = (f^{(i)}, v^{(i)}) \qquad i = 1, 2 \tag{4.8}$$

for all such v. It now follows from the interior regularity theory for strongly elliptic equations (cf. [4, 5, 6, 7, 11]) that the $g^{(i)}$ are in $C^\infty(G^{(i)})$ after correction on a set of measure zero. Integration by parts then shows that $u^{(i)} \equiv A_i^* g^{(i)}$ satisfies $A_i u^{(i)} = f^{(i)}$ in $G^{(i)}$.

We next consider the special cases when $v = (v^{(1)}, 0)$, $v = (0, v^{(2)})$ where the $v^{(i)}$ are in $C^\infty(\overline{G^{(i)}})$ and vanish near Σ_0. Then

$$(A_i^* g^{(i)}, A_i^* v^{(i)}) + \sum_{j=1}^{r} \langle B'_{ij} g^{(i)}, B'_{ij} v^{(i)} \rangle_{2r - m'_{ij}, i} = (f^{(i)}, v^{(i)}) \tag{4.9}$$

for all such v. It now follows from the regularity theory of [15] that $g^{(i)}$ is in $C^\infty(G^{(i)} \cup \Sigma_i)$. Moreover, (4.9) implies that

$$(u^{(i)}, A_i^* v^{(i)}) = (A_i u^{(i)}, v^{(i)})$$

for all $v^{(i)} \varepsilon C^\infty(\overline{G^{(i)}})$ satisfying $B'_{ij} v^{(i)} = 0$ on Σ_i and vanishing near $\bar{\Sigma}_0$. By Remark 2.1 this implies that $B_{ij} u^{(i)} = 0$ on Σ_i.

Finally, we consider the case when $v = (v^{(1)}, v^{(2)})$ with $v^{(1)} \varepsilon C^\infty(\overline{G^{(i)}})$ vanishing near Σ_i. This time we have

$$(4.10) \quad (A^*g, A^*v) = \sum_{j=1}^{2r} \langle C'_{1j} g^{(1)} + C'_{2j} g^{(2)}, C'_{1j} v^{(1)} + C'_{2j} v^{(2)} \rangle_{2r-m'_j,0} = (f, v).$$

By working with the pair of functions $g^{(1)}, g^{(2)}$ simultaneously, We can follow the reasoning of [15] step by step and show that each $g^{(i)}$ is in $C^\infty(G^{(i)} \cup \Sigma_0)$. Since no new ideas are involved, we do not provide the details. Once the regularity is known, (4.10) implies that

$$(u, A^* v) = (Au, v)$$

whenever $v^{(i)} \varepsilon C^\infty(\overline{G^{(i)}})$ vanishes near Σ_i and satisfies

$$(4.11) \qquad C'_1 v^{(1)} + C'_{2j} v^{(2)} = 0 \qquad \text{on} \quad \Sigma_0, j = 1, 2, \dots, 2r,$$

where $u = (u^{(1)}, u^{(2)})$. Hence, by (2.16),

$$\sum_{j=1}^{2r} \int_{\Sigma_0} C_{ij} u^{(1)} \overline{C'_{ij} v^{(1)}} \, ds + \sum_{j=1}^{2r} \int_{\Sigma_0} C_{2j} u^{(2)} \overline{C'_{2j} v^{(2)}} \, ds = 0$$

for all such v. Thus by (4.11)

$$\sum_{j=1}^{2r} \int_{\Sigma_0} (C_{1j} u^{(1)} - C_{2j} u^{(2)}) \overline{C'_{1j} v^{(1)}} \, ds = 0.$$

Since $v^{(1)}$ is otherwise arbitrary, this means that

$$C_{1j} u^{(1)} = C_{2j} u^{(2)} \qquad \text{on} \quad \Sigma_0, j = 1, 2, \dots, 2r.$$

Hence u is a solution of problem $\Pi\,(A, f, B_j, C_j)$.

Since our argument was based on assumption (4.5), our proof will be complete if we can show that N^* is contained in the set of solutions of $\Pi\,(A^* \cdot O, B'_j, C'_j)$. This latter fact, however, has essentially been proved. For we have shown that if g satisfies (4.7) for all $v \varepsilon H^*$, then each $g^{(i)} \varepsilon C^\infty(G^{(i)} \cup \Sigma_i \cup \Sigma_0)$, provided $f \varepsilon C^\infty(\overline{G})$. This is the case, in particular, if we take $f = 0$. Moreover, $[g, g]^* = 0$ implies

$$\| A_i^* g^{(i)} \|_0 = 0 \qquad\qquad i = 1, 2$$

$$\langle B'_{ij}\, g^{(i)} \rangle_{2r - m'_{ij},\, i} = 0 \qquad\qquad i = 1, 2\,;\ j = 1, 2, \dots, r,$$

$$\langle C'_{1j}\, g^{(1)} + C'_{2j}\, g^{(2)} \rangle_{2r - m'_j, 0} = 0, \qquad\qquad j = 1, 2, \dots, 2r.$$

But when the $g^{(i)}$ have the above mentioned regularity properties, these conditions imply that g is a solution of problem $\pi\,(A^*, 0, B'_j, C'_j)$. Hence every element of N^* is a solution of $\pi\,(A^*, 0, B'_j, C'_j)$, and the proof is complete.

REMARK 4.1. In general, not every solution of $\pi\,(A^*, 0, B'_j, C'_j)$ is in N^*. For every element of N^* is in $H^{2r}(G)$, while no such requirement is made on solutions of $\pi\,(A^*, 0, B'_j, C'_j)$. This suggests one way in which our theorem can be strengthened.

5. Points of $G^{(i)} \cup \Sigma_i$.

From the cousideration of Section 3, we see that it remains only to prove (3.10) for each point $x^0 \varepsilon \overline{G^{(1)}} \cup \overline{G^{(2)}}$. In this section we shall show that for points of $G^{(i)} \cup \Sigma_i$, (3.10) follows from known results (cf. [1, 2, 14]). The main difficulty lies in considering points of Σ_0 and S, and these cases will be treated seperately in Section 7 and 8.

First suppose $x^0 \varepsilon G^{(1)}$. We can take the neighborhood $\mathcal{N}(x^0)$ so small that its closure does not intersect $\partial G^{(1)}$. For such points the ellipticity of A_1^* implies

$$\| \zeta u^{(1)} \|_{2r}^2 \leq K\,(\| A_1^* \zeta u^{(1)} \|_0^2 + \| \zeta u^{(1)} \|_0^2) \tag{5.1}$$

for all $\zeta \varepsilon C_0^\infty(\mathcal{N}(x^0))$ and $u^{(1)} \varepsilon C^\infty(\overline{G^{(1)}})$ (cf. [13]). By standard procedures this can be transformed into

$$\| \zeta u^{(1)} \|_{2r}^2 - \varepsilon \| u^{(1)} \|_{2r}^2 \leq K'\,(\| A_1^* u^{(1)} \|_0^2 + \| u^{(1)} \|_0^2), \tag{5.2}$$

where K' depends also on ζ and ε. Inequality (5.2) immediately implies (3.10) for x^0. A similar argument holds for points of $G^{(2)}$.

Next assume that $x^0 \varepsilon \Sigma_1$. We take $\mathcal{N}(x^0)$ so small that its closure does not intersect $\Sigma_0 \cup \Sigma_2$. We then have

$$(5.3) \qquad \| \zeta u^{(1)} \|_{2r}^2 \leq K (\| A_1^* \zeta u^{(1)} \|_0^2 + \sum_{j=1}^{r} \langle B'_{1j} \zeta u^{(1)} \rangle_{x^0, 2r - m'_{1j}, 1}^2 + \| \zeta u_0^{(1)} \|_0^2)$$

for all $\zeta \varepsilon C_0^\infty(\mathcal{N}(x^0))$ and $u^{(1)} \varepsilon C^\infty(\overline{G^{(1)}})$. This follows from [14] once we know the fact, proved in [15], that the set $\{B'_{1j}\}_{j=1}^r$ convers A_1^* whenever $\{B_{1j}\}_{j=1}^r$ covers A_1. That $\{B_{1j}\}_{j=1}^r$ covers A_1, was assumed (Hypothesis 5). Hence (5.3) holds. Again we may transform the inequality into

$$(5.4) \qquad \begin{aligned} &\| \zeta u^{(1)} \|_{2r}^2 - \varepsilon \| u^{(1)} \|_{2r}^2 \leq K' (\| A_1^* u^{(1)} \|_0^2 \\ &+ \sum_{j=1}^{r} \langle \zeta B'_{1j} u^{(1)} \rangle_{x^0, 2r - m'_{1j}, 1}^2 + \| u^{(1)} \|_0^2) \end{aligned}$$

which implies (3.10) for x^0. A similar inequality holds for points of Σ_2.

In Section 7 we shall show that for points $x^0 \varepsilon \Sigma_0$ there is a neighborhood (x^0) such that

$$(5.5) \qquad \| \zeta u \|_{2r}^2 \leq K \left(\| A^* \zeta u \|_0^2 + \sum_{j=1}^{2r} \langle C'_{1j} \zeta u^{(1)} + C'_{2j} \zeta u^{(2)} \rangle_{x^0, 2r - m'_j, 0} + \| \zeta u \|_0^2 \right)$$

when $\zeta \varepsilon C_0^\infty(\mathcal{N}(x^0))$ and $u^{(i)} \varepsilon C^\infty(\overline{G^{(i)}})$.

In addition, we shall prove in Section 8 that

$$(5.6) \qquad \begin{aligned} &\| \zeta u \|_{2r}^2 \leq K \Big(\| A^* \zeta u \|_0^2 \\ &+ \operatorname{Re} \sum_{s=1}^{2} \sum_{j=1}^{2r} \sum_{k=1}^{r} \int E_{sjk}(\xi') F[C'_{1j} \zeta u^{(1)} + C'_{2j} \zeta u^{(2)}] \overline{F}[B'_{sk} \zeta u^{(s)}] \, d\xi' \\ &+ \| \zeta u \|_0^2 \Big) \end{aligned}$$

at points $x^0 \varepsilon S$, where the $E_{sjk}(\xi')$ are the polynomials mentioned in Hypothesis 9. By (3. 7) and (3.9) it follows by standard methods that each of the inequalities (5.5) and (5.6) implies (3.10) on their respective portions of the boundary (cf. [15, 16]). Moreover, by the usual trick of transforming $\mathcal{N}(x^0) \cap \overline{G^{(i)}}$ into a semisphere and approximating the A_i^*, B'_{ij}, C'_{ij} by homogeneous operators with constant coefficients which equal their principle parts at x^0, it is easily shown that either (5.5) or (5.6) holds in $\mathcal{N}(x^0) \cap \overline{G^{(i)}}$ if and only if it holds for homogeneous constant coefficient operators in a semisphere. It therefore remains to prove (5.5) and (5.6) in such « canonical » situations. This will be carried out in Sections 7 and 8 after we degress in the next section to the study of certain algebraic theorems which will be needed.

6. Preliminaries.

We now consider a few unrelated topics which present themselves in the proofs of the next few sections. We state them here for future reference.

LEMMA 6.1. *Let $P(z)$ be a polynomial of degree m with leading coefficient a_0 and having no real roots. Let $H(z)$ be a polynomial of degree $m-1$ which has all the roots of $P(z)$ which lie above the real axis. If b_0 is the leading coefficient of $H(z)$, then*

$$\int_{-\infty}^{\infty} \frac{H(x)}{P(x)}\, dx = -\pi i \frac{b_0}{a_0},$$

where the integral is taken in the Cauchy principle value sense.

Next consider the polynomials

$$R_{ij}(z) = \sum_{k=1}^{2r} c_{ijk} z^{k-1}, \quad i = 1, 2 ; j = 1, 2, \dots, 2r$$

$$P_{ij}^{+}(z) = \sum_{k=1}^{2r} a_{ijk} z^{k-1}, \quad i = 1, 2 ; j = 1, 2, \dots, r,$$

where the a_{ijk} are such that

$$P_{ij}^{+}(z) = z^{j-1} P_{i1}^{+}(z), \; i = 1, 2 ; j = 1, 2, \dots, r.$$

If $\omega^{(1)}, \omega^{(2)}$ are two complex vectors,

$$\omega^{(i)} = (\omega_0^{(i)}, \omega_1^{(i)}, \dots, \omega_{2r-1}^{(i)}), \quad i = 1, 2,$$

and $H(z) = \sum_{k=1}^{2r} a_k z^{k-1}$, we employ the notation

$$H(\omega^{(i)}) = \sum_{k=1}^{2r} a_k \omega_{k-1}^{(i)}.$$

LEMMA 6.2. *If*

$$\sum_{j=1}^{2r} \lambda_j R_{ij}(z) \equiv 0 \bmod P_{i1}^{+}(z), \ i = 1, 2$$

implies that each $\lambda_j = 0$, *then*

$$R_{1j}(\omega^{(1)}) + R_{2j}(\omega^{(2)}) = 0, \ j = 1, 2, \dots, 2r$$

$$P_{ij}^{+}(\omega^{(i)}) = 0, \ i = 1, 2; \ j = 1, 2, \dots, r,$$

imply $\omega^{(1)} = \omega^{(2)} = 0$, *and vice versa.*

The proof of Lemma 6.2 follows from the fact that the following three statements are equivalent.

1.
$$\sum_{j=1}^{2r} \lambda_j R_{1j}(z) + \sum_{j=2r+1}^{3r} \lambda_j P_{1, j-2r}^{+}(z) = 0$$

$$\sum_{j=1}^{2r} \lambda_j R_{2j}(z) + \sum_{j=3r+1}^{4r} \lambda_j P_{2, j-3r}(z) = 0$$

implies $\lambda_j = 0, j = 1, 2, \dots, 4r$.

2.
$$\sum_{j=1}^{2r,} \lambda_j c_{1jk} + \sum_{j=3r+1}^{2r} \lambda_j a_{1, j-2r, k} = 0, \ k = 1, 2, \dots, 2r,$$

$$\sum_{j=1}^{2r} \lambda_j c_{2jk} + \sum_{j=3r+1}^{4r} \lambda_j a_{2. j-r, k} = 0$$

implies $\lambda_j = 0, \ j = 1, 2, \dots, 4r$.

3.
$$\sum_{k=1}^{2r} c_{1jk} \omega_{k-1}^{(1)} + \sum_{k=1}^{2r} c_{2jk} \omega_{k-1}^{(2)} = 0, \ j = 1, 2, \dots, 2r,$$

$$\sum_{k=1}^{2r} a_{1jk} \omega_{k-1}^{(1)} = 0, \ j = 1, 2, \dots, r$$

$$\sum_{k=1}^{2r} a_{2jk} \omega_{k-1}^{(2)} = 0, \ j = 1, 2, \dots, r$$

implies $\omega^{(1)} = \omega^{(2)} = 0$.

Next consider the polynomial

$$P(z) = \sum_{k=0}^{m} a_k z^k$$

and set

$$T(z,\zeta)=\frac{P(z)-P(\zeta)}{z-\zeta}=\sum_{k=1}^{m} a_k \sum_{s=1}^{k} z^{k-s}\zeta^{s-1}$$

(6.1)

$$=\sum_{s=1}^{m}\zeta^{s-1}\sum_{k=s}^{m} a_k z^{k-s}=\sum_{s=1}^{m}\zeta^{s-1} V_s(z),$$

where

$$V_s(z)=\sum_{k=s}^{m} a_k z^{k-s}, \quad s=1,2,\dots,m.$$

Now assume that $P(z)=P^+(z)\,P^-(z)$, where $P^+(z)$ and $P^-(z)$ are polynomials of degree $r\leq m$ and $m-r$, respectively. Set

(6.2) $$H_j(z)=z^{j-1}P^+(z),\ j=1,2,\dots,m-r.$$

For each polynomial $H_j(z)$ we can define $T_j(z,\zeta)$ corresponding to $T(z,\zeta)$ by means of a formula similar to (6.1). Moreover, it is easily checked that for any complex vector $\omega=(\omega_0,\omega_1,\dots,\omega_{m-1})$ the polynomial $H_j(z)$ $T(z,\omega)-P(z)\,T_j(z,\omega)$ is of degree $m-1$. Furthermore, the coefficient of z_{m-1} is $a_m H_j(\omega)$. Hence, by Lemma 6.1 we have

LEMMA 6.3. *If $P(z)$ has no real roots and $P^+(z)$ has all the roots of $P(z)$ (with multiplicities) which lie above the real axis, then*

$$\int_{-\infty}^{\infty}\{P^{-1}(x)\,H_j(x)\,T(x,\omega)-T_j(x,\omega)\}\,dx=-\pi i H_j(\omega),$$

$$j=1,2,\dots,m-r$$

where the integral is taken in the Cauchy principle value sense.

LEMMA 6.4. *The relation*

(6.3) $$T(z,\omega)\equiv 0 \bmod P^-(z)$$

is equivalent to

(6.4) $$H_j(\omega)=0,\ j=1,2,\dots,m-r.$$

Proof: It is easily shown by induction that

(6.5) $$\frac{\partial^t T}{\partial z^t}(z_0,\zeta)=\frac{t\,!\,P(\zeta)}{(\zeta-z_0)^{t+1}}$$

when z_0 is a root of $P(z)$ of multiplicity greater than t. Moreover, if z_k, $k=1,2,\dots,p$, are the distinct roots of $P^-(z)$ with multiplicities ν_k, then (6.3) implies

$$\frac{\partial^t T}{\partial z^t}(z_k, \omega) = 0, \; k = 1, 2, \dots, p; \; 0 \leq t < \nu_k. \tag{6.6}$$

Set

$$T_{kt}(z) = \frac{P^-(z)}{(z - z_k)^t}, \; k = 1, 2, \dots, p; \; 0 \leq t < \nu_k.$$

It is easily checked that the $T_{kt}(z)$ are linearly independent and since there are r of them, there are numbers α_{jkt} such that

$$z^{k-1} = \sum_{k,t} \alpha_{jkt} T_{kt}(z), \; j = 1, 2, \dots, r. \tag{6.7}$$

Now set

$$P_{kt}(\zeta) = P^+(\zeta)\, T_{kt}(\zeta) = \frac{P(z)}{(\zeta - z_k)^t} = \frac{1}{t!} \frac{\partial^t T}{\partial z^t}(z_k, \zeta)$$

by (6.5). Thus (6.6) implies

$$P_{kt}(\omega) = 0, \; k = 1, 2, \dots, p; \; 0 \leq t < \nu_k. \tag{6.8}$$

But from (6.2) and (6.7), we have

$$H_j(z) = \sum_{k,t} \alpha_{jkt} T_{kt}(z) P^+(z) = \sum_{k,t} \alpha_{jkt} P_{kt}(z).$$

Hence (6.8) implies

$$H_j(\omega) = \sum_{k,t} \alpha_{jkt} P_{kt}(\omega) = 0, \; j = 1, 2, \dots, m - r.$$

Conversely, (6.4) implies (6.8), which gives in turn (6.6) and (6.3). This completes the proof.

Now consider the polynomials

$$R_j(z) = \sum_{s=1}^{j} \nu_{js} z^{s-1}, \; j = 1, 2, \dots, m,$$

where $\nu_{ij} \neq 0$, $j = 1, 2, \dots, m$. One easily finds coefficients β_{sj} such that

$$z^{s-1} = \sum_{j=1}^{s} \beta_{sj} R_j(z). \tag{6.9}$$

Thus

$$\sum_{s=t}^{j} \nu_{js}\, \beta_{st} = \delta_{jt}\,, \qquad 1 \leq t \leq j \leq m, \tag{6.10}$$

where δ_{jt} is the Kronecker delta.

LEMMA 6.5. *For any given set $\lambda_1, \lambda_2, \dots, \lambda_m$ of complex numbers, there is a complex vector $\omega = (\omega_0, \omega_1, \dots, \omega_{m-1})$ such that*

$$R_j(\omega) = \lambda_j\,, \qquad j = 1, 2, \dots, m.$$

Proof: We set

$$\omega_{s-1} = \sum_{t=1}^{s} \lambda_t\, \beta_{st}\,, \qquad s = 1, 2, \dots, m.$$

Then

$$R_j(\omega) = \sum_{s=1}^{j} \nu_{js} \sum_{t=1}^{s} \lambda_t\, \beta_{st} = \sum_{t=1}^{j} \lambda_t \sum_{s=t}^{j} \nu_{js}\, \beta_{st} = \sum_{t=1}^{j} \lambda_t\, \delta_{jt} = \lambda_j,$$

which was to be proved.

LEMMA 6.6. *If we set*

$$R'_{m-j}(z) = \sum_{s=j}^{m} \overline{\beta}_{sj}\, \overline{V}_s(z), \qquad j = 1, 2, \dots, m$$

where $\overline{Q}(z)$ denotes the polynominal with coefficients which are the complex coefficients of those of $Q(z)$, then

$$T(z, \zeta) = \sum_{j=1}^{m} R_j(\zeta)\, \overline{R}'_{m-j}(z).$$

Proof: by (6.1) and (6.9)

$$T(z, \zeta) = \sum_{s=1}^{m} \zeta^{s-1}\, \nu_s(z) = \sum_{s=1}^{m} \sum_{j=1}^{s} \beta_{st}\, R_j(\zeta)\, \nu_s(z)$$

$$= \sum_{j=1}^{m} R_j(\zeta) \sum_{s=j}^{m} \beta_{sj}\, \nu_s(z) = \sum_{j=1}^{m} R_j(\zeta)\, \overline{R}'_{m-j}(z).$$

Finally, we consider two polynomials

$$P_i(z) = \sum_{k=0}^{m} a_{ik}\, z^k, \qquad i = 1, 2.$$

For each of then we can define $T^{(i)}(z, \zeta)$ and the $V_{is}(z)$ by means of formula (6.1). Also, if we are given two sets of polynomials

$$R_{ij}(z) = \sum_{s=1}^{j} \gamma_{ijs} z^{s-1}, \qquad i = 1, 2,\ j = 1, 2, \ldots, m,$$

we can find coefficients β_{isj} such that

$$z^{s-1} = \sum_{j=1}^{s} \beta_{isj} R_{ij}(z), \qquad i = 1, 2\,;\ s = 1, 2, \ldots, m.$$

$$\sum_{s=1}^{j} \gamma_{isj} \beta_{ist} = \delta_{jt}\,. \qquad i = 1, 2\,;\ 1 \leq t \leq j \leq m.$$

Note that in these and future formulas we shall not employ the summation convention. Assume that $P_i(z) = P_i^{+}(z) P_i^{-}(z)$, where the degree of $P_i^{+}(z)$ is r. Then define

$$H_{ij}(z) = z^{j-1} P_i^{+}(z), \qquad i = 1, 2,\ j = 1, 2, \ldots, m - r,$$

and

$$R'_{i,m-j}(z) = \sum_{s=j}^{m} \bar{\beta}_{isj} \bar{V}_{is}(z), \qquad i = 1, 2\,;\ j = 1, 2, \ldots m.$$

We now have

THEOREM 6.1. *Assume $m = 2r$. If*

$$\sum_{j=1}^{m} \lambda_j R_{ij}(z) = 0 \quad mod \quad P_i^{-}(z), \qquad i = 1, 2 \tag{6.11}$$

implies that all the λ_j vanish, then

$$\sum_{j=1}^{m} \lambda'_j R'_{ij}(z) \equiv 0 \quad mod \quad P_i^{-}(z), \qquad i = 1, 2 \tag{6.12}$$

implies that all the λ'_j vanish, and vice versa.

Proof: Assume that (6.11) implies that all of the λ_j vanish and that (6.12) holds. We shall prove that all of the λ'_j vanish. By Lemma 6.5

$$T^{(i)}(z, \zeta) = \sum_{j=1}^{m} R_{ij}(\zeta) \bar{R}'_{i,m-j}(z). \tag{6.13}$$

Moreover, Lemma 6.5 shows that there are complex vectors $\omega^{(1)}$ and $\omega^{(2)}$ such that

$$R_{1j}(\omega^{(1)}) = R_{2j}(\omega^{(2)}) = \bar{\lambda}'_{m-j}, \qquad j = 1, 2, \ldots, m. \tag{6.14}$$

Thus

$$T^{(i)}(z, \omega^{(i)}) = \sum_{j=1}^{m} \lambda'_j R'_{ij}(z) \equiv 0 \quad \text{mod} \quad P_i^-(z), \qquad i = 1, 2,$$

by (6.12) and (6.13). But this is equivalent, by Lemma 6.4, to

$$H_{ij}(\omega^{(i)}) = 0, \qquad i = 1, 2 \,;\, j = 1, 2, \dots, r. \tag{6.15}$$

But it follows from Lemma 6.2 that (6.14) and (6.15) imply $\omega^{(1)} = \omega^{(2)} = 0$. Hence the λ'_j vanish and the first stratement is proved. The converse is proved in similar fashion.

7. Points of Σ_0.

In this section we shall prove (5.5) when $G^{(1)}$ is the semisphere $x_n > 0$, $|x| < 1$, $G^{(2)}$ is the semisphere $x_n < 0$, $|x| < 1$, and the coefficients of the A_i and the C_{ij} are constants. In such a case (5.5) readily follows form

$$\| v \|_{2r}^2 \leq K (\| A^* v \|_0^2 + \Sigma \langle C'_{1j} v^{(1)} + C'_{2j} v^{(2)} \rangle^2_{0, 2r - m'_j, 0} + \| v \|_0^2) \tag{7.1}$$

holding for all $v^{(i)} \varepsilon C^\infty(G^{(i)})$ which vanish near $|x| = 1$. Here the boundary norm is taken over Σ_0, which in this case is the set $|x| < 1, x_n = 0$. To convert (7.1) into (5.5), we substitute $v = \zeta u$, where $u^{(i)} \varepsilon C^\infty(G^{(i)})$ and ζ is a C^∞ function which vanishes near $|x| = 1$. The error terms are the handley by standard techniques (cf. [14, 16]).

We next note that Theorem 6.1 shows that the C'_{ij} satisfy Hypothesis 8 with respect to the A_i^*. Hence (7.1) will be proved if we can show, employing only Hypothesis 8, that a similar inequality holds for the A_i and C_{ij}. Thus the asterisks may be dropped in (7.1).

Define $v^{(i)}$ to be identically zero outside $G^{(i)}$ and consider $v = (v^{(1)}, v^{(2)})$ as a vector function defined on E^n. Let

$$\tilde{v}(\xi', \eta) = \int e^{i(x'\xi' + x_n \eta)} v(x', x_n)\, dx'\, dx_n$$

be the Fourier transform of v, where $\xi' = (\xi_1, \xi_2, \dots, \xi_{n-1})$ corresponds to $x' = (x_1, x_2, \dots, x_{n-1})$, and η corresponds to x_n. Set

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, P(\xi', \eta) = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}, P^+(\xi', \eta) = \begin{pmatrix} P_1^+ & 0 \\ 0 & P_2^+ \end{pmatrix}$$

(cf. (2.7)), and

$$H_j(\xi', \eta) = |\xi'|^{r-j}\, \eta^{j-1}\, P^+(\xi', \eta), \qquad j = 1, 2, \dots, r. \tag{7.2}$$

In addition, we put

$$\Omega^{(i)} = (\Omega_0^{(i)}, \Omega_1^{(i)}, \dots, \Omega_{2r-1}^{(i)}), \qquad i = 1, 2,$$

$$W^{(i)} = (W_0^{(i)}, W_1^{(i)}, \dots, W_{2r-1}^{(i)}), \qquad i = 1, 2,$$

$$\omega^{(i)} = (\omega_0^{(i)}, \omega_1^{(i)}, \dots, \omega_{2r-1}^{(i)}), \qquad i = 1, 2,$$

where

$$\Omega_s^{(i)} = D_n^s\, v^{(i)} = \eta^s\, \tilde{v}^{(i)} + \sum_{\nu=0}^{s-1} \eta^{s-\nu-1}\, F[D_n^\nu\, v^{(i)}], \tag{7.3}$$

$$W_s^{(i)} = F[D_n^s\, v^{(i)}], \tag{7.4}$$

and

$$\omega_s^{(i)} = |\xi'|^{-s-1}\, W_s^{(i)}, \tag{7.5}$$

(cf. [14]). It was proved in [14] that for any function $w \in C^\infty(\overline{G^{(i)}})$ vanishing near $|x| = 1$ and outside $G^{(i)}$

$$\int \tilde{w}\, d\eta = -\pi\, i F[w]. \tag{7.6}$$

Hence

$$\int \Omega^{(i)}\, d\eta = -\pi\, i W^{(i)}, \qquad i = 1, 2. \tag{7.7}$$

If $g(\xi', \eta) = \sum_{k=1}^{2r} g_k(\xi')\, \eta^{k-1}$ is a polynomial in η, we employ the notation

$$g(\xi', \Omega^{(i)}) = \sum_{k=1}^{2r} g_k(\xi')\, \Omega_{k-1}^{(i)}, \qquad i = 1, 2,$$

wit similar definitions for $g(\xi', W^{(i)})$ and $g(\xi', \omega^{(i)})$. If

$$U(\xi', \eta) = \begin{pmatrix} g_{11}(\xi', \eta) & g_{12}(\xi', \eta) \\ g_{21}(\xi', \eta) & g_{22}(\xi', \eta) \end{pmatrix}$$

is a matrix in which each $g_{il}(\xi', \eta)$ is a polynomial in η, we set

$$U(\xi', \Omega) = \begin{pmatrix} g_{11}(\xi', \Omega^{(1)}) & g_{12}(\xi', \Omega^{(2)}) \\ g_{21}(\xi', \Omega^{(1)}) & g_{22}(\xi', \Omega^{(2)}) \end{pmatrix},$$

with similar notation for $U(\xi' W)$, $U(\xi', \omega)$. From (7.7) we have

$$\int U(\xi', \Omega)\, d\eta = U(\xi', \int \Omega\, d\eta) = -\pi i U(\xi', W). \tag{7.8}$$

By (7.3) we have

$$\widetilde{Av} = P\tilde{v} + T, \tag{7.9}$$

where $T = (T^{(1)}, T^{(2)})$ and

$$T^{(i)} = \sum_{\nu=1}^{2r} \sum_{s=0}^{\nu-1} \eta^{\nu-s-1} W_s \sum_{|\mu|+\nu=2r} a_{i\mu} \xi'^{\mu}, \qquad i = 1, 2, \tag{7.10}$$

(cf. [14]). Similarly,

$$H_j(\xi', \Omega) = H_j(\xi', \eta)\tilde{v} + T_j, \qquad j = 1, 2, \dots, r, \tag{7.11}$$

where the T_j are similar to T. We therefore have

$$\begin{aligned} |\widetilde{Av}|^2 &= |P\tilde{v}|^2 + 2 \operatorname{Re} T^* P\tilde{v} + |T|^2 \\ &= |P\tilde{v}|^2 + 2 \operatorname{Re}(T^* P - \Sigma \lambda_j^* H_j)\tilde{v} + |T - \bar{P}^{-1} \Sigma H_j^* \lambda_j|^2 \\ &\quad - 2 \operatorname{Re} \Sigma \lambda_j^* T_j + 2 \operatorname{Re} \Sigma \lambda_j^* H_j(\xi', \Omega) + 2 \operatorname{Re} \Sigma \lambda_j^* H_j P^{-1} T \\ &\quad - |P^{-1} \Sigma H_j^* \lambda_j|^2 \\ &= |P\tilde{v} + M|^2 + 2 \operatorname{Re} \Sigma \lambda_j^* H_j(\xi', \Omega) \\ &\quad + 2 \operatorname{Re} \Sigma \lambda_j^* (H_j P^{-1} T - T_j) - |\bar{P}^{-1} \Sigma H_j^* \lambda_j|^2, \end{aligned} \tag{7.12}$$

where the asterisk denotes the conjugate transpose of a matrix, $H_j = H_j(\xi', \eta)$, and the λ_j are vector functions of ξ' to be chosen later. We have also set

$$M = T - P^{-1} \Sigma \lambda_j^* H_j$$

and made use of the fact that $(P^{-1})^* = \bar{P}^{-1}$.

Anticipating our next step, we note that

$$H_j\ P^{-1}\ T - T_j = \begin{pmatrix} H_j^{(1)} & 0 \\ 0 & H_j^{(2)} \end{pmatrix} \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2^{-1} \end{pmatrix} \begin{pmatrix} T^{(1)} \\ T^{(2)} \end{pmatrix} - \begin{pmatrix} T_j^{(1)} \\ T_j^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} P_1^{-1}\ H_j^{(1)}\ T^{(1)} - T_j^{(1)} \\ P_2^{-1}\ H_j^{(2)}\ T^{(2)} - T_j^{(2)} \end{pmatrix}.$$

Hence, by Lemma 6.3,

$$\int (H_j\ P^{-1}\ T - T_j)\, d\eta = -\pi\, i H_j\ (\xi', W). \tag{7.13}$$

Integrating (7.12) with respect to η, we have by (7.8) and (7.13)

$$\int | \widetilde{Av} |^2\, d\eta = \int | \widetilde{Pv} + M |^2\, d\eta$$

$$+ 4\, \mathrm{Re}\, (-\pi\, i)\, \Sigma\, \lambda_j^*\ H_j\ (\xi,\ W) - \int | \bar{P}^{-1}\, \Sigma\, H_j^*\ \lambda_j |^2\, d\eta.$$

We now pick

$$\lambda_j = -\, 2\pi\, i\varepsilon\, |\, \xi'\, |\, H_j\ (\xi,\ W), \qquad j = 1, 2, \dots, r,$$

where $\varepsilon > 0$ is a constant to be chosen later. We then have

$$\int | \widetilde{Av} |^2\, d\eta = \int | P\, \widetilde{v} + M |^2\, d\eta \tag{7.14}$$

$$+ 2\, \varepsilon^{-1}\, |\, \xi'\, |^{-1}\, \Sigma\, |\, \lambda_j |^2 - \int | \bar{P}^{-1}\, \Sigma\, H_j^*\, \lambda_j |^2\, d\eta.$$

Now the last term on the right hand side of (7.14) is a quadratic form in the λ_j. Moreover, each coefficient is a homogeneous function of ξ^0 of degree $-\ 1$ (cf. [14]). Hence there is a constant $\varepsilon > 0$ such that

$$|\, \xi'\, |^{-1}\, \Sigma\, |\, \lambda_j\, |^2 \geq \varepsilon \int | \bar{P}^{-1}\, \Sigma\, H_j^*\ \lambda_j |^2\, d\eta$$

for all possible values of the λ_j. Inserting this value of ε in (7.14) gives

$$\int | \widetilde{Av} |^2\, d\eta \geq \int | P\, \widetilde{v} + M |^2\, d\eta + \varepsilon^{-1}\, |\, \xi'\, |^{-1}\, \Sigma\, |\, \lambda_j\, |^2. \tag{7.15}$$

The stage is now set for applying Lemma 6.4 of [14]. We need only to recall that

$$F[C_{1j}\, v^{(1)} + C_{2j}\, v^{(2)}] = R_{1j}(\xi', W^{(1)}) + R_{2j}(\xi', W^{(2)}).$$

We then have

$$\begin{aligned}(7.16)\qquad &\int |\tilde{A} v|^2\, d\eta + |\xi'|\, \Sigma\, |F[C_{1j}\, v^{(1)} + C_{2j}\, v^{(2)}]|^2 \\ &\geq \int |P\tilde{v} + M|^2\, d\eta + \varepsilon^{-1}\, |\xi'|^{-1}\, \Sigma\, |\lambda_j|^2 \\ &+ |\xi'|\, \Sigma\, |R_{1j}(\xi', W^{(1)}) + R_{2j}(\xi', W^{(2)})|^2.\end{aligned}$$

All of the homogeneity requirements are satisfied. It only remains to show that the sum of the last two terms on the right hand side of (7.16) cannot vanish for $\xi' \neq 0$ unless $W^{(1)} = W^{(2)} = 0$. This is indeed so. For the vanishing of the λ_j implies

$$(7.17)\qquad H_j(\xi', W) = 0, \qquad j = 1, 2, \ldots, r.$$

The vanishing of the last term implies

$$(7.18)\qquad R_{1j}(\xi', W^{(1)}) + R_{2j}(\xi', W^{(2)}) = 0, \quad j = 1, 2, \ldots, 2r$$

But by Lemma 6. 2, Hypotesis 8 then implies $W^{(1)} = W^{(2)} = 0$. This completes proof.

8. Points on S.

In this section we shall prove (5.6) for points $x_0 \,\varepsilon\, S$. By hypothesis, for every such point x^0 there is a neighborhood $\mathcal{N}(x^0)$ such that $\overline{G^{(1)}} \cap \overline{\mathcal{N}(x^0)}$ can be mapped in a one-to-one C^∞ way onto the semisphere $x_n \geq 0, |x| \leq 1$. Similarly, $\overline{G^{(2)}} \cap \overline{\mathcal{N}(x^0)}$ can be mapped in such a way onto $x_h^n \leq 0, |x| \leq 1$. We may assume that points of Σ_0 have the same images under both mappings. By the usual procedure we reduce the problem to proving

$$\begin{aligned}(8.1)\qquad &\| v \|_{2r}^2 \leq K\,(\| A^* v \|_0^2 \\ &+ \mathrm{Re} \sum_{s=1}^{2} \sum_{j=1}^{2r} \sum_{k=1}^{r} \int E_{sjk}(\xi')\, F[C'_{1j}\, v^{(1)} + C'_{2j}\, v^{(2)}]\, F[B'_{sk}\, v^{(s)}]\, d\xi')\end{aligned}$$

for vector functions $v = (v^{(1)}, v^{(2)})$ which vanish near the image of $S \cap \mathcal{N}(x^0)$ and near $|x| = 1$, where A^* and the C'_{ij} and B'_{sk} have constant coefficients. The $E_{sjk}(\xi')$ are the homogeneous functions mentioned in Hypothesis 9.

We employ the analague of (7.15) for A^*, namely

$$(8.2) \qquad \int |\widetilde{A^* v}|^2 \, d\eta \geq \int |\bar{P}\tilde{v} + M'|^2 \, d\eta + \varepsilon^{-1} |\xi'|^{-1} \Sigma |\lambda'_j|^2,$$

where M' and the λ'_j have the same relationship to A^* as M and the λ_j have to A. We now obtain our result by showing that

$$(8.3) \qquad \alpha J + \varepsilon^{-1} |\xi'|^{-1} \Sigma |\lambda'_j|^2$$

never vanishes for $\xi' \neq 0$ unless $W^{(1)} = W^{(2)} = 0$, where J represents the second term on the right hand side of (8.1) and $\alpha > 0$ is to be chosen.

Let $\mathcal{L}$ be the compact set in $8r + n - 1$ Euclidean space for which

$$|\xi'|^2 = |\omega^{(1)}|^2 = |\omega^{(2)}|^2 = 1.$$

(The ξ_k are real while the $\omega^{(i)}_s$ are complex.) Let $\mathcal{L}'$ be the subset of $\mathcal{L}$ of those points for which $\Sigma |\lambda'_j|^2 = 0$, i. e. those points for which

$$\bar{H}_j(\xi', W) = 0, \qquad j = 1, 2, \dots, r.$$

For such points J is positive by Hypothesis 9. By continuity, $J > 0$ on some open set $\mathcal{M}$ containing $\mathcal{L}'$. Moreover, since $\mathcal{L} - \mathcal{M}$ is compact, there is a positive constant $\alpha > 0$ such that

$$\varepsilon\alpha |J| \leq \frac{1}{2} |\xi'|^{-1} \Sigma |\lambda'_j|^2$$

on $\mathcal{L} - \mathcal{M}$. Hence (8.3) is greater than $\frac{\varepsilon^{-1}}{2} |\xi'|^{-1} \Sigma |\lambda'_j|^2 > 0$ on $\mathcal{L} - \mathcal{M}$, while it is greater than $J > 0$ on $\mathcal{M}$. Hence (8.3) is positive on the whole compact set $\mathcal{L}$. By homogeneity (which is easily checked) it is positive for all $\xi' \neq 0$, $\omega^{(1)} \neq 0$ and $\omega^{(2)} \neq 0$. We now apply Lemma 6.4 of [14] and the proof is complete.

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Note added in proof. In the case of second order equations several authors have proved that $u^{(i)}$ satisfies a Hölder condition in $\overline{G^{(i)}}$ (Stampacchia, Campanato, Nikolsky) even when the $\partial G^{(i)}$ are not smooth.

Peetre (mimeographed notes) has also extended the problem to higher order equations (indeed he considers N equations in N domains). His method works for strongly elliptic equations and boundary conditions satisfying somewhat stronger hypotheses than those of the present paper.