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STABLE DISTRIBUTIONS AND LAPLACE TRANSFORMS

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1. A classical result of P. LÉVY enumerates the FOURIER transforms of all densities of probability

$$f = f(x) \qquad (f \geqq 0\,,\ -\infty < x < \infty) \tag{1}$$

which belong to stable distributions, as follows: After a normalization of the unit of length on the x-axis, the functions (1) depend on two real parameters, say α and β, which vary on the range

$$0 < \alpha \leqq 2\,,\ |\beta| \leqq \left|\tan\frac{1}{2}\pi\alpha\right| \tag{2}$$

(with the understanding that the second of the inequalities (2) must be replaced by $|\beta| < \left|\tan\frac{1}{2}\pi\alpha\right|$ if $\alpha = 1$) and, if (α, β) is any point of this parameter range, the corresponding density (1) has a FOURIER transform the inversion of which is

$$f(x) = \frac{1}{\pi}\int_0^\infty \exp(-t^\alpha)\cos(xt + \beta t^\alpha)\,dt\,. \tag{3}$$

Cf., e. g., [2], pp. 257-263 and [3], pp. 356-359.

The representation (3) holds for $-\infty < x < \infty$, and $f(x) = f(-x)$ is not true (except when $\beta = 0$). But it is sufficient to consider (3) on the half-line

$$0 < x < \infty \tag{4}$$

(and at the point $x = 0$, which can however be included by continuity). For, on the one hand, the range (2) goes over into itself if (α, β) is replaced by $(\alpha, -\beta)$ and, on the other hand, (3) shows that the half-line $-\infty < x < 0$ can be reduced to the half-line (4) if β is replaced by $-\beta$.

2. In the limiting case $\alpha = 2$, it follows from (2) that $\beta = 0$, hence (3) becomes the symmetric normal density. If $\alpha > 1$, and if the x in (3) is replaced by the complex variable z, then the resulting integral $f(z)$ is uniformly convergent on every fixed circle $|z| <$ const. and is, therefore, an entire function. This is no longer true in the limiting case $\alpha = 1$, since (3) reduces for $\alpha = 1$ to Cauchy's rational density, $\pi f(x - x_0) = (1 + x^2)^{-1}$, where x_0 corresponds to the parameter β (and is, therefore, arbitrary). In what follows only the remaining case, $\alpha < 1$, will be considered; so that (2) reduces to

$$0 < \alpha < 1 \tag{5}$$

and

$$-\tan\gamma \leqq \beta \leqq \tan\gamma \qquad (0 < \tan\gamma < \infty), \tag{6}$$

where $\gamma = \gamma(\alpha)$ is an abbreviation for

$$\gamma = \frac{1}{2}\pi\alpha \qquad \left(0 < \gamma < \frac{1}{2}\pi\right). \tag{7}$$

The abbreviation $\nu = \nu(\alpha)$ for

$$\nu = (1 - \alpha)/\alpha \qquad (0 < \nu < \infty) \tag{8}$$

will also be used.

It will first be shown that, *under the assumption* (5), *the stable density* (3) *can be represented as an absolutely convergent Laplace integral*

$$f(x) = \int_0^\infty e^{-xu} g(u)\, du \quad \text{on} \quad (4), \tag{9}$$

and that the Unterfunktion of (9) *is*

$$g(u) = \frac{1}{\pi} \exp(-\lambda u^\alpha) \sin(\mu u^\alpha), \tag{10}$$

where $\lambda = \lambda(\alpha, \beta)$ and $\mu = \mu(\alpha, \beta)$ are abbreviations for

$$\lambda = \cos\gamma + \beta\sin\gamma, \quad \mu = \sin\gamma - \beta\cos\gamma \qquad \left(\gamma = \frac{1}{2}\alpha\pi\right). \tag{11}$$

In the particular case of symmetry, $\beta = 0$, this Laplacean representation of $f(x)$ was obtained in [6], pp. 86-88 (even in the multidimensional case of radial symmetry). It will turn out that in the general case, where only (6) is assumed, the proof follows by an application of the same rotation of the complex plane as it did, *loc. cit.*, in the particular case $\beta = 0$.

3. If the integration variable t is replaced by $t^{1/\alpha}$ in (3) and if the $\cos(\ .\ .\ .\ )$ is written as $\operatorname{Re}\exp i(\ .\ .\ .\ )$, then (8) shows that (3) appears in the form

$$\alpha\,\pi\,f(x) = \operatorname{Re} h(x)\,, \tag{12}$$

where

$$h(x) = \int_0^\infty t^\nu \exp\{ixt^{1/\alpha} - (1 - i\beta)\,t\}\,dt \tag{13}$$

$(\nu > 0)$. Let t be thought of as a complex variable and let the half-line $\arg t = 0$, which is the path of integration in (13), be rotated into the position of the half-line $\arg t = \gamma$, where $\gamma = \frac{1}{2}\pi\alpha$. In order to see that, under the assumptions (4) and (5), this deformation of the path of integration is legitimate, it is sufficient to apply Cauchy's theorem to the integrand of (13) and to the positively oriented contour which consists of two segments and of two circular arcs, $t = r$, $t = r\,e^{i\gamma}$ and $t = A\,e^{i\theta}$, $t = B\,e^{i\theta}$, where $0 < A \leqq r \leqq B < \infty$ and $0 \leqq \theta \leqq \gamma\left(< \frac{1}{2}\pi\right)$, and to let $A \to 0$ and $B \to \infty$. The result is that the definition (13) of $h(x)$ is equivalent to

$$h(x) = i\int_0^\infty r\,\exp\{-x\,r^{1/\alpha} - (1 - i\beta)\,e^{i\gamma}\,r\}\,dr\,; \tag{14}$$

cf. (8) and (7) (the factor i in (14) results from $\exp(i\gamma/\alpha) = \exp(i\pi/2) = i$).

Put $u = r^{1/\alpha}$ in (14), where $0 < r < \infty$, hence $0 < u < \infty$, and insert the resulting representation of $h(x)$ into (12). In view of the definitions (12), (11) and (10), this leads to (9) after a trivial calculation.

4. In order to illustrate (9)-(11), consider first the upper limiting case, $\beta = \tan\gamma$, allowed by (6). In this case, $\mu = 0$, by (11). Hence (10) shows that (9) vanishes at every point x of the half-line (4) and, therefore (by continuity) at $x=0$ as well; so that

$$(15)\qquad f(x)\equiv 0 \text{ for } 0\leqq x<\infty \text{ if } \beta=\tan\frac{1}{2}\pi\alpha \qquad (0<\alpha<1).$$

This fact was observed already by LÉVY [2], pp. 261-262.

In order to obtain $f(x)$ for $-\infty<x<0$ in the case $\beta=\tan\gamma$ of (15), it is sufficient to know $f(x)$ on the half-line (4) in the case $\beta=-\tan\gamma$ (simply because (3) is valid for $-\infty<x<\infty$). But if $\beta=-\tan\gamma$, then (11) reduces

$$(16)\qquad \lambda\cos\gamma=\cos 2\gamma,\ \mu=2\sin\gamma \qquad \left(\gamma=\frac{1}{2}\pi\alpha\right).$$

Hence $f(-x)$ is given on (4) by the case (17) of (9)-(10) if $\beta=\tan\frac{1}{2}\pi\alpha$, as in (15).

The mid-point of the range (6), that is, $\beta=0$, belongs, by (3), to the symmetric case, $f(x)=f(-x)$. In this case, (11) reduces to

$$(17)\qquad \lambda=\cos\frac{1}{2}\pi\alpha,\quad \mu=\sin\frac{1}{2}\pi\alpha.$$

5. Another particular case, $\alpha=\frac{1}{2}$, turns out to lead to an elementary density function $f(x)$ for every choice of β on the range (6). This seems to be known only in the limiting cases, $\beta=\pm\tan\frac{1}{2}\pi\alpha$ and $\beta=0$, mentioned before; cf. LÉVY [5], p. 294. But owing to (9)-(11), what is actually involved for any β is hardly different from a calculation of DOETSCH [1], pp. 622-623, to which LÉVY [5], p. 284, refers. In fact, the situation is as follows:

It follows from (8) and (11) that $\nu=1$ and

$$(18)\qquad \lambda=2^{-\frac{1}{2}}(1+\beta),\quad \mu=2^{-\frac{1}{2}}(1-\beta) \text{ if } \alpha=\frac{1}{2}.$$

But (9) and (10) show that, if β is arbitrary an $\alpha=\frac{1}{2}$, then

$$\pi f(x)=\int_0^\infty e^{-xu^2}e^{-\lambda u}\sin\mu u\cdot 2u\,du\equiv -x^{-1}\int_0^\infty e^{-\lambda u}\sin\mu u\,de^{-xu^2},$$

and so a partial integration shows that

$$\pi x f(x) = \operatorname{Im} \int_0^\infty \exp\{-x u^2 - (\lambda - i \mu) u\} \, d u. \tag{19}$$

Since (6) and (7) show that $-1 \leqq \beta \leqq 1$ if $\alpha = \frac{1}{2}$, it follows from (18) and (4) that, by a substitution $u \to cu$ (where $c = c(\beta; x)$ is positive), the integral occurring in (19) can be reduced to

$$\int_0^\infty \exp\{\zeta u - u^2\} \, d u \tag{20}$$

(where $\zeta = \zeta(\beta; x)$ is complex). This proves the assertion, since, according to LAPLACE, the function (20) of ζ is elementary.

6. In what follows, the analytic continuation, say $f(z)$, where $z = x + i y$, of the function (3), where $0 < x < \infty$, will be investigated for arbitrary parameter values (α, β). As mentioned at the beginning of Section 2, the situation is trivial in this regard if $\alpha \geqq 1$. Hence (3) will be assumed. In the symmetric case, $\beta = 0$, the singularities of $f(z)$ have been determined in [7]. In what follows, the entire range, (5)-(7), of (α, β) will be considered.

First, from (9) and (10),

$$\pi f(z) = \int_0^\infty e^{-zu} e^{-\lambda u^\alpha} \sin \mu u^\alpha \, d u \tag{21}$$

if $z = x > 0$. Since both numbers (11) are real, (21) is absolutely, hence uniformly, convergent for $\operatorname{Re} z > \text{const.} > 0$. Consequently, the function $f(z)$ is regular in the half-plane $\operatorname{Re} z > 0$. A by-product of the following considerations will be that $f(z)$ admits of a direct analytic (regular) continuation across every point $z \neq 0$ of the line $\operatorname{Re} z = 0$. But the point $z = 0$ is singular. This can be seen as follows:

Since $\alpha > 0$, it is clear from (3), by uniform convergence, that $D^n f(x)$, where $D = d/d x$, exists for $-\infty < x < \infty$ and $n = 1, 2 \ldots$ (which, in view of (15), implies that $D^n f(0) = 0$ holds for every $n > 0$ if $\beta = \tan \frac{1}{2} \pi \alpha$; so that, since $f(x) \not\equiv 0$, the point $z = 0$ is surely a singularity if $\beta = \tan \frac{1}{2} \pi \alpha$).

It is also seen from (3) that

$$(22) \qquad (-1)^n \pi D^{2n} f(0) = \int_0^\infty e^{-t^\alpha} \cos \beta t \, dt, \qquad D^{2n+1} f(0) = 0 .$$

But the integral (22) can readily be dealt with, and an application of Stirling's formula shows that MacLaurin's series $f(0) + \ldots + D^n f(0)^n/n! + \ldots$ has the radius of convergence ∞, 1 or 0 according as $\alpha > 1$, $\alpha = 1$ or $\alpha < 1$. Since (5) is assumed, this proves that $z = 0$ is a singularity of $f(z)$.

7. It turns out that the determination of the function-theoretical character of $f(z)$ in the large (that is, beyond the half-plane $\operatorname{Re} z > 0$) can be based on (22) if use is made of a device applied in [7] to the particular case $\beta = 0$ of symmetry. The device consists in first replacing f by the function e which, on the half-line (4), is defined by

$$(23) \qquad e(x) = x^{1+\alpha} f(x^\alpha),$$

and then defining $e(z)$, where $z = x + iy$, by means of (21) and (23).

The result will be that $e(1/2)$ *is a transcendental entire function of* z (it is understood that, when $x\,(>0)$ in (23) is replaced by the unrestricted complex variable z, then that initial determination $z^{1/\alpha}$ is meant which is positive for $z = x > 0$). In view of (23) and (5), the italicized assertion concerning the function $e = e_{\alpha\beta}$ implies that, for a fixed α on the interval (5) an for every β compatible with (5) and (6), *the function* $f(z) = f_{\alpha\beta}(z)$ *is a single-valued regular function on* R_∞ *or on some* R_k, *where* $k = 1, 2, \ldots$, *and* $k = k(\alpha)$, *according as* α *is irratonial or rational.* Here R_∞ denotes the Riemann surface of log z (where $z \neq 0$), and R_k, where $k = 1, 2, 3, \ldots$, is the surface which results if the point $z = 0$ is excluded from the Riemann surface of $z^{1/k}$.

In all three cases, *the transcendental nature of the singularity of* $f(z)$ *at* $z = 0$ is not described by (23) (where $e(1/z)$ is entire in z) but is revealed by the fact proved at the end of Section 7, along with the fact that $f(x)$ has derivatives of arbitrarily high order for $-\infty < x < \infty$ (hence, whether $x \to +0$ or $x \to -0$); cf. (22).

REMARK. If $x\,(> 0)$ is replaced by $x^{-1/\alpha}$ in (23), then what results will be the relation into which (23) as it stands now, goes over if the letters e, f are interchanged in it and, at the same time, α is replaced by $1/\alpha$; so that, by virtue of the replacement of $0 < \alpha < 1$ by $1 < \alpha^* < \infty$, where $\alpha^* = 1/\alpha$, the connection between the two functions $f(z)$, $e(z)$ is *involutory.*

For the symmetric case, $\beta = 0$, this reciprocity between $f(z)$ and $e(z)$ was emphasized in [7], p. 681, where the connection of these transcendents with those of Mittag-Leffler was also indicated.

8. The proof of the assertion italicized after (23) proceeds as follows:

Suppose first that x is on the half-line (4) and replace z by $x^{-1/\alpha} > 0$ in (21). Then, if u is replaced by $t = u^{1/\alpha}$, it is seen that

$$\alpha \pi f(x^{-1/\alpha}) = \int_0^\infty t^{-1+1/\alpha} \exp\{-\lambda t - (t/x)^{1/\alpha}\} \sin \mu t \, dt.$$

Hence, if t is replaced by xt (when $x > 0$ is fixed), it follows that

$$\alpha \pi f(x^{-1/\alpha}) = x^{1/\alpha} \int_0^\infty t^\nu \exp(-t^{1/\alpha}) \varphi(xt) \, dt, \tag{24}$$

where the index $\nu = \nu(\alpha) > 0$ and the function φ are defined by (8) and

$$\varphi(t) = e^{-\lambda t} \sin \mu t. \tag{25}$$

If x is replaced by $1/x$ in (24), the definition (23) shows that

$$\alpha \pi z e(1/z) = \int_0^\infty \varphi(zt) t^\nu \exp(-t^{1/\alpha}) \, dt \tag{26}$$

if $z = x > 0$. But let z now be complex and let R be any positive number. Then, if z is in the circle $|z| < R$, it is clear from (25) that the integral (26) is majorized by

$$\int_0^\infty t^\nu \exp(Pt - t^{1/\alpha}) \, dt, \tag{27}$$

where $P = (|\lambda| + |\mu|) R$. Since the convergence of the integral (27) is assured by (5) for every $P > 0$, it follows that the integral (26) is uniformly convergent on every circle $|z| < R$. Consequently, (26) represents an entire function of z.

Finally, it is clear from (25) that the function (26) vanishes as $z = 0$ and remains therefore entire if it is divided by z. This completes the proof of the assertion italicized after (23).

9. It may finally be mentioned that (9), with (10) and (11), can be used in order to obtain a representation for the *angular* stable density, say $f_0(x)$, which, in terms of the *linear* stable density $f(x)$, is defined by

$$f_0(x) = \sum_{n=-\infty}^{\infty} f(x+n) \tag{28}$$

(so that $f_0(x)$ is periodic, of period 1). For the symmetric case, $\beta = 0$, of $f_0(x)$, cf. [8] and [9]. (*)

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(*) Formula (6) of [9] contains an error $(m+1+\lambda$ instead of $(m+1)\lambda+1)$, having been copied incorrectly from the (correct) formulae (3), (7) of [7]. Correspondingly, the function which multiplies c_m in formula (18) of [9] must be correctedto

$$p^{-a} \sum_{n=0}^{\infty} (n+x)^{-b} + \sum_{n=1}^{\infty} (n-x)^{-b},$$

where

$$a = (m+1)^{\lambda} \text{ and } b = (m+1)\lambda + 1 .$$

This correction was kindly communicated to me by Professor S. C. van Veen soon after the appearance of [9].