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SOME FORMULAS IN A RIEMANNIAN SPACE

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In this paper a number of theorems and formulas involving two arbitrary affine connections in a Riemannian space V_n have been established by imposing certain conditions on the affine connections. In section I it has been assumed that the covariant derivatives of the metric tensor of the V_n with respect to the affine connections are the same while in section 2 the torsions of the affine connections have been taken to be the same.

1. Let Γ_{jk}^i and L_{jk}^i be the coefficients of two arbitrary affine connections in a Riemannian space V_n with metric tensor g_{ij} and let a comma and a semicolon denote the covariant derivatives of the g_{ij} 's with respect to the two connections. Then

$$g_{ij,k} - g_{ij;k} = -g_{is}(\Gamma_{jk}^s - L_{jk}^s) - g_{js}(\Gamma_{ik}^s - L_{ik}^s)$$

Putting $T_{jk}^i = \Gamma_{jk}^i - L_{jk}^i$, it follows that

$$g_{ij,k} = g_{ij;k}$$

if and only if

$$(I. 1) \quad g_{is} T_{jk}^s + g_{js} T_{ik}^s = 0$$

Hence we have the following theorem:

THEOREM 1. The covariant derivatives of the g_{ij} 's with respect to two affine connections with coefficients Γ_{jk}^i and $\Gamma_{jk}^i + T_{jk}^i$ are the same if and only if the tensor T_{jk}^i satisfies (I. 1).

As an example it is easy to verify that the above result holds with respect to the coefficients of affine connections

$$\Gamma_{jk}^t \quad \text{and} \quad \Gamma_{jk}^t \pm g^{st}(g_{jk,s} - g_{ks,j})$$

This example is an application of Sen's sequence (3) which is defined as follows:

$$\text{Put } a = \Gamma_{ij}^t, \quad a^* = \Gamma_{ij}^t + g^{mt} g_{im,j}, \quad a' = \Gamma_{ji}^t$$

Then it is known that for every affine connection a there exist uniquely two others a^* and a' which are respectively called the associate and the conjugate of a having the property

$$a^{**} = a'' = a$$

In particular, a is self-associate if $a = a^*$ and self-conjugate if $a = a'$. Now if we construct the sequence

$$(I. 2) \quad a_1 = a, a_2 = a^*, a_3 = a^{*'}, a_4 = a^{**}, a_5 = a^{*''}, \dots$$

then the sequence is a finite cyclic sequence of twelve terms and it is Sen's sequence. In the sequence if we put

$$\begin{aligned} \alpha &= g^{mt} g_{im,j}, & \alpha_c &= g^{mt} g_{jm,i}, & \gamma &= g^{mt} g_{ij,m} = \gamma_c \\ \beta &= g^{mt} g_{is} (\Gamma_{mj}^s - \Gamma_{jm}^s), & \beta_c &= g^{mt} g_{js} (\Gamma_{mi}^s - \Gamma_{im}^s) \end{aligned}$$

and suppose that a is self-conjugate, then we have

$$\begin{aligned} a_1 &= a_{12} = a, & a_2 &= a_{11} = a' + \alpha, & a_3 &= a_{10} = a + \alpha_c \\ a_4 &= a_9 = a + \alpha - \gamma, & a_5 &= a_8 = a + \alpha_c - \gamma, & a_6 &= a_7 = a + \alpha + \alpha_c - \gamma \end{aligned}$$

It follows that

$$(I. 3) \quad a_1 - a_5 = a_2 - a_6 = g^{st} (g_{jk,s} - g_{ks,j})$$

Further, if the covariant derivatives of the g_{ij} 's with respect to two affine connections are the same and if one of them is self-associate then the other is also self-associate, because the covariant derivatives of the g_{ij} 's must vanish. Now when $a_1 = a_2$ is self-associate, then $a_7 = a_8$ is also self-associate. Hence we have the following theorem:

THEOREM 2. If a_1 be coefficients of a self-conjugate affine connection, then the g_{ij} 's have the same covariant derivatives with respect to the pairs of affine connections (a_1, a_5) and (a_2, a_6) of Sen's sequence. And if a_1 be self-associate then the g_{ij} 's have the same covariant derivatives with respect to the pair (a_1, a_7) .

Further, let Γ_{ij}^t and $\Gamma_{ij}^t + T_{ij}^t$ be the coefficients of two affine connections. Then their associates are

$$\Gamma_{ij}^{*t} \quad \text{and} \quad \Gamma_{ij}^{*t} - g^{mt} g_{is} T_{mj}^s$$

and their conjugates are

$$\Gamma_{ji}^t \quad \text{and} \quad \Gamma_{ji}^t + T_{ji}^t$$

It follows immediately from Theorem 1 that

THEOREM 3. If the g_{ij} 's have the same covariant derivatives with respect to two arbitrary affine connections, then the same is true with respect to their associates and conjugates.

Now, let Γ_{ij}^t and L_{ij}^t be the coefficients of two affine connections and

$$\Delta_{ij}^t = \frac{1}{2} (\Gamma_{ij}^t + L_{ij}^t), \quad T_{ij}^t = \Gamma_{ij}^t - L_{ij}^t$$

Also, let Γ_{ijk}^t , L_{ijk}^t and Δ_{ijk}^t denote curvature tensors formed with Γ_{ij}^t , L_{ij}^t and Δ_{ij}^t respectively.

Then it is known that

$$(I. 4) \quad \Delta_{ijk}^t - \frac{1}{2} (\Gamma_{ijk}^t + L_{ijk}^t) = \frac{1}{4} (T_{sk}^t T_{ij}^s - T_{sj}^t T_{ik}^s)$$

Now if the g_{ij} 's have the same covariant derivatives with respect to Γ_{ij}^t and L_{ij}^t then by (I. 1)

$$T_{sk}^t T_{ij}^s = g^{mt} g_{sp} T_{mk}^p g^{ns} g_{iq} T_{nj}^q = g^{mt} g_{iq} T_{mk}^n T_{nj}^q$$

Therefore

$$g_{ht} (T_{sk}^t T_{ij}^s - T_{sj}^t T_{ik}^s) = g_{ht} g^{mt} g_{iq} (T_{mk}^n T_{nj}^q - T_{mj}^n T_{nk}^q) = g_{it} (T_{sj}^t T_{hk}^s - T_{sk}^t T_{hj}^s)$$

Or

$$\Delta_{hijk} - \frac{1}{2} (\Gamma_{hijk} + L_{hijk}) = \frac{1}{4} g_{ht} (T_{sk}^t T_{ij}^s - T_{sj}^t T_{ik}^s) = \frac{1}{4} g_{it} (T_{sj}^t T_{hk}^s - T_{sk}^t T_{hj}^s)$$

Hence

$$(I. 5) \quad \Delta_{hijk} - \frac{1}{2} (\Gamma_{hijk} + L_{hijk}) = - [\Delta_{ihjk} - \frac{1}{2} (\Gamma_{ihjk} + L_{ihjk})]$$

Thus we have the following theorem:

THEOREM 4. If the g_{ij} 's have the same covariant derivatives with respect to Γ_{ij}^t and L_{ij}^t and if $\Delta_{ij}^t = \frac{1}{2} (\Gamma_{ij}^t + L_{ij}^t)$, then the curvature tensors formed with them satisfy the relation (I. 5).

This result is easily verified in the case when Γ_{ij}^t and therefore L_{ij}^t , Δ_{ij}^t are self-associate. For in this case the curvature tensors are skew in the first two indices (4).

As before, suppose that the covariant derivatives of the g_{ij} 's with respect to Γ_{ij}^t and L_{ij}^t are the same. Forming the second covariant derivatives it is seen that

$$(I. 6) \quad g_{ij,kl} - g_{ij,kl} = -[g_{sj,k} T_{il}^s + g_{is,k} T_{jl}^s + g_{ij,s} T_{kl}^s]$$

where

$$\Gamma_{ij}^t - L_{ij}^t = T_{ij}^t = -g^{mt} g_i \cdot T_{mj}^n$$

Therefore

$$\begin{aligned} \text{by (I. 6)} \quad g^{ij} (g_{ij,kl} - g_{ij,kl}) &= g^{ms} g_{sj,k} T_{il}^s + g^{ms} g_{is,k} T_{jl}^s - g^{ij} g_{ij,s} T_{kl}^s = \\ &= g^{ij} [g_{sj,k} T_{il}^s + g_{si,k} T_{jl}^s + g_{ij,s} T_{kl}^s - 2 g_{ij,s} T_{kl}^s] = -g^{ij} (g_{ij,kl} - g_{ij,kl}) - 2 g^{ij} g_{ij,s} T_{kl}^s \end{aligned}$$

Therefore

$$g^{ij} (g_{ij,kl} - g_{ij,kl}) = -g^{ij} g_{ij,s} T_{kl}^s$$

Interchanging k and l and subtracting

$$g^{ij} [(g_{ij,kl} - g_{ij,kl}) - (g_{ij,kl} - g_{ij,kl})] = g^{ij} g_{ij,s} (T_{lk}^s - T_{kl}^s)$$

Finally using Ricci's identity

$$(I. 7) \quad g^{ij} [(g_{it} \Gamma_{jkl}^t + g_{jt} \Gamma_{ikl}^t) - (g_{it} L_{jkl}^t + g_{jt} L_{ikl}^t)] = g^{ij} g_{ij,s} [(\Gamma_{lk}^s - L_{lk}^s) - (\Gamma_{kl}^s - L_{kl}^s)]$$

Let us further suppose that Γ_{ij}^t and L_{ij}^t are both self-associate or both self-conjugate. Then the right hand side of (I. 7) vanishes. We have therefore

$$g^{ij} [(\Gamma_{ij,kl} + \Gamma_{jkl}) - (L_{ij,kl} + L_{jkl})] = 0$$

whence

$$(I. 8) \quad g^{ij} (\Gamma_{jkl} - L_{jkl}) = 0$$

As said before, this result is obvious when both the affine connections are self-associate,

Hence we have the following theorem :

THEOREM 5. If the g_{ij}' s have the same covariant derivatives with respect to two self-conjugate affine connections with coefficients Γ_{ij}^t, L_{ij}^t and if Γ_{ijk}, L_{ijk} be the corresponding covariant curvature tensors, then (I. 8) holds.

2. The torsion of an affine connection with coefficients Γ_{ij}^t is defined to be the tensor $\frac{1}{2} (\Gamma_{ij}^t - \Gamma_{ji}^t) (2)$. It follows that two arbitrary affine connections with coefficients Γ_{ij}^t and $\Gamma_{ij}^t + T_{ij}^t$ have the same torsion if and only if T_{ij}^t is symmetric in i and j . It is now easy to see that if two affine connections have the same torsion the same is true of their conjugates. E. g., in Sen's sequence each of the pairs $(a_1, a_6), (a_2, a_9), (a_4, a_{11})$ and therefore their conjugates $(a_{12}, a_7), (a_3, a_8), (a_5, a_{10})$ have the same torsion.

Again, let $a = \Gamma_{ij}^t, b = L_{ij}^t$ be the coefficients of two affine connections and $a - b = T_{ij}^t$. Their associates a^* and b^* will have the same torsion if the tensor

$$T_{ij}^t + g^{mt} (g_{im,j} - g_{im;j}) = -g^{mt} g_{is} T_{mj}^s$$

is symmetric in i, j i. e., if

$$(2. 1) \quad g_{is} T_{mj}^s = g_{js} T_{mi}^s$$

Putting $g_{is} T_{mj}^s = T_{imj}$ we have the following theorem:

THEOREM 6. Let a and b have the same torsion; then their associates will also have the same torsion if the tensor T_{ijk} is symmetric in all the indices.

Let $e_0 = (a, b) = \frac{1}{2} (a + b) + (a - b), \bar{e}_0 = (b, a) = \frac{1}{2} (a + b) + (b - a)$

$$e_1 = (e_0, \bar{e}_0) = \frac{1}{2} (a + b) + 2(a - b), \bar{e}_1 = \frac{1}{2} (a + b) + 2(b - a)$$

Similarly for $e_2, \bar{e}_2$ etc.

Then

$$e_r = \frac{1}{2} (a + b) + 2^r (a - b), \bar{e}_r = \frac{1}{2} (a + b) + 2^r (b - a)$$

Therefore

$$(2. 2) \quad e_r - \bar{e}_r = 2^r (a - b)$$

It follows that if a and b have the same torsion then the same is true of e_r and $\bar{e}_r$.

Further, we have the following theorem:

THEOREM 7. If the associates of a and b have the same torsion, the same is true of the associates of e_r and $\overline{e_r}$.

Let Γ_{ij}^t and L_{ij}^t have the same torsion. Then applying the condition that T_{ij}^t is symmetric in i, j , we obtain from (I. 4) the cyclical property, namely

$$(2.3) \quad \Delta_{ijk}^t - \frac{1}{2}(\Gamma_{ijk}^t + L_{ijk}^t) + \Delta_{jki}^t - \frac{1}{2}(\Gamma_{jki}^t + L_{jki}^t) + \Delta_{kij}^t - \frac{1}{2}(\Gamma_{kij}^t + L_{kij}^t) = 0$$

This result is obvious if Γ_{ij}^t and therefore L_{ij}^t , Δ_{ij}^t are self-conjugate.

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