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# INVERSE PROCESSES IN INVARIANTS, WITH APPLICATIONS TO THREE PROBLEMS IN MECHANICS.

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*(Jubilee memoir. Fortieth anniversary of the author's Philosophical Doctorate).*

After a mathematical subject has been developed from various points of view, by different writers, there may have arisen a problem in criticism. One may be able to work synthetically with these viewpoints, to assert the predominance of one of them, perhaps, and to advance the subject, consequently, in a new direction. Criticism then becomes a phase of origination and more than a mere review of shortcomings in published work.

In a century and one third, invariant theory has been developed from many points of view. Without reciting its history we may say that, within the last quarter of this time, the bases of the subject have been materially broadened, by the introduction of novel groups of transformations, by the use of invariantive concepts in the field of analysis and by the introduction of the method of invariants into mechanics. Among the leading algorithms of invariant theory is that which makes the subject a branch of the theory of linear, partial differential equations and by extending this algorithm farther than others have done, we have unified some diverse viewpoints and originated a theory of inverse processes. An inverse problem, in general, is one in which the ground-quantities and the transformations are the unknowns, to be determined when the invariant is given.

## I. - Introduction.

If a function of two sets,

$$f(x_1, x_2; a_0, \dots, a_e),$$

is homogeneous in each set, that is, if

$$\left[ x \frac{\partial}{\partial x} \right] f = mf, \quad \left[ a \frac{\partial}{\partial a} \right] f = f; \quad \left[ x \frac{\partial}{\partial x} \right] = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2},$$

$m$  a positive integer, and if a second function of the sets of degree  $g$  in the  $a_i$ ,

$$D(x_1, x_2; a_0, \dots, a_e),$$

is invariant when the  $x_i$  are transformed by a scheme  $Q$ ,  $f$  can be derived from  $D$  by a linear operator under certain conditions. Assuming that the  $\xi_i$  in,

$$\Delta = \xi_0 \frac{\partial}{\partial a_0} + \xi_1 \frac{\partial}{\partial a_1} + \dots + \xi_e \frac{\partial}{\partial a_e} \equiv \left[ \xi \frac{\partial}{\partial a} \right],$$

are cogredient to the  $\alpha_i$ , under the transformations  $S$  which  $Q$  induces upon the  $\alpha_i$ ,  $\Delta^\sigma D$ , ( $\sigma=0, 1, \dots, g$ ), is invariant if different from zero. End for end symmetry in the variables  $x_1, x_2$  is postulated for  $\Delta$ . With  $\sigma=g-1$ ,  $\Delta^\sigma D$ , being linear in the  $\alpha_i$  must be the ground-form  $f$  of the invariant problem, but not uniquely, because the cogredient set  $\xi_i$  is not unique in general and  $D$  is only one of many invariants, here one of a fundamental system [1].

Assuming  $Q$  linear,

$$(1) \quad Q: \quad x_i = \lambda_i x_1' + \mu_i x_2', \quad (i=1, 2)$$

we may state the problem more inclusively by allowing  $f$  and therefore  $D$  to involve the parameters of the transformation, assumed at first to be arbitrary variables.

The essential fact about  $\Delta^{g-1}D$  is that it is invariant and linear in  $a_0, \dots, a_e$ . With  $m$  left arbitrary,  $\Delta^{g-1}D$  (also  $\Delta^g D \neq 0$ ) will contain as factors powers of universal covariants of  $Q$ . There may be involved in these factors a case of the covariant [2],

$$\lambda_2 x_1^2 + (\mu_2 - \lambda_1) x_1 x_2 - \mu_1 x_2^2,$$

whose roots are the poles of  $Q$  and from which  $Q$  can be determined, and there may be among the factors, covariants properly universal. These will involve essentially only the variables; will be covariantive for all admitted values of the parameters  $\lambda_i, \mu_i$ , whatever group or set  $Q$  represents. The latter universally covariant factors serve to determine  $Q$ .

This will seem to be, at first, mostly a binary theory. It is true that an invariant operator like  $\Delta$  can be constructed on the basis of the determinant (invariant under linear transformations  $Q'$ ) [3],

$$u_x = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = (yz)_1 x_1 + (yz)_2 x_2 + (yz)_3 x_3 = u_1 x_1 + u_2 x_2 + u_3 x_3.$$

This operator will be of the form,

$$\Delta' = u_1^m \frac{\partial}{\partial a_{m00}} + \binom{m}{1} u_1^{m-1} u_2 \frac{\partial}{\partial a_{m-110}} + \dots,$$

where  $a_{ijk}$  are the actual coefficients of a ternary quantic  $a_x^m$  or any set of parameters which could be considered as such coefficients. However, repeated operations  $\Delta'$  upon any mixed form, such as,

$$(abu)^2 a_x^{m-2} b_x^{m-2}, \quad (b \sim a),$$

often leads to a vanishing result before it leads to a form of the first degree in  $a_{ijk}$ . We shall show that the problem in three variables is, however, often solvable in important special forms.

## II. - The transformation as unknown.

Thus, in  $\Delta^{g-1}D$ , we look to the factor linear in  $a_0, \dots, a_e$  for a form of the ground-function, and to the other factors for a means of determining  $Q$ . Consider the expression  $D_1$ , assumed invariant,

$$D_1 = (a_0 + a_1 + a_2 + a_3)(a_0 - a_1 + a_2 - a_3)(a_0^2 - a_0a_2 + a_1a_3 - a_3^2)^2.$$

If the four  $a_i$  in the order of their subscripts are regarded as cogredient to the coefficients of  $x_1^3, x_1^2x_2, x_1x_2^2, x_2^3$ , in the covariant,

$$(xy)^3 = (x_1y_2 - x_2y_1)^3,$$

an invariant operator is,

$$\Delta_1 = y_2^3 \frac{\partial}{\partial a_0} - 3y_2^2y_1 \frac{\partial}{\partial a_1} + 3y_2y_1^2 \frac{\partial}{\partial a_2} - y_1^3 \frac{\partial}{\partial a_3}.$$

We find,

$$\Delta_1^3 D_1 = 720 (y_2^3 - y_1^3)^2 (a_0y_2^3 + a_1y_2^2y_1 + a_2y_2y_1^2 + a_3y_1^3).$$

Hence the ground-form  $f_1$  is the arbitrary cubic, and the transformation is the cyclic  $Q_1$  (or  $Q_2$ ),

$$\begin{aligned} Q_1 : \quad y_1 &= \mu y_1' + \nu y_2', & y_2 &= \nu y_1' + \mu y_2', \\ Q_2 : \quad y_1 &= \mu y_1' + \nu y_2', & y_2 &= -\nu y_1' - \mu y_2'. \end{aligned}$$

The expression  $s = a_0^2 - a_0a_2 + a_1a_3 - a_3^2$  is also a cyclic invariant of  $f_1$  but the ground-form corresponding is

$$\Delta_1 s = (2a_0 - a_2)y_2^3 - 3a_3y_2^2y_1 - 3a_0y_2y_1^2 + (-a_1 + 2a_3)y_1^3.$$

Since the group is cyclic,  $y_1, y_2$  can be interchanged.

It is usually implied, when  $(y_1, y_2)$  is cogredient to  $(x_1, x_2)$ , that  $y_1, y_2$  are replaceable by  $x_1, x_2$ , respectively, but for cogredience, it is not necessary that  $y_1, y_2$  should be linear in  $x_1, x_2$ .

Suppose that we wish to pass from  $y_i$  to  $x_i$  by the substitutions,

$$y_1 = x_1^2 + x_2^2, \quad y_2 = x_1^2 - x_2^2,$$

and that  $e=2$ . If the function assumed invariant is,

$$D^1 = \lambda a_1^2 + a_0 a_2, \quad (\lambda \text{ constant}),$$

$\Delta^1$  being the operator,

$$\Delta^1 = (x_1^2 - x_2^2)^2 \frac{\partial}{\partial a_0} - 2(x_1^4 - x_2^4) \frac{\partial}{\partial a_1} + (x_1^2 + x_2^2)^2 \frac{\partial}{\partial a_2},$$

we get,

$$(2) \quad \begin{aligned} \Delta^1 D^1 &= (\alpha_0 - 4\lambda\alpha_1 + \alpha_2) x_1^4 + 2(\alpha_0 - \alpha_2) x_1^2 x_2^2 + (\alpha_0 + 4\lambda\alpha_1 + \alpha_2) x_2^4 = f^1, \\ (\Delta^1)^2 D^1 &= (8\lambda + 2)(x_1^4 - x_2^4)^2. \end{aligned}$$

The latter form, as a universal covariant, determines the transformation,

$$Q^1: \quad x_1 = i\mu x_2', \quad x_2 = \mu x_1', \quad (i = \sqrt{-1}).$$

The problem fails when  $\lambda = -1/4$ .

The choice of the functions  $y_1(x)$ ,  $y_2(x)$ , cogredient to  $x_1$ ,  $x_2$ , is in fact arbitrary, but unless these (i. e.  $\xi_i$ ) and the expression  $D$ , assumed invariant, are fortunately chosen within the bounds of some essential invariant problem, the final function  $\Delta^g D (\neq 0)$  will prove to be invariant only with respect to the identical transformation.

#### Formal $n$ -ary groups in less than $n^2$ parameters.

Under the ternary transformation with arbitrary <sup>(1)</sup> coefficients,

$$S_1: \quad x_i = \lambda_i x_1' + \mu_i x_2' + \nu_i x_3', \quad (i = 1, 2, 3)$$

which we write more simply in the form,

$$S_1: \quad (x) = \begin{pmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{pmatrix} (x'),$$

the determinant,

$$d = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_2 & y_3 & y_1 \\ 1 & 1 & 1 \end{vmatrix},$$

may be assumed invariant, the relation of cogredience,

$$(y_1, y_2, y_3) \text{ Co. } (x_1, x_2, x_3),$$

being also assumed, but a result of these hypotheses is a particularization of  $S_1$  to a four-parameter sub-group of the general, algebraic, homogeneous group.

This is proved as follows. Since

$$d = (y_3 - y_1) x_1 + (y_1 - y_2) x_2 + (y_2 - y_3) x_3,$$

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<sup>(1)</sup> An invariant is *algebraic* if the group is  $Q$ , or  $S_1$ . A *ground-quantic* is any covariant of degree unity in  $\alpha_0, \dots, \alpha_e$ . An algebraic *process* is an analytic process which can be carried out rationally.

the differences  $(y_3 - y_1)$ ,  $(y_1 - y_2)$ ,  $(y_2 - y_3)$  are contragredient to  $(x_1, x_2, x_3)$ , under  $S_1$ . Hence we have the identities,

$$\begin{aligned} y_3 - y_1 &= \lambda_1(y_3' - y_1') + \lambda_2(y_1' - y_2') + \lambda_3(y_2' - y_3') = (\lambda_3 - \lambda_1)y_1' + (\mu_3 - \mu_1)y_2' + (\nu_3 - \nu_1)y_3', \\ y_1 - y_2 &= \mu_1(y_3' - y_1') + \mu_2(y_1' - y_2') + \mu_3(y_2' - y_3') = (\lambda_1 - \lambda_2)y_1' + (\mu_1 - \mu_2)y_2' + (\nu_1 - \nu_2)y_3', \\ y_2 - y_3 &= \nu_1(y_3' - y_1') + \nu_2(y_1' - y_2') + \nu_3(y_2' - y_3') = (\lambda_2 - \lambda_3)y_1' + (\mu_2 - \mu_3)y_2' + (\nu_2 - \nu_3)y_3'. \end{aligned}$$

These lead to the relations,

$$\begin{aligned} \lambda_2 - \lambda_1 &= \lambda_3 - \lambda_1, & \lambda_3 - \lambda_2 &= \mu_3 - \mu_1, & \lambda_1 - \lambda_3 &= \nu_3 - \nu_1, \\ \mu_2 - \mu_1 &= \lambda_1 - \lambda_2, & \mu_3 - \mu_2 &= \mu_1 - \mu_2, & \mu_1 - \mu_3 &= \nu_1 - \nu_2, \\ \nu_2 - \nu_1 &= \lambda_2 - \lambda_3, & \nu_3 - \nu_2 &= \mu_2 - \mu_3, & \nu_1 - \nu_3 &= \nu_2 - \nu_3, \end{aligned}$$

whence  $S_1$  takes the form,

$$S: (x) = \begin{pmatrix} \lambda_2 + \mu_2 - \mu_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_1 \\ \lambda_2 & \mu_1 & \nu_1 + \mu_2 - \mu_1 \end{pmatrix} (x').$$

**THEOREM.** - *The transformations of the type  $S$  form a group, represented by  $(S)$ .*

In proof, assume that the determinant  $\delta$  of  $S$  is not zero,

$$\delta = (\lambda_2 + \mu_2 + \nu_1)(\mu_1 - \mu_2)^2 \neq 0.$$

Then  $S$  is non-singular and its inverse is easily shown to be of the same type as  $S$ . Let  $T$  be an independent transformation of this type,

$$T: (x) = \begin{pmatrix} L_2 + M_2 - M_1 & M_1 & N_1 \\ L_2 & M_2 & N_1 \\ L_2 & M_1 & N_1 + M_2 - M_1 \end{pmatrix} (x'),$$

then, if we combine  $S$  and  $T$ , we find,

$$S' = ST: (x) = \begin{pmatrix} q + t - r & r & s \\ q & t & s \\ q & r & s + t - r \end{pmatrix} (x''),$$

in which,

$$\begin{aligned} q &= L_2(\lambda_2 + \mu_2 - \mu_1) + (M_2 + N_1)\lambda_2, & r &= (L_2 + M_2 - M_1)\mu_1 + M_1\mu_2 + N_1\mu_1, \\ s &= (L_2 + M_2)\nu_1 + N_1(\nu_1 + \mu_2 - \mu_1), & t &= L_2\mu_1 + M_2\mu_2 + N_1\mu_1. \end{aligned}$$

Hence the product of two transformations of the set  $(S)$  is within the set  $(S)$ ; also the identical transformation,  $(q = r = s = 0, t = 1)$ , is within  $(S)$ , q. e. d.

If we substitute  $y_i = x_i$  in  $d$ , the resulting covariant  $d_x$ ,

$$d_x = -x_1^2 - x_2^2 - x_3^2 + x_1x_2 + x_1x_3 + x_2x_3,$$

is linearly factorable,

$$d_x = d_1 d_2 = (x_1 - w x_2 + w^2 x_3)(x_1 + w^2 x_2 - w x_3),$$

$w$  being an imaginary cube root of  $-1$ . The forms  $d_1, d_2$  are left invariant individually by the typical transformation of  $(S)$ , (taken as  $S'$ ). The invariant relations are,

$$d_1 = (t-r) d_1', \quad d_2 = (t-r) d_2'.$$

There is a third (real) covariant of order one, viz.,

$$d_3 = q x_1 + r x_2 + s x_3; \quad d_3 = (q+s+t) d_3'.$$

It is a covariant of a subset of  $(S)$ , (vary  $t$ ). The vertices of the covariant triangle  $d_1 d_2 d_3$  are the poles of  $S'$ , one being real.

If we substitute from the inverse of  $T'$ ,

$$T' : (d) = \begin{pmatrix} 1 & -w & w^2 \\ 1 & w^2 & -w \\ q & r & s \end{pmatrix} (x),$$

for  $x_1, x_2, x_3$ , in the arbitrary ternary quantic,

$$f_m = \alpha_x^m = \sum a_{ijk} x_1^i x_2^j x_3^k,$$

the result is an expansion of  $f_m$  in the arguments  $d_1, d_2, d_3$ ,

$$f'_m = A_d^m = \sum A_{ijk} d_1^i d_2^j d_3^k.$$

The  $\frac{1}{2}(m+1)(m+2)$  coefficients  $A_{ijk}$  are invariants of degree one in the  $a_{ijk}$  and appertain to a domain  $\Omega$  which includes  $w, q, r, s$ . They are thus invariants of the subset of  $(S)$  and, with  $d_1, d_2, d_3$ , are called invariant elements. In  $\Omega$  the invariant elements are a fundamental system of invariants and covariants. This is evident since the inverse of  $T'$  and the inverse of

$$A_{ijk} = A_{ijk}(a_{m00}, a_{m-110}, \dots, a_{00m}),$$

serve to express any concomitant of  $f_m$  under the subset, as a rational, integral function, in  $\Omega$ , of  $d_1, d_2, d_3, A_{ijk}$ . The  $A_{ijk}$  ( $=0$ ) are invariant hyperplanes expressed in the  $a_{ijk}$  as variables.

Concomitants of  $f_m$  under  $(S)$  and therefore of a domain  $\Omega_1$  free from  $w, q, r, s$ , are polynomials in the invariant elements of structure such that an algebraic simplification from  $\Omega$  to  $\Omega_1$  is always possible. Such a system of polynomials is a HILBERT system within which there exists a set which forms a finite basis. The basis theory will be considered further in a later section.

A fundamental system for  $f_m = \alpha_x^m$  under the ternary algebraic group of nine parameters, *e. g.* the system of a cubic (GORDAN) and that of a quartic (E.

NOETHER), is involved with advanced formalism, but in any mixed form or contravariant of such a system,  $u_1, u_2, u_3$  can be replaced by  $x_3 - x_1, x_1 - x_2, x_2 - x_3$  respectively. The result will be a covariant of  $f_m$  under  $(S)$ . The method furnishes, as far as proved, only parts of complete systems under  $(S)$  but it gives new life to the algebra and geometry by reduction of the formal complication and an increase in the number of covariant configurations.

Formulae for the conic-form,

$$f_2 = ax_1^2 + bx_2^2 + cx_3^2 + 2hx_1x_2 + 2gx_1x_3 + 2fx_2x_3,$$

are added. The invariant elements are  $d_1, d_2, d_3$ , and the following expressions, in which  $\zeta = (w^2 + w)(q + r + s)$ .

$$\begin{aligned} \zeta^2 A = & (w^2r^2 - ws^2 - 2rs)a + (w^2q^2 + 2wqs + s^2)b + (-wq^2 - 2w^2qr + r^2)c \\ & + 2(qs - w^2qr - w^2s^2 - wrs)h + 2(qr + w^2rs + wqs + wr^2)g \\ & + 2(-rs + w^2qs - wqr - q^2)f, \end{aligned}$$

$$\begin{aligned} \zeta^2 B = & (w^2s^2 - wr^2 - 2rs)a + (-2w^2qs - wq^2 + s^2)b + (w^2q^2 + 2wqr + r^2)c \\ & + 2(qs + w^2rs + wqr + ws^2)h + 2(qr - w^2qs - w^2r^2 - wrs)g \\ & + 2(w^2qr - wqs - rs - q^2)f, \end{aligned}$$

$$\zeta^2 C = -3(a + b + c + 2h + 2g + 2f),$$

$$\begin{aligned} \zeta^2 H = & (-r^2 - rs - s^2)a + (-q^2 - qs - s^2)b + (-q^2 - qr - r^2)c \\ & + (2qr + qs + rs - s^2)h + (2qs + qr + rs - r^2)g + (2rs + qr + qs - q^2)f, \end{aligned}$$

$$\begin{aligned} \zeta^2 G = & ((w-2)r - (w+1)s)a + (-(2w-1)s - (w-2)q)b \\ & + ((w+1)q + (2w-1)r)c + (-(w-2)q + (w-2)r - 3ws)h \\ & + ((w+1)q + 3(w-1)r - (w+1)s)g + (3q + (2w-1)(r-s))f, \end{aligned}$$

$$\begin{aligned} \zeta^2 F = & (-(w+1)r + (w-2)s)a + ((w+1)q + (2w-1)s)b \\ & + (-(w-2)q - (2w-1)r)c + (3(w-1)s + (w+1)(q-r))h \\ & + (-3wr + (w-2)(s-q))g + (3q + (2w-1)(s-r))f. \end{aligned}$$

The line-equation contravariant  $(abu)^2$  now becomes a covariant conic-form expressible as,

$$\Gamma = \begin{vmatrix} a & h & g & x_3 - x_1 \\ h & b & f & x_1 - x_2 \\ g & f & c & x_2 - x_3 \\ x_3 - x_1 & x_1 - x_2 & x_2 - x_3 & 0 \end{vmatrix}.$$

The coefficients of  $d_x$  are cogredient to those of  $f_2$ , hence there exists an invariant operator  $O$ ,

$$(3) \quad O = -2 \frac{\partial}{\partial a} - 2 \frac{\partial}{\partial b} - 2 \frac{\partial}{\partial c} + \frac{\partial}{\partial h} + \frac{\partial}{\partial g} + \frac{\partial}{\partial f}.$$

If  $D$  is the discriminant of  $f_2$ , the following concomitants exist, (Cf.  $\zeta^2 C$  above);

$$D, \quad OD, \quad O^2D=6(q+r+s)^2C, \quad f_2, \quad \Gamma, \quad O\Gamma=2(q+r+s)^2Cd_x, \quad O^2\Gamma=0.$$

In the twenty known concomitants of the simultaneous system of  $f_2$  and a second arbitrary conic-form  $g_2$  under  $S_1$ , both ground-quantities have the standing of covariants, but if we make  $g_2$  the same as  $d_x$ ,  $g_2$  loses this property unless, at the same time, we particularize the group to  $(S)$ . When this is done, the twenty quantities, (i. e. those of them that do not vanish), become concomitants of  $f_2$  under  $(S)$ , wherein also  $u_1, u_2, u_3$ , become, respectively,  $x_3-x_1, x_1-x_2, x_2-x_3$ .

\* \* \*

If we proceed from the bi-quaternary form,

$$k=(y_4-y_1)x_1+(y_1-y_2)x_2+(y_2-y_3)x_3+(y_3-y_4)x_4,$$

and follow the method we have used in the case of  $d$ , there is obtained a six-parameter quaternary group  $(U)$ , where,

$$U: \quad (x) = \begin{pmatrix} u+w-v & v & x & z \\ u & w & y & z+x-y \\ u+x-y & v & y+w-v & z \\ u & v+x-y & y & z+w-v \end{pmatrix} (x').$$

The transformation  $U$  leaves invariant the conicoid,

$$k_x = -x_1^2 - x_2^2 - x_3^2 - x_4^2 + x_1x_2 + x_3x_4 + x_1x_4 + x_2x_3.$$

There is, probably, a corresponding group for any number,  $n$ , of variables  $x_i$  but the difficulties increase with  $n$ .

We can now readily write  $\Delta$ -operators and algorithms to correspond with § I for quantics  $a_x^m$  (ternary) or  $b_x^m$  (quaternary) under the respective groups  $(S)$  and  $(U)$ .

### III. - Instances of determination of ground-quantics.

Consider the class of expansions obtained when the arbitrary quantic  $f_M$  of order,

$$M=(m+1)n-1, \quad (n>0),$$

$$f_M = a_x^M = a_0x_1^M + a_1x_1^{M-1}x_2 + \dots + a_Mx_2^M,$$

is developed in the homogeneous form,

$$f_M = \varphi_{0n-1} f_1^m + \binom{m}{1} \varphi_{1n-1} f_1^{m-1} f_2 + \dots + \binom{m}{m} \varphi_{mn-1} f_2^m,$$

$f_1, f_2$  being arbitrary forms of order  $n$ ,

$$\begin{aligned} f_1 &= \beta_0 x_1^n + \beta_1 x_1^{n-1} x_2 + \dots + \beta_n x_2^n = f_1(x), \\ f_2 &= \gamma_0 x_1^n + \gamma_1 x_1^{n-1} x_2 + \dots + \gamma_n x_2^n = f_2(x). \end{aligned}$$

The expansion has a symbolic basis, by which is meant that it can be expressed also in the form,

$$f_M = \Phi f_1^m + \frac{\nabla \Phi}{\underline{1}} f_1^{m-1} f_2 + \frac{\nabla^2 \Phi}{\underline{2}} f_1^{m-2} f_2^2 + \dots + \frac{\nabla^m \Phi}{\underline{m}} f_2^m,$$

wherein,

$$\Phi = \varphi_{0n-1}, \quad -\nabla = \left( \beta_0 \frac{\partial}{\partial \gamma_0} + \beta_1 \frac{\partial}{\partial \gamma_1} + \dots + \beta_n \frac{\partial}{\partial \gamma_n} \right) = \left[ \beta \frac{\partial}{\partial \gamma} \right],$$

and therefore in the form of a symbolic power,

$$f_M = (\Xi_1 f_1 + \Xi_2 f_2)^m = \Xi_f^m = \Xi_f'^m = \dots$$

The point is that the symbolism makes sense, for we have not only,

$$\Xi_1^{m-i} \Xi_2^i = \varphi_{in-1}, \quad (i=0, \dots, m),$$

but also a symbolical equivalent for  $\nabla$ ,

$$\nabla = \Xi_2 \frac{\partial}{\partial \Xi_1}.$$

Suppose there is given a product  $P$ ,

$$P = (\Xi \Xi')^p (\Xi \Xi'')^q \dots (\Xi' \Xi'')^r \dots \Xi_{f(x)}^s \Xi_{f'(x)}^t \dots,$$

wherein the number of symbols  $\Xi, \Xi', \dots$  is  $g$ , the degree of  $P$  in  $\varphi_{0n-1}, \varphi_{1n-1}, \dots, \varphi_{mn-1}$ , and that the sum of all the exponents which affect a chosen symbol  $\Xi$  of  $P$  is  $m$ . Assume  $P$  to be invariant under a transformation of  $f_M$ , for which there exist the following cogrediencies,

$$(\Xi_2, -\Xi_1) \text{ Co. } (\partial/\partial f_2, -\partial/\partial f_1) \text{ Co. } (f_1(y), f_2(y)) \text{ Co. } (y_1, y_2) \text{ Co. } (x_1, x_2).$$

Then an operator  $\Delta$  of § I can be chosen in the form,

$$(4) \quad \Delta = \left( f_2(y) \frac{\partial}{\partial \Xi_1} - f_1(y) \frac{\partial}{\partial \Xi_2} \right)^m \equiv \left( - \left( f(y) \frac{\partial}{\partial \Xi} \right) \right)^m.$$

Then,

$$\begin{aligned} - \left( f(y) \frac{\partial}{\partial \Xi} \right) (\Xi \Xi') &= \Xi'_{f(y)}, \\ - \left( f(y) \frac{\partial}{\partial \Xi} \right) \Xi_{f(x)} &= (f(x) f(y)) = f_1(x) f_2(y) - f_2(x) f_1(y). \end{aligned}$$

Hence  $\Delta P$  is free from the symbols  $\Xi_1, \Xi_2$ ;  $\Delta^2 P$  is free from  $\Xi_1, \Xi_2, \Xi_1', \Xi_2'$ , while,

$$\Delta^{g-1} P = K(f(x) f(y))^E \Xi_{f(y)}^{(g)M-F} \Xi_{f(x)}^{(g)F},$$

$K$  being a numerical constant. The ground-quantic is thus determined uniquely by  $\Delta^{g-1}P$ , as

$$f_M = \Xi_{f(y)}^{(g)M-F} \Xi_{f(x)}^{(g)F} \Big|_{y_1=x_1, y_2=x_2}.$$

The covariant  $(f(x), f(y))^E$  shows that  $P$  is a combinant [4] of  $f_1, f_2$  and its factor  $(xy)^E$  shows that the variables  $x_1, x_2$  are subject to the general linear algebraic transformation  $Q$  of (1).

#### IV. - Modular instances and theory.

When the equalities dealt with in the above sections are changed to identical congruences with respect to a prime modulus  $p$ , much of the formulary remains intact. The group  $G$  (or  $G(p)$ ) being the homogeneous total group (mod.  $p$ ), two independent universal covariants exist from which cogrediencies may be obtained. These are due to DICKSON, as follows [5]

$$\begin{aligned} L &= x_1^p x_2 - x_1 x_2^p, \\ N &= x_1^{p(p-1)} + x_1^{(p-1)(p-1)} x_2^{p-1} + x_1^{(p-2)(p-1)} x_2^{2(p-1)} + \dots + x_2^{p(p-1)}. \end{aligned}$$

The determinant  $(xy)$ , with  $(y_1, y_2)$  Co.  $(x_1, x_2)$ , and  $y_i$  arbitrary, is the all-inclusive form of universal covariant when the group is the general linear algebraic  $Q$  (cf. (1)), in which case  $y_i$  may be considered an arbitrary function of  $x_1, x_2$ . If the group is special, e. g.  $G(p)$ ,  $y_i$ , in an invariant  $(xy)$ , will be a particular function of  $x_1, x_2$ .

For construction of  $\Delta$  all cogrediencies are obtainable from universal covariants. From  $L, y_i = x_i^p$  Co.  $x_i$ , whence is obtained,

$$\Delta = x_2^{pe} \frac{\partial}{\partial a_0} - \binom{e}{1} x_2^{p(e-1)} x_1^p \frac{\partial}{\partial a_1} + \dots + (-1)^e x_1^{pe} \frac{\partial}{\partial a_e},$$

( $f = a_0 x_1^e + a_1 x_1^{e-1} x_2 + \dots$ ). Other operators may be obtained by writing  $N$  in the form  $(xy)$ . It may be written in either of two forms,

$$N = \begin{vmatrix} x_1^{(p-1)i+1} & x_2^{(p-1)i+1} \\ \psi_1/x_2^{(p-1)i} & \psi_2/x_1^{(p-1)i} \end{vmatrix}, \quad (i=0 \text{ or } 1; p > 2),$$

in which,

$$\begin{aligned} \psi_1 &= - (x_1^{[p-(p+1)/2](p-1)} x_2^{[(p+1)/2](p-1)-1} + x_1^{[p-(p+1)/2-1](p-1)} x_2^{[(p+1)/2+1](p-1)-1} + \dots + x_2^{p(p-1)-1}) \\ \psi_2 &= x_1^{p(p-1)-1} + x_1^{(p-1)(p-1)-1} x_2^{p-1} + \dots + x_1^{[p-(p+1)/2+1](p-1)-1} x_2^{[(p+1)/2-1](p-1)}. \end{aligned}$$

The cogrediency under  $G(p)$ ,

$$(5) \quad (\psi_{1i}, \psi_{2i}) \text{ Co. } (x_1^{(p-1)i+1}, x_2^{(p-1)i+1}),$$

leads to the operator  $\Delta_i$  invariant with respect to the quantic,

$$f_e = a_0 x_1^e + a_1 x_1^{e-1} x_2 + \dots + a_e x_2^e,$$

viz.,

$$\Delta_i = \psi_{2i}^e \frac{\partial}{\partial a_0} - \binom{e}{1} \psi_{2i}^{e-1} \psi_{1i} \frac{\partial}{\partial a_1} + \dots + (-1)^e \binom{e}{e} \psi_{1i}^e \frac{\partial}{\partial a_e},$$

( $\psi_{1i} = \psi_1/x_2^{(p-1)i}$ ,  $\psi_{2i} = \psi_2/x_1^{(p-1)i}$ ;  $i=0, 1$ ).

There is no simple formula analogous to  $P$  which would represent definitively all concomitants of  $f$  under  $G(p)$ , but particular cases raise important questions. Consider the formal invariant, ( $p=3$ ,  $e=2$ ), first brought to light by DICKSON,

$$R = a_0^2 a_2 + a_0 a_2^2 + a_0 a_1^2 + a_1^2 a_2 - a_0^3 - a_2^3.$$

We find,

$$\Delta_i R = (x_1^2 + x_2^2)^2 [(a_0^2 + a_1^2 - a_0 a_2) x_2^{2(3-2i)} + (a_0 a_1 + a_1 a_2) x_1^{(3-2i)} x_2^{(3-2i)} + (a_1^2 + a_2^2 - a_0 a_2) x_1^{2(3-2i)}],$$

$$\Delta_i^2 R = -(x_1^2 + x_2^2)^4 [a_0 x_2^{4(3-2i)} + a_1 x_2^{3(3-2i)} x_1^{3-2i} + a_1 x_2^{3-2i} x_1^{3(3-2i)} + a_2 x_1^{4(3-2i)}].$$

The quantities in brackets will be covariants of the whole group  $G(3)$  after transformation by  $x_1 = x_2'$ ,  $x_2 = x_1'$ , but the group of  $x_1^2 + x_2^2$  is the orthogonal  $q$  (or  $q'$ ),

$$q: x_1 = \mu x_1' - \nu x_2', \quad x_2 = \nu x_1' + \mu x_2', \quad q': x_1 = \mu x_1' - \nu x_2', \quad x_2 = -\nu x_1' - \mu x_2',$$

where  $\mu, \nu$  are integral residues (mod. 3).

The bracketed covariant in  $\Delta_i^2 R$  is reducible if  $i=0$ , and if  $i=1$  it is the ground-form,

$$f_4 = a_0 x_1^4 + a_1 x_1^3 x_2 + a_1 x_1 x_2^3 + a_2 x_2^4,$$

and different from  $f_2 = a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2$ . Note that  $f_2 = \left[ x \frac{\partial}{\partial x^3} \right] f_4$ .

Operators  $\Delta_{\gamma\delta}$  analogous to (3) in § 2 exist for arbitrary quantities  $f_e$  whose orders can be composed from the orders of  $L$  and  $N$ ,

$$(6) \quad e = \gamma p(p-1) + \delta(p+1), \quad (\gamma, \delta \text{ integers}),$$

for example,

$$\Delta_{10} = \frac{\partial}{\partial a_0} + \frac{\partial}{\partial a_{p-1}} + \frac{\partial}{\partial a_{2(p-1)}} + \dots + \frac{\partial}{\partial a_{p(p-1)}}.$$

The above theory in which special situations are emphasized, prepares for the following,

PROPOSITION. - *For the case of a typical, special group of transformations, viz.  $G(p)$ , to base a new theory of fundamental systems upon the total set of ground-quantics.*

The quantic  $f_e$  will be as employed above,  $a_0, \dots, a_e$  being arbitrary variables, and the term concomitant, where not otherwise defined, will mean an invariant or covariant, modulo  $p$ , in  $a_0, \dots, a_e, x_1, x_2$ . Equalities, except where further explained, will be congruences, modulo  $p$ , existing identically in the  $a_i, x_j$ . Occasionally, as a reminder, the modulus will be indicated. The Proposition is solved by means of a series of lemmas.

LEMMA 1. - *Every covariant  $f$ , (mod.  $p$ ), of degree unity in the coefficients  $a_i$  of  $f_e$  can be derived from one of degree  $> 1$  in the  $a_i$  by a succession of operations by a  $\Delta$  of the modular type.*

Let the coefficients of  $f$  be  $\beta_0, \dots, \beta_c$ . They are linear in the  $a_i$ . Form any definite algebraic concomitant  $R$  of  $f$  of degree  $g > 1$  in the  $\beta_j$ . Then  $R$  is also a concomitant of  $f_e$  under  $G(p)$ . By § 3, ( $n=1$ ), there exists a  $\Delta_1$ , and

$$\Delta_1^{g-1} R = J(xy)^b f_{y^r}, \quad (J \text{ an integer; } f_y \text{ a polar}).$$

The constant  $J$  is of non-essential type and may always be discarded but no essential numerical factor  $\rho$  in  $R$  can be discarded. If  $\rho \equiv 0 \pmod{p}$ ,  $R$  vanishes and a concomitant of  $f$  different from  $R$  must be chosen. The operator  $\Delta_1$  is of the form,

$$\Delta_1 = y_2^c \frac{\partial}{\partial \beta_0} - \binom{c}{1} y_2^{c-1} y_1 \frac{\partial}{\partial \beta_1} + \dots + (-1)^c \binom{c}{c} y_1^c \frac{\partial}{\partial \beta_c},$$

however,

$$\frac{\partial}{\partial \beta_j} = \frac{\partial a_0}{\partial \beta_j} \frac{\partial}{\partial a_0} + \frac{\partial a_1}{\partial \beta_j} \frac{\partial}{\partial a_1} + \dots + \frac{\partial a_e}{\partial \beta_j} \frac{\partial}{\partial a_e}, \quad (j=0, \dots, c).$$

The derivatives  $\partial a_i / \partial \beta_j$  are numerical. They are not all zero, else  $\Delta_1$  as an algebraic operator would be illusory, which it never is. Substituting the values of the  $\partial / \partial \beta_j$ ,  $\Delta_1$  becomes an operator  $\Delta$  in the  $\partial / \partial a_i$  for which, both algebraically and modulo  $p$ .

$$\Delta^{g-1} R / J(xy)^b \Big|_{y_k=x_k} = f, \quad \text{q. e. d.}$$

Note that, if the algebraic concomitant chosen as  $R$  is of the functional form,

$$\varphi(a_0^p, a_1^p, \dots, a_e^p),$$

it should be replaced, in this theory, by

$$R = \varphi(a_0, a_1, \dots, a_e).$$

This is possible, since, under the induced group,

$$(a_0^p, a_1^p, \dots, a_e^p) \text{ Co. } (a_0, a_1, \dots, a_e).$$

LEMMA 2. - *Every concomitant  $D$ , modulo  $p$  of the formal type, of  $f_e$ , is a simultaneous algebraic concomitant of the set consisting of  $L, N$ , and*

the totality of linearly independent formal modular concomitants,  $l_1, l_2, \dots$ , of the first degree in  $a_0, \dots, a_e$ .

The proof results from the succession of algorithms (a), (b), ..., (f). Let  $D$  be of degree  $g$  in  $a_0, \dots, a_e$ .

(a) We are to prove first the converse of Lemma 1, that  $D$  of Lemma 2 can be differentiated to an  $l_i \neq 0$  by successive operations by a  $\Delta$  operator of one of two types, the first being,

$$\Delta(z) = y_2(z)^e \frac{\partial}{\partial a_0} - \binom{e}{1} y_2(z)^{e-1} y_1(z) \frac{\partial}{\partial a_1} + \dots + (-1)^e y_1(z)^e \frac{\partial}{\partial a_e},$$

where  $y_i(z)$  is an arbitrary function of  $z_1, z_2$  particularizable only in such ways as will make the cogredency,

$$(z_1, z_2) \text{ Co. } (y_1(x), y_2(x)) \text{ Co. } (x_1, x_2),$$

in any case, one which represents the total group,  $G(p) \pmod{p}$ , as distinguished from a subgroup. If  $\Delta(z)D=0$ , it is because this polynomial in  $x_1, x_2$  vanishes identically [6]. If  $y_1 = z_1^{pt}$ ,  $y_2 = z_2^{pt}$ , with  $t$  any chosen integer  $> 0$ , and if  $C_j$  is any coefficient of the covariant  $D$ , ( $C_j = D$  if  $D$  is an invariant), then,

$$\Delta(z)C_j = z_2^{ept} \frac{\partial C_j}{\partial a_0} - \binom{e}{1} z_2^{(e-1)pt} z_1^{pt} \frac{\partial C_j}{\partial a_1} + \dots + (-1)^e z_1^{ept} \frac{\partial C_j}{\partial a_e},$$

vanishes identically. Hence  $C_j$  is free from all  $a_i$ , (or contains all of its  $a_i$  in the form  $a_i^{gp}$ . Cf. (b)), except possibly the  $a_i$  for which  $\binom{e}{i}$  is divisible by  $p$ . Thus the hypothesis,  $\Delta(z)D=0$  is untenable with  $y_1, y_2$  as assumed, except possibly in certain special cases. In the special case, choose  $\Delta_1(z)$  instead of  $\Delta(z)$ ,  $\Delta_1(z)$  being the result of substituting  $\partial/\partial a_0, \partial/\partial a_1, \dots$ , for  $x_1^e, x_1^{e-1} x_2, \dots$ , respectively, in the covariant,

$$\Psi = (zx)(z^p x)(z^{p^2} x) \dots (z^{p^{e-1}} x).$$

No coefficient of a term of  $\Psi$  contains the modulus. Hence in the special cases being considered,  $\Delta_1(z)C_j=0$  reduces  $\Delta_1(z)D=0$  to an absurdity (Cf. (b) however).

Let  $\Delta(z)D \neq 0$ , for some  $y_1, y_2$  considered. Then <sup>(2)</sup>,

$$(7) \quad \Delta(z)D = d_1, \quad \Delta(z)d_1(z, x) = d_2 \neq 0.$$

Continuing we reach,

$$\Delta(z)d_{g-2}(z, x) = d_{g-1} = K_1 A(z, x) \alpha_2^r b_x^s \neq 0,$$

$\alpha_x^r b_x^s$  being a symbolical product equal to a form of degree one in  $a_0, \dots, a_e$ ,

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<sup>(2)</sup> The  $\Delta(z) \Delta(z)D$  is found most easily by operating the formal square of  $\Delta(z)$  upon  $D$ . The expression  $K_1$  is constant or (if  $r=s=0$ ) an invariant of degree unity in  $a_0, \dots, a_e$ .

and  $A(z, x)$  a universal covariant which becomes a polynomial in  $L, N$  when  $z_k = x_k$ , the possibility  $A(x, x) = 0$  not being excluded if  $A(z, x)$  contains the factor  $(zx)$ . We have  $\alpha_x^r b_x^s = l_i \neq 0$ . Inversely  $\alpha_z^r b_x^s$  is formally a term of the  $r$ -th polar of  $l_i$ , but, since  $\alpha_z^r b_x^s$  is a covariant, it is the whole polar of  $l_i$ .

(b) There is one and only one class of exceptions to (a), viz. when  $D$  has the functional form, like

$$\varphi(\alpha_0^p, \dots) = [\alpha^p \partial / \partial \alpha]^h \psi(\alpha_0, \dots), \quad (0 < h < g + 1),$$

of a formal modular concomitant of  $f_e$  taken simultaneously with a quantic,

$$f_{te} = \alpha_0^{p^t} x_1^e + \alpha_1^{p^t} x_1^{e-1} x_2 + \dots + \alpha_e^{p^t} x_2^e.$$

Represent the process of algebraic transvection between two quantics  $A, B$ , by  $\{A, B\}^r$ . Then, using only known elementary lemmas in number theory, such as

$$\binom{e p^t}{i p^t} \equiv \binom{e}{i} \pmod{p},$$

we find,

$$f_{te} \equiv \{f_e^{p^t}, L_t^e\}^{ep^t} \pmod{p} \quad (L_t = x_1 x_2^{p^t} - x_2 x_1^{p^t}).$$

Since  $L_t$  is reducible in terms of  $L$  and  $N$ ,  $f_{te}$  is in the algebraic system of the set  $(L, N, l_1, l_2, \dots)$ . There is a like formula for  $F_{te}$ ,  $F_c$  being any covariant of order  $c$ . It is known also that  $N$  is an algebraic covariant of  $L$ , being given by the formula,

$$\{ \dots \{ \{ L, L \}^2, L \}, L \}, \dots, L \} \equiv N \pmod{p}.$$

Our series of differential equations which ended with  $\Delta(z) d_{g-2}(z, x) \neq 0$  will now end before any  $d_i(z, x)$  linear in  $\alpha_0, \dots, \alpha_e$  is reached, but not before we reach a form  $D'$  which is a concomitant of  $f_{te}$  alone. Replace, in  $\Delta$ ,  $\partial / \partial \alpha_i$  by  $\partial / \partial \alpha_i^{p^t}$ , ( $i = 0, \dots, e$ ) and call the new operator  $\Delta(\partial / \partial \alpha_i^{p^t})$ . It will reduce  $D'$  to a covariant linear in  $\alpha_0^{p^t}, \dots, \alpha_e^{p^t}$  and the latter will be an algebraic transvectant of one of the  $l_i$ . Integrating back to  $D'$  the latter is seen to be an algebraic concomitant of the set  $(L, N, l_1, l_2, \dots)$ . Using  $\Delta(\partial / \partial \alpha_i)$  again we can integrate from  $D'$  to  $D$ , but the nature of the integration processes is to be described in (d). There is an obvious extension of this algorithm to the case where  $D$  is a simultaneous concomitant of any number of forms  $f_{te}$  including  $f_e$ .

(e) The maximum order in  $x_1, x_2$  of an irreducible formal modular concomitant  $D$  is  $p^2 - 1$ . In the writer's paper in the *Annals of Mathematics* volume 19, this theorem was not stated in its full generality, but in an earlier paper, a more general view-point had been discussed. In relation (1) of the *Annals* article, if  $m > p^2$ , so there is an excess in the number of coefficients of  $\varphi_1, \varphi_2$  over the

number in  $f_m$ , the excess number of coefficients can be arbitrarily assigned if the congruences afterward available are left consistent. Solution of the congruences determine  $\varphi_1, \varphi_2$  in terms of which  $f_m, m > p^2 - 1$ , is reduced. Also  $D$  is reduced if its order is  $> p^2 - 1$ . For,

$$D' = (\lambda\mu)^k D, \quad D = L\varphi_1 + N\varphi_2, \quad D' = L'\varphi_1' + N'\varphi_2'.$$

Hence  $\varphi_2' = (\lambda\mu)^j \varphi_2$ ,  $(x_1, x_2)$  being an integral root of  $L$ . But if  $\varphi_2$  is thus modular and homogeneous in  $x_1$  and  $x_2$  it is a formal covariant.

(d) Let  $\Delta(z)$  be as originally chosen in (a). Then the partial differential equation in  $\alpha_0, \dots, \alpha_e$ ,

$$\Delta(z)d_{g-2}(z, x) = d_{g-1}(z, x),$$

can be solved for  $d_{g-2}$  as the dependent variable. The solution is known to exist and to be a concomitant. The process of solution can be regarded as unique, is analytic, and in effect algebraic (Cf. § V). The equalities are equations, the modulus not being applied either in forming equations (7) or in integrating them. Then we can integrate the equation,

$$\Delta(z)d_{g-3}(z, x) = d_{g-2}(z, x).$$

Continuing we can solve for  $D$  from,

$$\Delta(z)D = d_1(z, x).$$

The argument where  $D'$  is involved has already been stated. The main point is that  $D$  is thus obtained by algebraic processes based originally upon an  $l_i$  taken with  $L$  and  $N$ , (Cf. (f)).

(e) Some details concerning integration processes may be mentioned. We have always LAGRANGE'S method, but it need not be used. If the formal modular concomitant  $D$  of  $f_e$ , with which we start is also an algebraic concomitant of  $f_e$ ,  $D$  is a linear combination of symbolic monomials of the type of  $T$ ,

$$T = (ab)^e (ac)^\sigma \dots (bc)^\tau \dots a_x^q b_x^s \dots, \quad (f_e = a_x^e = b_x^e = \dots).$$

Now,

$$\Delta T = \left( y_2 \frac{\partial}{\partial a_1} - y_1 \frac{\partial}{\partial a_2} \right)^e T = k(bc)^\tau (bd)^r \dots b_x^s \dots (xy)^q b_y^e c_y^\sigma \dots,$$

and inversely if the right hand quantic  $kT_1$  is given and  $T$  is unknown in this relation, we can solve for  $T$ . It is a term of the transvectant <sup>(3)</sup>,

$$t_1 = \{ T^{(1)}, a_x^e \}^{e-q},$$

---

<sup>(3)</sup> The letters  $k, \beta, \gamma, \delta$  represent constants.

$T^{(4)}$  being what  $T_1$  becomes when  $(xy)^q$  is deleted and  $y_j$  changed to  $x_j$ , ( $j=1, 2$ ). Hence by a known theorem,  $T$  is a linear expression in algebraic transvectants,

$$T = \sum_{i=1, 2, \dots} h_i \{ T^{(i)}, \alpha_x^e \}^{r_i} = \sum_i h_i t_i, \quad (h_i \text{ numerical}),$$

the forms  $T^{(i)}$  being obtained from the leading one,  $T^{(4)}$ , by convolution. If  $d_1$ , in our equation  $\Delta(z)D = d_1$ , is a sum;  $d_1 = \Sigma \beta T_1$ ,  $D$  is a sum  $\Sigma \gamma T = \Sigma \Sigma \delta t_i$ . This formulary is applicable to all equations (7) if  $D$  is algebraic.

Let  $D$  in  $\Delta(z)D = d_1$  be formal modular without being regarded, at first, as algebraic. Some expressions based on modular cogrediencies will occur in the analysis which we were able to dispense with in the discussion just given. Among these are the polars,

$$P' = x_1^{p^t} \frac{\partial F}{\partial x_1} + x_2^{p^t} \frac{\partial F}{\partial x_2} = m \{ L_t, F \}^4 \pmod{p},$$

$m$  being the order of  $F$ , but the transvectant is in the algebraic system of the set  $(L, N, l_1, l_2, \dots)$  if  $F$  is, ( $m \not\equiv 0 \pmod{p}$ ). In the work of calculating any  $\{ A, B \}^r$ , the last step is to change  $y_j$  into  $x_j$ . By means of the change  $y_j$  equal to  $x_j^{p^t}$ , ( $j=1, 2$ ), we obtain a modular transvectant  $\mu = \{ A, B \}_{p^t}^r$ , but the writer has proved that any  $\mu$  is a linear expression in modular polars of algebraic transvectants, the domain of the expression to include  $L$  and  $N$  [7].

The cogrediencies following (and certain contragrediencies),

$$(\alpha_0^{p^t}, \alpha_1^{p^t}, \dots, \alpha_e^{p^t}) \text{ Co. } (a_0, a_1, \dots, a_e),$$

under the induced linear group, are represented simply by certain pure invariants due to W. L. G. WILLIAMS [9]. The induced group is special but, for it, we can copy the invariants designated by WILLIAMS as the fundamental invariants of the arbitrary linear form in  $q = e + 1$  variables under the total linear group whose coefficients are marks of the  $GF(p^n)$ . In the present case  $n=1$ , and his system becomes, for us, any  $q$  invariants whose leading terms are respectively,

$$\alpha_0 \alpha_1^p \alpha_2^{p^2} \alpha_3^{p^3} \dots \alpha_e^{p^e}, \quad \alpha_1^{p^s} \alpha_2^{p^{2s}} \alpha_3^{p^{3s}} \dots \alpha_e^{p^{es}}, \dots, \quad \alpha_e^{p^{es}}, \quad (s = p - 1).$$

The invariants meant are now evidently simultaneous invariants, for certain indicated values of  $t$ , of forms  $f_{te}$  which we have constructed by algebraic transvection between  $f_e$  and  $L$  and  $N$ .

Transvection, as is well known, is the definitive process for construction of complete systems of algebraic concomitants of any set of binary quantics.

( $f$ ) The analytic integration of equations (7) described in ( $d$ ), is, in effect, algebraic and is a master ( $4$ ) process which includes implicitly all of the

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(4) MACMAHON: *Combinatory Analysis*, Ch. II.

formularies which we have described in (e). At first  $D$  seems to be derivable algebraically only from  $l_1, l_2, \dots$ , taken with the universal covariants <sup>(5)</sup> of the two cogredient sets  $z_i, x_i$ . However the variables are  $a_0, \dots, a_e$ , in  $\Delta$ ;  $z_1, z_2, x_1, x_2$  figuring as constants. The final integral  $D$  is free from  $z_1, z_2$  and is unaffected by any admissible <sup>(6)</sup> particularization, as  $z_i = x_i^{p\nu}$ ,  $\nu$  an arbitrary integer. Hence, in fact,  $D$  is an algebraic concomitant of the set  $(L, l_1, l_2, \dots)$ . Since  $D$  is derivable algebraically from this set,  $D$  had the algebraic <sup>(7)</sup> character *ab initio* and was a simultaneous algebraic concomitant of this ground-set  $(L, l_1, l_2, \dots)$ , q. e. d.

LEMMA 3. - *The number of linearly independent concomitants, modulo  $p$ , of  $f_e$ , of degree unity in  $a_0, \dots, a_e$ , is finite. Consequently the system of all formal modular concomitants of  $f_e$  is finite.*

It is known that the algebraic system of any finite set  $(L, N, l_1, l_2, \dots)$  is finite. Hence we need to prove only the first part of the Lemma 3.

If  $V$ ,

$$V: \quad x_1 = \gamma_1 x_1' + \gamma_2 x_2', \quad x_2 = \delta_0 x_1' + \delta_1 x_2',$$

is the arbitrary transformation from the total group  $G(p)$ , its poles are the roots of the congruence,

$$\delta_0 x_1^2 + (\delta_1 - \gamma_1) x_1 x_2 - \gamma_2 x_2^2 = \delta_0 (x_1 - r_1 x_2)(x_1 - r_2 x_2) \equiv 0 \pmod{p}.$$

The  $r_i$  are integral residues for some transformations  $V$  and Galoisian imaginaries for other  $V$ , according to the current integral values of  $\delta_0, \delta_1, \gamma_1, \gamma_2$ . With this understanding  $r_1, r_2$ , may be treated as parameters. The forms  $\varrho_1, \varrho_2$  are covariants under  $V$ ;

$$\varrho_1 = x_1 - r_1 x_2, \quad \varrho_2 = x_1 - r_2 x_2,$$

and the relations exist,

$$\begin{aligned} \varrho_1' &= a_1^{-1} \varrho_1, & \varrho_2' &= a_2^{-1} \varrho_2, \\ 2a_1 &= \gamma_1 + \delta_1 + (r_1 - r_2)\delta_0, & 2a_2 &= \gamma_1 + \delta_1 - (r_1 - r_2)\delta_0, & a_1 a_2 &= \gamma_1 \delta_1 - \gamma_2 \delta_0. \end{aligned}$$

$$W: \quad x_1 = (r_2 \varrho_1 - r_1 \varrho_2) / (r_2 - r_1), \quad x_2 = (\varrho_1 - \varrho_2) / (r_2 - r_1).$$

Transformed by  $W$ ,  $f_e$  becomes a polynomial in  $e+3$  concomitants known as invariant elements, viz.,  $\varrho_1, \varrho_2$  and  $e+1$  invariants linear in  $a_0, \dots, a_e$ , (the coefficients of  $f_e'$ ),  $I_0, \dots, I_e$ .

If  $l$  is a typical first degree covariant described in Lemma 3,  $l$  is isobaric (mod.  $p-1$ ), and its covariant relation under  $W$  exhibits it as a rational integral

<sup>(5)</sup> KRATHWOHL: Amer. Journal of Math., vol. 36 (1914), p. 449.

<sup>(6)</sup> A value  $z_i$  is admissible if  $z_i$  Co.  $x_i$  and  $z_i$  contains an arbitrary parameter.

<sup>(7)</sup> Paragraph (f) (clarified by e) advances the argument mainly by the truism that a concomitant derived by integration of (7) is derived by an algebraic process.

polynomial in  $\varrho_1, \varrho_2, I_0, \dots, I_e$ . As such it will simplify back to its primary form as a concomitant in  $x_1, x_2, a_0, \dots, a_e$ , with the aid of FERMAT'S congruence (generalized),

$$r_i^{p^2} - r_i \equiv 0 \pmod{p}, \quad (i=1, 2).$$

This as a structural law defines a system of polynomials in  $\varrho_1, \varrho_2, I_j$  in the sense of HILBERT'S basis theorem [8] Among all polynomials  $l$  built according to this law, there exists a finite basis-set  $(l_1, \dots, l_\nu)$  such that the typical  $l$  of degree unity can be written,

$$(8) \quad l = P_1 l_1 + P_2 l_2 + \dots + P_\nu l_\nu,$$

wherein the  $P_i$  are also polynomials in the invariant elements but do not, unless by additional proof, simplify as aforesaid to concomitants in  $x_1, x_2, a_0, \dots, a_e$ . But, since all  $l$  are linear in  $a_0, \dots, a_e$ , no  $P_i$  involves  $a_0, \dots, a_e$ . When (8) is given its expression in  $\varrho_1, \varrho_2, I_j$ , the  $P_i$  will be term-wise invariant expressions in  $\varrho_1, \varrho_2$  alone. Since

$$l' = (r_2 - r_1)^h l,$$

a relation exists,

$$(9) \quad P_1' (r_2 - r_1)^{\mu_1} l_1 + \dots + P_\nu' (r_2 - r_1)^{\mu_\nu} l_\nu = (r_2 - r_1)^h (P_1 l_1 + \dots + P_\nu l_\nu).$$

But the  $\alpha_i$  can now be given particular values without affecting the  $P_i$ . Hence, from (8), we can have  $\nu$  linear equations to determine the  $P_i$  in forms free from  $r_1, r_2$ . From their expressions in  $\varrho_1, \varrho_2$  or from (9) they are covariants. Hence they are universal covariants and, if not constant, are polynomials in  $L$  and  $N$ , q. e. d.

We have proved therefore that a formal modular fundamental system of  $f_e$  is an algebraic simultaneous system of a terminating ground-set of quantities,  $(L, N, l_1, l_2, \dots)$ . The algebraic system is got by transvection (8) and the ARONHOLD symbolism, with algorithm I, § IV, Lemma 2, (e), suffices if we write  $L = \lambda_x^{(a)} \dots \lambda_x^{(p+1)}$ ,  $N = \nu_x^{(a)} \dots \nu_x^{(p(p-1))}$ , certain combinations of symbols  $\lambda_1^{(j)}$ ,  $\lambda_2^{(j)}$ , and also of  $\nu_1^{(k)}$ ,  $\nu_2^{(k)}$ , being zero. If  $e \geq p^2$ ,  $\beta < a + 1$ ,

$$f_e = N^\alpha \eta_e + N^{\beta-1} Lx + \dots + NL^{\gamma-1} \psi + L^\delta \omega \pmod{p},$$

whence it is seen that the  $l_i$  include the forms  $\eta_e, x, \dots, \omega$ . There are also various types of first degree covariants in case  $e < p^2$ .

In § 6 we show that some covariants, at least, among the  $\eta_e, x, \dots, \omega$ , are algebraic covariants of the set  $(L, N, f_e)$ . On the question, whether the latter

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(8) The converse statement forces the conclusion that any transvectant which does not turn out to be a formal modular concomitant has  $p$  as an essential factor, because, to make  $L$  a ground-quantic is to limit the group to  $G(p)$ .

set would always suffice as a ground-set, we say that we obtain a typical  $l_i$  from  $a_z^r b_x^s$  of (7) by changing  $z_j$  into  $x_j$ , ( $j=1, 2$ ). The result is  $B=a_x^r b_x^s$ , and,

$$\Delta(z)B=E(z, x) \neq 0,$$

$E(z, x)$  being a universal covariant in the two cogredient sets  $(z_1, z_2)$ ,  $(x_1, x_2)$ , and  $E(x, x)$  a polynomial in  $L$  and  $N$ . With  $E(z, x)$  expressed as a polynomial in fundamental universal covariants, the integral  $B$  of the equation is determined algebraically from these covariants and  $f_e$  alone. Since the variables in the equation are  $a_0, \dots, a_e$ ;  $B$  is unaffected by any admissible expression of  $z_i$  in terms of  $x_i$ . Hence  $B$  is a simultaneous algebraic concomitant of the set  $(f_e, L)$ , and is obtainable by transvection between  $f_e$  and  $L$ .

Finally every formal modular concomitant (mod.  $p$ ) of  $f_e$  is a simultaneous algebraic concomitant of  $f_e$  and  $L$ .

## V. – Polynomial solutions in general of linear, partial differential equations.

### Almost invariative functions.

The usual general methods of solving a linear, partial differential equation of the first order often leads to a solution in such transcendental form as is not easy to convert into a rational, integral polynomial even when it is known that a solution exists in polynomial form. We prove however that if the variables may be restricted, each to a segment of variation, any solution may be approximately expressed by a polynomial. If the latter is then a covariant, it is an almost-covariant ( $\epsilon$ -covariant) of a determinate ground-function. The idea of an  $\epsilon$ -covariant can be illustrated simply if we add infinitesimal increments to the coefficients of  $f_e = a_0 x_1^e + \dots$ , replacing  $a_i$  by  $a_i + \delta_i a_i$ , ( $i=0, \dots, e$ ). A covariant  $I$  of  $f_e$  then receives an increment, and  $I$  itself will be an  $\epsilon$ -covariant of,

$$f_e + \delta f_e = (a_0 + \delta_0 a_0) x_1^e + (a_1 + \delta_1 a_1) x_1^{e-1} x_2 + \dots + (a_e + \delta_e a_e) x_2^e.$$

Let,

$$\theta = f(r, a_1, \dots, a_e)$$

be a function which is expansible into a power series absolutely convergent in a region  $\rho$  of  $r$ -values.

Let

$$\theta = F_e(r, a_1, \dots, a_e),$$

be a given covariant of  $f$  subject to a transformation  $q$ . This problem exists: To determine  $f$  and  $q$  with  $r$  restricted to  $\rho$ . We reduce this question, relating to functions, also to a problem on polynomials.

If  $f(r, a_1, \dots, a_e)$  be expanded into a power series by ordinary methods, the history of the  $a_i$  may be largely lost but the requirement of convergency restricts to some region the variation of each  $a_i$ . We first prove the following,

**THEOREM.** - *Any  $e$ -parameter manifold (for example the solution of a differential equation),*

$$(10) \quad \theta = F_e(r, a_1, \dots, a_e),$$

*$r$  being restricted to  $\varrho$  and each  $a_i$  properly restricted, can be expressed rationally in  $e$  parameters, approximately, and the limits of error of the approximation will be bounded and controllable.*

A preliminary discussion of postulates is given.

*Regions, fields, and frontiers.* In the polar plane, two fixed angles ( $\theta$ ) and the two corresponding finite radii ( $r$ ) delimit a region  $\varrho_0$  shaped like a keystone. Let  $C$  be a continuous segment which passes from the lower to the upper bounding arc, ( $e_1, e_2$ ), without intersecting the bounding radii. Assume  $C$  to be single valued with respect to  $\theta$ , each  $\theta$  between the radii being a coordinate of one and only one point of  $C$ . The radial width of  $\varrho_0$  will be taken to be  $\varrho$ , the range of variation of  $r$ . We say then that  $C$  is properly contained in  $\varrho_0$ .

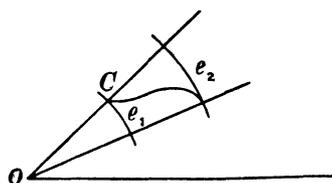


Fig. 1.

A set of curves  $C_1, \dots, C_t$ , ( $t > 1$ ), properly contained (p. c.) in the  $\varrho_0$  of minimum angular width, and radial width  $\varrho$ , is to be called a field  $\tau$ .

*Proof of the theorem.* If a curve  $\theta = F(r)$  has a continuous branch p. c. in  $\varrho_0$ ,  $n$  determinations (points) in  $\varrho_0$ , viz.,  $d_i: (\theta_i, r_i)$ , ( $i = 1, \dots, n$ ), are necessary and sufficient to determine  $F(r)$  in the approximate form, ( $r$  on  $\varrho$ ),

$$(11) \quad \theta = \alpha r^{n-1} + \beta r^{n-2} + \dots + \kappa.$$

The exactness of the approximation depends upon the number and law of distribution of the  $d_i$  on the branch. The law may be described by a symbol  $l_0 = (\lambda_1, \dots, \lambda_n)$ , the  $\lambda_j$  being the respective apothems from  $O$  when perpendiculars are drawn from the  $d_i$  to the polar axis. The inaccuracy of the representation of  $\theta = F(r)$  by (11) depends upon our choice of the points or, as we may say, is bounded, and the frontier is the point-set  $(d_1, \dots, d_n)$ .

Let  $T_1$  be a one-parameter system (field) of curves,  $\theta = F_1(r, a_1)$ , all p. c. in  $\varrho_0$ . The curve-set obtained by assigning  $q$  values to  $a_1$ , ( $r$  on  $\varrho$ ), may be represented by

$$(12) \quad \theta = \sum_{i=1, n} \alpha_{ij} r^{n-i}, \quad (j = 1, \dots, q).$$

Assume that the  $\alpha_{ij}$ , ( $j=1, \dots, q$ ), are the determinations of a function  $\zeta_i$  of  $\mu$ ,

$$\zeta_i = \zeta_{i1}\mu^{q-1} + \zeta_{i2}\mu^{q-2} + \dots + \zeta_{iq},$$

for assigned values  $\mu = \mu^{(1)}, \dots, \mu = \mu^{(q)}$ , ( $\mu^{(g)} \neq \mu^{(h)}$ ). With none but obvious restrictions these values of  $\mu$  may be chosen arbitrarily, for, whatever set  $\mu^{(g)}$  is chosen, the parameters  $\zeta_{ij}$  are given by CRAMER'S rule. Hence the one-parameter system,

$$(13) \quad \theta = \sum_{i=1, n} (\zeta_{i1}\mu^{q-1} + \zeta_{i2}\mu^{q-2} + \dots + \zeta_{iq})r^{n-i}, \quad [= \varphi(\zeta_{i, i_2}; \mu; r); i_1 = i],$$

with  $\mu$  continuous from the smallest to the largest  $\mu^{(g)}$ , is in approximate coincidence with  $T_1$  or with a sub-field of  $T_1$ . The inaccuracy (limits of error) of the coincidence depends upon the law  $l_1$  of distribution of the curve-set (12) upon  $T_1$ , the curve-set being therefore the frontier of the bounded inaccuracy.

Before we generalize (13), by induction, we consider certain details of the generalization from  $\theta = F_1(r, \alpha_1)$  to the two-parameter case  $\theta = F_2(r, \alpha_1, \alpha_2)$ . Assume a  $\varrho_0$  which p. c.  $\infty^2$  curves  $\theta = F_2$ , and assign a well-spaced set of  $q_2$  values  $\alpha_2^{(1)}, \dots, \alpha_2^{(q_2)}$  in succession to  $\alpha_2$ . Represent each single parameter family obtained, in a form (13). We can use (13) itself after attaching the superscript  $k$  to each  $\zeta_{i, i_2}$ ,  $k$  to take the integral values  $1, \dots, q_2$ . Referring to continuous variation of a real parameter from an abscissa  $a$  to an abscissa  $b$  as a segment (a, b), we have seen that the set  $\mu^{(g)}$  may be chosen from any segment consistent with convergency. Hence when,

$$(14) \quad \theta = F_2(r, \alpha_1, \alpha_2^{(1)}), \dots, \quad \theta = F_2(r, \alpha_1, \alpha_2^{(q_2)}),$$

are represented rationally as stated, the sets like  $\mu^{(g)}$ , corresponding to the respective  $\alpha_2^{(1)}, \dots, \alpha_2^{(q_2)}$ , may all be chosen from the same segment. Hence one and the same parameter  $\mu$ , (viz.,  $\mu_1$ ) can be used in all  $q_2$  rational representations, which then become

$$(15) \quad \theta = \varphi_1(\zeta_{i, i_2}^{(k)}; \mu_1; r), \quad (k=1, \dots, q_2; \varphi_1 = \varphi).$$

We can now generalize to the case of  $e$  parameters. Let the curves of the manifold,

$$\theta = F_e(r, \alpha_1, \dots, \alpha_e),$$

be all p. c. in a  $\varrho_0$ . Assign to  $\alpha_e$ ,  $q_e$  particular values  $\alpha_e^{(1)}, \dots, \alpha_e^{(q_e)}$ , thus obtaining  $q_e$  determinations in  $e-1$  parameters, of  $\theta = F_e$ . The hypothesis of induction is that each determination, in  $\varrho_0$ , has a rational representation. The determinations are,

$$(16) \quad \theta = F_e(r, \alpha_1, \dots, \alpha_{e-1}, \alpha_e^{(1)}), \dots, \quad \theta = F_e(r, \alpha_1, \dots, \alpha_{e-1}, \alpha_e^{(q_e)}),$$

and the respective rational representations are,

$$(17) \quad \theta = \sum_{i_1=1, n} \sum_{i_2=1, q_1} \dots \sum_{i_e=1, q_{e-1}} \zeta_{i_1 \dots i_e}^{(l)} \mu_1^{q_1 - i_2} \mu_2^{q_2 - i_3} \dots \mu_{e-1}^{q_{e-1} - i_e} r^{n - i_1}, \quad (l=1, \dots, q_e),$$

equation (17) being (13) when  $l=1$ ,  $e=2$ . The abbreviation of (17) is,

$$\theta = \varphi_{e-1}(\zeta_{i_1 \dots i_e}^{(l)}; \mu_1, \dots, \mu_{e-1}; r), \quad (l=1, \dots, q_e).$$

The  $q_e$  determined numbers  $\zeta_{i_1 \dots i_e}^{(l)}$ , ( $l=1, \dots, q_e$ ), may be regarded as  $q_e$  determinations of a function  $f_{i_1 \dots i_e}$  of  $\mu_e$ ,

$$f_{i_1 \dots i_e} = \zeta_{i_1 \dots i_e} \mu_e^{q_e - 1} + \zeta_{i_1 \dots i_e, 2} \mu_e^{q_e - 2} + \dots + \zeta_{i_1 \dots i_e, q_e} \mu_e^0$$

for the respective assignments  $\mu_e = \mu_e^{(1)}, \dots, \mu_e = \mu_e^{(q_e)}$ , chosen from a segment independent of the segments preempted by previous  $\mu_i$ , ( $i < e$ ). Replacing in (17),  $\zeta_{i_1 \dots i_e}^{(l)}$  by  $f_{i_1 \dots i_e}$ , we get,

$$(18) \quad \theta = \varphi_e(\zeta_{i_1 \dots i_{e+1}}; \mu_1, \dots, \mu_e; r).$$

Allowing  $\mu_e$  in (18) to vary continuously over its segment from the least to the greatest numbers of the set  $\mu_e^{(k)}$  we obtain not merely the  $q_e$  particular equations (17), but the whole manifold,

$$\theta = F_e(r, a_1, \dots, a_e),$$

as defined in  $\varrho_0$ , within an inaccuracy (limits of error) which depends upon the law  $l_e$  of the frontier manifold-set (16), q. e. d.

If  $\theta = F_e$  is a general solution of a partial differential equation like (7),  $\theta = \varphi_e$  is an almost accurate determination of the solution in polynomial form, with  $r, \mu_1, \dots, \mu_e$  restricted to segments.

If  $F_e$  is a covariant, as originally described,  $\varphi_e$  is an  $\varepsilon$ -covariant. If  $\varphi_e$  is made homogeneous in the  $\mu_j$  by the adjunction of one  $\mu$ , as  $\mu_0$ ,  $\varphi_e$  has the form of a covariant of a binary quantic. Assuming that, under  $q$ ,

$$(\zeta_0, \zeta_1, \dots, \zeta_e) \text{ Co. } (\mu_0, \mu_1, \dots, \mu_e),$$

and that  $g$  is the degree of  $\varphi_e$  in  $\mu_0, \dots, \mu_e$ , it follows that, except for  $\varepsilon$ -terms,

$$\Delta^{g-1} \varphi_e = \left( \zeta_0 \frac{\partial}{\partial \mu_0} + \zeta_1 \frac{\partial}{\partial \mu_1} + \dots + \zeta_e \frac{\partial}{\partial \mu_e} \right)^{g-1} \varphi_e,$$

if not zero, is one of a set of admissible ground-functions  $f(r, \mu_0, \dots, \mu_e)$ , possibly multiplied by a universal covariant of  $q$ , while  $\Delta^g \varphi_e$  (if  $\neq 0$ ) is such a universal covariant, from which  $q$  may be determined.

We can also generalize (17) so the number of  $r$ -variables,  $(r_0, r_1, \dots, r_n)$ , is  $n+1$  instead of two.

## VI. – A formal modular system algebraically interpreted.

An important recent problem on complete systems has been the direct one of determining the fundamental covariants of various orders of quantities,

$$f_e = a_0 x_1^e + a_1 x_1^{e-1} x_2 + \dots + a_e x_2^e,$$

with arbitrary coefficients, under the total linear group  $G(p)$ ,  $p$  prime, of order,

$$(p^2 - p)(p^2 - 1).$$

When  $p=2$  the group, of order 6, may be represented as follows;

$$x_1 = \lambda_1 x_1' + \mu_1 x_2', \quad x_2 = \lambda_2 x_1' + \mu_2 x_2' \pmod{2},$$

and its independent universal covariants are,

$$L = x_1^2 x_2 + x_1 x_2^2, \quad Q = x_1^2 + x_1 x_2 + x_2^2, \quad (=N).$$

As a construction process the method given in § IV, where such a system is seen to be an algebraic simultaneous system of  $(L, N, l_1, l_2, \dots, l_n)$ , is hardly satisfactory because some of the  $l_i$  are of high order in  $x_1, x_2$  and moreover a purely algebraic method neglects useful number-theoretic facts, <sup>(9)</sup> but, after we have determined the simultaneous system (mod. 2) of  $(f_1, f_2, f_3)$ , we shall express the fundamental covariants in terms of algebraic transvectants and thus bring to light specifically the unity between the invariant theory of quantities under the algebraic group (1), and that under a modular group. It becomes clear that contributions to the modular theory have greatly advanced general invariant theory.

The algebra here is a type of algebra of the odd numbers. Nature makes the separation into even and odd in various ways. Some species of simple flowers have nearly always an even number of petals, others an odd number.

### Method for seminvariant systems.

A fundamental system of seminvariants of  $(f_1, f_2, f_3)$  is found as the culmination of an inductive process based on the simultaneous system of universal covariants (mod. 2), of the three transformations  $S_1, S_2, S_3$ ,

$$\begin{aligned} S_1: \quad x_1 &= x_1' + x_2', & x_2 &= x_2', & S_2: \quad y_1 &= y_1' + y_2', & y_2 &= y_2', \\ S_3: \quad z_1 &= z_1' + z_2', & z_2 &= z_2'. \end{aligned}$$

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<sup>(9)</sup> DICKSON elaborated a Theory of Classes in invariant theory which is primarily number-theoretic. He brought it to a high degree of generality Cf. [5].

The system of  $(S_1, S_2)$  had been previously determined and consists of [5],

$$(19) \quad x_2, \quad y_2, \quad \psi_1 = x_1^2 + x_1x_2, \quad \psi_2 = y_1^2 + y_1y_2, \quad (xy) = x_1y_2 + x_2y_1.$$

The writer has shown previously that the following trilinear is irreducible,

$$\Omega = x_1y_2z_1 + x_1y_1z_2 + x_2y_1z_1 + x_2y_1z_2.$$

THEOREM. - *A fundamental system of universal covariants, (mod. 2), of  $(S_1, S_2, S_3)$  is composed of,*

$$(20) \quad x_2, \quad y_2, \quad z_2, \quad \psi_1, \quad \psi_2, \quad \psi_3 = z_1^2 + z_1z_2, \quad (xy), \quad (xz), \quad (yz), \quad \Omega.$$

The proof is as follows. A covariant may be assumed homogeneous in the three pairs of variables, and of the form,

$$F = \sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^n a_{jki} (x_1^{l-i} x_2^i) (y_1^{m-j} y_2^j) (z_1^{n-k} z_2^k),$$

( $a_{jki} = 0$  or 1). The transformed of  $F$  by  $(S_1, S_2, S_3)$  is

$$F' = \sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^n a_{jki} \left[ \sum_{\lambda} \binom{l-i}{\lambda} x_1^{l-i-\lambda} x_2^{i+\lambda} \right] \left[ \sum_{\mu} \binom{m-j}{\mu} y_1^{m-j-\mu} y_2^{j+\mu} \right] \left[ \sum_{\nu} \binom{n-k}{\nu} z_1^{n-k-\nu} z_2^{k+\nu} \right],$$

( $\lambda = 0, \dots, l-i$ ;  $\mu = 0, \dots, m-j$ ;  $\nu = 0, \dots, n-k$ ). The expression for  $F' - F$  may be obtained by deletion of the zero value of each letter  $\lambda, \mu, \nu$ , from  $F'$ . Since  $F' - F \equiv 0 \pmod{2}$ , identically in the variables, we obtain, as the necessary and sufficient conditions for the covariancy of  $F$ , a set of linear congruences in the  $a_{jki}$ . These congruences do not determine the coefficients  $a_{jki}$  uniquely.

The coefficient of  $y_1^m z_1^n$  in  $F' - F$  is,

$$\binom{l}{1} a_{000} x_1^{l-1} x_2 + \left[ \binom{l}{2} a_{000} + \binom{l-1}{1} a_{001} \right] x_1^{l-2} x_2^2 + \dots,$$

hence  $a_{000} \equiv 0 \pmod{2}$  except possibly when  $l$  is even. Likewise  $a_{000} \equiv 0$  except possibly when  $l, m, n$  are all even.

Let  $n$  be even,  $n = 2a$ . Then  $F$  can be written as,

$$(21) \quad F = \varphi(x_1, x_2; y_1, y_2) z_1^{2a} + \psi z_2$$

$\varphi$  being a covariant of  $(S_1, S_2)$  and a polynomial in the quantics (19) alone. Therefore,

$$F = \varphi \psi_3^a + (\psi + x) z_2 \pmod{2},$$

and  $\psi + x$ , ( $=\Delta_1$ ), is a covariant of  $(S_1, S_2, S_3)$  of the next lower order  $n-1$  in  $z_1, z_2$ ; (All equalities are congruences mod. 2).

To obtain the analogous reduction formula for the case of an odd  $n$ , ( $=2\alpha+1$ ) we note that, since  $a_{000}$  is then zero,  $\varphi$  contains no term without one of the variables  $x_2, y_2$  as a factor. Hence,

$$\varphi = x_2\varphi_1 + y_2\varphi_2 + (xy)\varphi_3,$$

and  $\varphi_1, \varphi_2, \varphi_3$  are covariants. Therefore  $F$  is of the form,

$$\begin{aligned} F &= [x_2z_1\varphi_1 + y_2z_1\varphi_2 + (xy)z_1\varphi_3]\psi_3^\alpha + (\psi + \xi)z_2 \\ &= [(xz)\varphi_1 + (yz)\varphi_2 + \Omega\varphi_3]\psi_3^\alpha + \Delta_2z_2 \pmod{2}, \end{aligned}$$

in which  $\Delta_2$  is a covariant of order  $n-1$  in  $z_1, z_2$ . If the same schemes of reduction are applied to the residuals,  $\Delta_1, \Delta_2$ , successively, the orders in  $z_1, z_2$  are ultimately reduced to zero, so that  $F$  is reduced completely to a rational integral polynomial in the covariants (20), q. e. d.

By the same methods of reduction, or by direct verification, the following syzygies may be established.

$$(22) \quad \left\{ \begin{aligned} (xy)^2 + x_2y_2(xy) + y_2^2\psi_1 + x_2^2\psi_2 &= 0, \\ (xz)^2 + x_2z_2(xz) + z_2^2\psi_1 + x_2^2\psi_3 &= 0, \\ (yz)^2 + y_2z_2(yz) + z_2^2\psi_2 + y_2^2\psi_3 &= 0, \\ \Omega^2 + z_2(xy)\Omega + z_2^2\psi_1\psi_2 + (xy)^2\psi_3 &= 0. \end{aligned} \right.$$

It follows that no reduced covariant of  $(S_1, S_2, S_3)$  need involve any properly simultaneous fundamental covariant to a higher power than the first.

**The seminvariants of  $(f_1, g_1, f_2)$ .**

Consider the set,

$$f_1 = a_0x_1 + a_1x_2, \quad g_1 = b_0x_1 + b_1x_2, \quad f_2 = c_0x_1^2 + c_1x_1x_2 + c_2x_2^2.$$

The seminvariants of any one of these quantities are its invariants under the transformation  $S_1$ . The three induced transformations under  $S_1$ , are

$$\begin{aligned} \sigma_1: \quad a_0' &= a_0, & a_1' &= a_0 + a_1, & \sigma_2: \quad b_0' &= b_0, & b_1' &= b_0 + b_1, \\ \sigma_3: \quad c_0' &= c_0, & c_1' &= c_1, & c_2' &= c_0 + c_1 + c_2. \end{aligned}$$

Writing  $c_0=0$  temporarily, we note that  $\sigma_1, \sigma_2, \sigma_3$ , respectively, are identical in structure with  $S_1, S_2, S_3$ . Therefore if

$$F(a_0, a_1, b_0, b_1, c_0, c_1, c_2),$$

is the general seminvariant of the set  $(f_1, g_1, f_2)$  we have,

$$F(a_0, a_1, b_0, b_1, 0, c_1, c_2) = \Phi_1(a_0, b_0, c_1, P_1, P_2, P, D_1, D', D, W'),$$

wherein,

$$P_1 = a_1^2 + a_0 a_1, \quad P_2 = b_1^2 + b_0 b_1, \quad P = c_2^2 + c_1 c_2, \quad D_1 = a_1 b_0 + a_0 b_1, \quad D' = a_1 c_1 + a_0 c_2, \\ D = b_1 c_1 + b_0 c_2, \quad W' = a_1 b_0 c_2 + a_1 b_1 c_1 + a_0 b_1 c_2 + a_0 b_1 c_1.$$

The quantic  $\Phi_1$  is integral in its arguments. The quantities  $P, D', D, W'$  are not seminvariants of the induced transformations  $\sigma_i$ , but they are, respectively, residues (mod.  $c_0$ ), of the following seminvariants,

$$P_3 = P + c_0 c_2, \quad D_2 = D' + a_1 c_0, \quad D_3 = D + b_1 c_0, \quad W = W' + c_0(a_1 b_0 + a_1 b_1).$$

Hence there exists a relation of the following type,

$$F(a_0, a_1, b_0, b_1, c_0, c_1, c_2) = \Phi_1(a_0, b_0, c_1, P_1, P_2, P_3, D_1, D_2, D_3, W) \\ + c_0 \Psi_1(a_0, a_1, b_0, b_1, c_0, c_1, c_2) \pmod{2},$$

and evidently  $\Psi_1$  is a seminvariant of order, in  $c_0, c_1, c_2$ , one less than the order of  $F$  in  $c_0, c_1, c_2$ .

We can apply this reduction to  $\Psi_1$  and its successors  $\Psi_2, \Psi_3, \dots$  until a vanishing  $\Psi_n$  is reached, or as long as a  $\Psi_k$  contains  $c_0$ , and as long as the last  $\Phi_i$  being considered contains one or more of  $P, D', D, W'$ . If this  $\Phi_i$  contains none of these four expressions, (even though it may contain the invariant  $c_1$ ), the process of reduction can no longer be used to reduce the order in  $c_0$ , but then the last  $\Psi_i$  is a seminvariant which involves only  $a_0, a_1, b_0, b_1, c_1$ , and is reducible in terms of  $a_0, b_0, c_1, P_1, P_2, D_1$ . Therefore  $F$  is reduced, that is, we have proved;

**THEOREM.** - *A fundamental system of seminvariants of  $(f_1, g_1, f_2)$  is composed of the eleven quantics,*

$$a_0, b_0, c_0, c_1, P_1, P_2, P_3, D_1, D_2, D_3, W.$$

**The seminvariants of  $(f_1, f_2, g_2)$ .**

Assume,

$$f_1 = a_0 x_1 + a_1 x_2, \quad f_2 = b_0 x_1^2 + b_1 x_1 x_2 + b_2 x_2^2, \quad g_2 = c_0 x_1^2 + c_1 x_1 x_2 + c_2 x_2^2.$$

The transformations induced by  $S_1$  are,

$$s_1: \quad a_0' = a_0, \quad a_1' = a_0 + a_1, \quad s_2: \quad b_0' = b_0, \quad b_1' = b_1, \quad b_2' = b_0 + b_1 + b_2, \\ s_3: \quad c_0' = c_0, \quad c_1' = c_1, \quad c_2' = c_0 + c_1 + c_2.$$

With  $b_0=0$ ,  $s_1, s_2, s_3$  are, except for notation, identical in structure with  $\sigma_1, \sigma_2, \sigma_3$ , respectively; therefore if

$$G(a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2)$$

is the arbitrary seminvariant of  $(f_1, f_2, g_2)$ , we have,

$$G(a_0, a_1, 0, b_1, b_2, c_0, c_1, c_2) = H_1(a_0, b_1, c_0, c_1, p_1', p_2', p_3', d_1', d_2', d_3', z'),$$

wherein abbreviations have the following meanings,

$$\begin{aligned} p_1' &= a_1^2 + a_0 a_1, & p_2' &= b_2^2 + b_1 b_2, & p_3' &= c_0 c_2 + c_1 c_2 + c_2^2, \\ d_1' &= a_1 b_1 + a_0 b_2, & d_2' &= a_1 c_0 + a_1 c_1 + a_0 c_2, & d_3' &= b_2 c_0 + b_2 c_1 + b_1 c_2, \\ z' &= a_1 b_1 c_2 + a_1 b_2 c_1 + a_0 b_2 c_2 + a_0 b_2 c_1 + a_1 b_1 c_0 + a_1 b_2 c_0. \end{aligned}$$

The latter seven quantities are respectively residual, (mod.  $b_0$ ), to the following seminvariants of  $(f_1, f_2, g_2)$ ,

$$\begin{aligned} p_1 &= p_1', & p_2 &= p_2' + b_0 b_2, & p_3 &= p_3', & d_1 &= d_1' + a_1 b_0, & d_2 &= d_2', & d_3 &= d_3' + b_0 c_2, \\ z &= z' + b_0(a_1 c_1 + a_0 c_2 + a_1 c_2). \end{aligned}$$

Hence,

$$\begin{aligned} G(a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2) &= H_1(a_0, b_1, c_0, c_1, p_1, p_2, p_3, d_1, d_2, d_3, z) \\ &\quad + b_0 G_1(a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2), \end{aligned}$$

and  $G_1$  is a seminvariant of degree in  $b_0$  one less than the degree of  $G$  in  $b_0$ . Reducing  $G_1$  and the other succeeding residuals by the same process, we reach a residual seminvariant  $G_n$  which is of degree 0 in  $b_0$ . Therefore, except when  $G_{n+1}=0$ ,  $G_n$  does not contain  $p_2, d_1, d_3$ , or  $z$ , that is, a repetition of the argument with which the preceding section closes gives the following,

**THEOREM.** - *A fundamental system of seminvariants of  $(f_1, f_2, g_2)$  is composed of,*

$$a_0, b_0, b_1, c_0, c_1, p_1, p_2, p_3, d_1, d_2, d_3, z.$$

**The seminvariants of  $(f_1, f_2, f_3)$ .**

We now consider the ground-quantities of the main Problem,

$$f_1 = a_0 x_1 + a_1 x_2, \quad f_2 = b_0 x_1^2 + b_1 x_1 x_2 + b_2 x_2^2, \quad f_3 = c_0 x_1^3 + c_1 x_1^2 x_2 + c_2 x_1 x_2^2 + c_3 x_2^3.$$

The transformations induced by  $S_1$  are,

$$\begin{aligned} t_1: & a_0' = a_0, & a_1' &= a_0 + a_1, & t_2: & b_0' = b_0, & b_1' &= b_1, & b_2' &= b_0 + b_1 + b_2, \\ t_3: & c_0' = c_0, & c_1' &= c_0 + c_1, & c_2' &= c_0 + c_2, & c_3' &= c_0 + c_1 + c_2 + c_3. \end{aligned}$$

If  $c_0=0$ ,  $t_1, t_2, t_3$  are identical with  $s_1, s_2, s_3$ , respectively, except for notation in the case of  $t_3$ . If  $F(c_0)$  is the arbitrary seminvariant of  $(f_1, f_2, f_3)$ ,

$$F(0) = \Phi(a_0, b_0, b_1, c_1, e, q_1', q_2', q_3', e_1', e_2', e_3', \alpha'),$$

wherein  $e$  is the invariant  $e=c_1+c_2$ , and

$$\begin{aligned} q_1' &= a_1^2 + a_0 a_1, & q_2' &= (b_0 + b_1 + b_2) b_2, & q_3' &= (c_1 + c_2 + c_3) c_3, \\ e_1' &= a_0 b_2 + a_1 (b_0 + b_1), & e_2' &= a_0 c_3 + a_1 (c_1 + c_2), & e_3' &= b_2 (c_1 + c_2) + (b_0 + b_1) c_3, \\ \alpha' &= a_1 b_1 c_3 + a_1 b_2 c_2 + a_0 b_2 c_3 + a_0 b_2 c_2 + a_1 b_1 c_1 + a_1 b_2 c_1 + a_1 b_0 c_2 + a_0 b_0 c_3 + a_1 b_0 c_3. \end{aligned}$$

These quantities respectively are residual (mod.  $c_0$ ) to the seven seminvariants of  $(f_1, f_2, f_3)$ ,

$$\begin{aligned} q_1 &= q_1', & q_2 &= q_2', & q_3 &= q_3' + c_0 c_3, \\ e_1 &= e_1', & e_2 &= e_2' + a_1 c_0, & e_3 &= e_3' + b_2 c_0, & \alpha &= \alpha' + a_1 b_0 c_0. \end{aligned}$$

The letter  $c_1$  is exceptional in  $\Phi$ , and, for this reason, we write  $\Phi$  explicitly as far as  $c_1$  is concerned and obtain,

$$F(0) = G_0 + G_1 c_1^2 + \dots + G_v c_1^{2v} + c_1 (H_0 + H_1 c_1^2 + \dots + H_u c_1^{2u}) \pmod{2}.$$

All  $G, H$  functions are like

$$G_0 = G_0(a_0, b_0, b_1, e, q_1', q_2', q_3', e_1', e_2', e_3', \alpha').$$

In the latter value of  $F(0)$  we next make the replacements,

$$\left( \begin{array}{l} a_0, b_0, b_1, e, q_1', q_2', q_3', e_1', e_2', e_3', \alpha' \\ a_0, b_0, b_1, e, q_1, q_2, q_3, e_1, e_2, e_3, \alpha \end{array} \right),$$

also  $c_1^{2j}$ , ( $j=0, 1, \dots$ ), is replaced by  $\lambda^j$ ,  $\lambda$  being the seminvariant  $\lambda=c_1^2+c_0c_1$ . The grand result is,

$$(23) \quad F(c_0) = I' + c_1 J + c_0 K \pmod{2},$$

where  $I'$  and  $J$  are seminvariants reduced in terms of,

$$a_0, b_0, b_1, e, q_1, q_2, q_3, e_1, e_2, e_3, \alpha, \lambda.$$

We transform the congruence (23) by  $(t_1, t_2, t_3)$  and obtain,

$$F(c_0) = I' + (c_0 + c_1) J + (K + K_1) c_0 \pmod{2},$$

whence  $K_1=J$ , that is, the seminvariant  $J$  is the increment of a polynomial  $K$  in the nine coefficients of  $f_1, f_2, f_3$ , under  $(t_1, t_2, t_3)$ . The terms of such an increment can be grouped so each group contains, as a factor, one of the following expressions;  $a_0, b_0 + b_1, c_0, e (=c_1 + c_2)$ . Hence,

$$F(c_0) = I' + a_0 c_1 J_1 + (b_0 + b_1) c_1 J_2 + e c_1 J_3 + c_0 K',$$

wherein  $J_1, J_2, J_3$  are reduced seminvariants. Consider next the new seminvariants, (new except  $\Delta$ ),

$$\Delta = c_0 c_3 + c_1 c_2, \quad \Gamma = a_0 c_1 + a_1 c_0, \quad E = (b_0 + b_1) c_1 + b_2 c_0.$$

Since  $\Delta = c_1 e + \lambda + c_0 c_1 + c_0 c_3$ ,

$$(24) \quad F(c_0) = I' + \Gamma J_1 + E J_2 + (\Delta + \lambda) J_3 + c_0 L_1, \quad (= M_1 + c_0 L_1).$$

All letters on the right are now reduced seminvariants excepting  $L_1$ . Though  $L_1$  is not reduced it, also, is a seminvariant. Its order in  $c_0, \dots, c_3$  is one less than the order of  $F(c_0)$ . We apply an analogous reduction to  $L_1, L_2, \dots$  in succession. If a vanishing  $L_n$  is reached before the order in  $c_0$  is reduced to zero,  $F(c_0)$  is reduced in terms of fundamental seminvariants. If we reach an  $L_k \neq 0$  free from  $c_0$ , and  $L_{k+1} \neq 0$ ,  $M_{k+1}$  will not contain  $\Delta, \lambda, \Gamma, E, q_3, e_2, e_3$  or  $a$ , otherwise  $L_{k+1}$  would be of order  $-1$  in  $c_0$ , which is absurd. Hence  $M_{k+1}$  is a polynomial in  $e$  whose coefficients are seminvariants of  $f_1, f_2$ , whence  $F(c_0)$  is reduced. Therefore,

**THEOREM.** - *A fundamental system of seminvariants of  $(f_1, f_2, f_3)$  is composed of the quantics,*

$$(25) \quad a_0, b_0, b_1, c_0, e, \lambda, q_1, q_2, q_3, e_1, e_2, e_3, a, \Delta, \Gamma, E.$$

### Syzygies.

The first five of the following syzygetic relations were given in previous papers by the author. Of the rest, the last is the most complicated. Its derivation was based upon the last relation (22), supplemented by some empiricism.

$$\begin{aligned} a_0^2 + q_1 + Q' &= 0, & (Q' &= a_0^2 + a_0 a_1 + a_1^2), \\ a_0^3 + a_0 Q' + L' &= 0, & (L' &= a_0 q_1), \\ b_0^2 + b_0^2 b_1 + b_0 q + k &= 0, & (q &= q_2 + b_0 b_1 + b_0^2, \quad k = b_0 q_2), \\ c_0^3 + c_0^2 e + c_0 I + K &= 0, & (I &= c_0^2 + c_0 e + q_3, \quad K = c_0 q_3), \\ \lambda^2 + \lambda(c_0^2 + c_0 e + I + e^2) &+ (c_0^2 + c_0 e + I + \Delta)(c_0 e + e^2) + I\Delta + \Delta^2 + g_1 &= 0, \end{aligned}$$

where  $g_1$  is an invariant,

$$\begin{aligned} e_1 + a_0 b_1 + j_1 &= 0, & \{j_1 &= a_0(b_1 + b_2) + a_1(b_0 + b_1)\}, \\ e_2 + a_0 e + j_2 &= 0, & e_3 + b_0 e + c_0 b_1 + j_3 &= 0, \end{aligned}$$

where  $j_1, j_2, j_3$  are invariants,

$$\begin{aligned} \Gamma^2 + \Gamma a_0 c_0 + \lambda a_0^2 + (a_0^2 + Q') c_0^2 &= 0, \\ E^2 + E(b_0 + b_1) c_0 + \lambda(b_0^2 + b_1^2) &+ (b_0^2 + b_0 b_1 + q) c_0^2 = 0. \end{aligned}$$

For the quantities  $f_1, g_1, f_2$  there exists the syzygy,

$$W^2 + W(a_0 b_0 c_0 + a_0 D_3) + P_1(b_0^2 c_0^2 + D_3^2) + a_0^2 c_0 c_1 P_2 + a_0^2 P_2 P_3 = 0,$$

and for  $f_1, f_2, g_2$ , the syzygy,

$$z^2 + z(a_0 b_0 c_1 + a_0 b_1 c_0 + a_0 d_3) + p_1(b_0^2 c_1^2 + b_1^2 c_0^2 + d_3^2) + a_0^2 c_0 c_1 p_2 + \\ + a_0^2 b_0 b_1 p_3 + a_0^2 p_2 p_3 + a_0^2 b_0 c_1 d_3 = 0.$$

For  $f_1, f_2, f_3$  there exists the syzygy,

$$\alpha^2 + \alpha \{ (b_0 + b_1)\Gamma + a_0 b_0 e + a_0 b_1 c_0 + a_0 e_3 \} + q_1 \{ E^2 + E(b_0 c_0 + b_1 c_0) \\ + (b_0^2 c_0 + b_0 b_1 c_0)e + b_0^2 e^2 + b_0 b_1 c_0^2 + (b_0 c_0 + b_1 c_0)e_3 + e_3^2 \} + a_0^2 b_0 e_3 e \\ + a_0 b_0 e_3 \Gamma + (a_0^2 \Delta + a_0^2 c_0 e + a_0 c_0 \Gamma + a_0 c_0 e_2) q_2 + a_0^2 b_0 b_1 q_3 + a_0^2 q_2 q_3 = 0.$$

The arbitrary seminvariant of the set  $(f_1, f_2, f_3)$  can be reduced, by means of these syzygies, to a finite, rational and integral polynomial  $f$  which is linear in each fundamental seminvariant  $\alpha, \lambda, \Gamma, E$ , quadratic in  $\alpha_0, b_0, c_0$ , and free from  $q_1, q_2, q_3, e_1, e_2, e_3$ . The coefficients of  $f$  are arbitrary polynomials in the invariants (26). It is not claimed that this set is a fundamental system of invariants.

$$(26) \quad \Delta, e, q, Q', L', k, I, K, b_1, g_1, j_1, j_2, j_3.$$

#### Technique of covariant reduction.

Any formal covariant modulo 2, of a set  $f_m, f_n, \dots$ ,

$$K_s = C_0 x_1^s + C_1 x_1^{s-1} x_2 + \dots + C_s x_2^s,$$

gives rise to a scale of concomitants,

$$K_s, w^i K_s, \quad (i=1, \dots, s) \quad w^j [K_s], \quad (j=0, 1), \\ \langle K_s \rangle, \quad \{ K_s \}, \quad \left( w = x_1^2 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2} \right).$$

These exist for all orders  $s$ ; (Def.,  $[g] = g$  if  $g$  is linear), where <sup>(10)</sup>,

$$\{ K_s \} = C_0 x_1^2 + \langle K_s \rangle x_1 x_2 + C_s x_2^2, \quad \langle K_s \rangle = C_1 + C_2 + \dots + C_{s-1}, \\ [K_s] = (C_0 + \langle K_s \rangle) x_1 + (\langle K_s \rangle + C_s) x_2,$$

and if  $s$  is odd, ( $s=2\alpha+1$ ), the scale contains also a cubic,

$$\{ \overline{K_s} \} = C_0 x_1^3 + I_1 x_1^2 x_2 + I_2 x_1 x_2^2 + C_s x_2^3, \quad (i_j \neq i_k \neq 0 \text{ or } s; I_1 + I_2 = \langle K_s \rangle),$$

<sup>(10)</sup> « Bull. Amer. Math. Soc. », vol. 30 (1924), p. 135. Also, « Trans. Amer. M. S. », vol. 17 (1916). The models of the copied forms  $\{ \overline{K_s} \}$  are tabulated on p. 550.

( $I_1 = C_{i_1} + \dots + C_{i_a}$ ,  $I_2 = C_{i_{a+1}} + \dots + C_{i_{2a}}$ ). If  $S$  is a seminvariant and  $S_1$  is its conjugate under the substitution on the coefficients of  $f_1, f_2, f_3$ ,

$$t = (a_0 a_1)(b_0 b_2)(b_1)(c_0 c_3)(c_1 c_2),$$

a necessary and sufficient condition that  $S$  should have the property that it can lead a covariant is, that  $S + S_1'$  should be an invariant, where  $S_1'$  is the increment of  $S_1$  under  $(t_1, t_2, t_3)$ . In fact, if  $S$  leads a  $K_s$ , it leads,

$$R(S) = \{K_s\} = Sx_1^2 + \varphi x_1 x_2 + S_1 x_2^2,$$

$\varphi$  being the scale invariant. The relation which expresses the covariancy of  $R(S)$  gives  $S + S_1' = \varphi$ . With  $S$  given we can construct  $R(S)$  uniquely.

If  $S$  and  $T$  are seminvariants which lead covariants  $K_s$  and  $A$  respectively,  $A$  being cubic, and if  $H$  is a cubic led by  $ST$ ,  $H$  is reducible in any set which contains  $A, \{A\}, \varphi, [K_s], L$  and invariants. The reduction formula is,

$$H = [K_s] \{A\} + \varphi A + \psi L \pmod{2}, \quad ([K_s] = (S + \varphi)x_1 + (\varphi + S_1)x_2),$$

$\psi$  being an invariant.

If  $S$  and  $T$  are seminvariants which lead respective quadratic covariants, viz.,

$$\{C\} = Sx_1^2 + \varphi x_1 x_2 + S_1 x_2^2, \quad \{D\} = Tx_1^2 + \alpha x_1 x_2 + T_1 x_2^2,$$

and  $G$  is a quadratic led by  $ST$ ,  $G$  is reducible in any set which contains  $\{C\}, \{D\}, [C], [D], \varphi, x$ , and  $Q$ . The formula is,

$$G = [C][D] + \{C\}x + \{D\}\varphi + \varphi x Q \pmod{2}.$$

Some of the syzygetic relations among leading seminvariants, implied by congruences of the latter type, are consequences of a class of secondary syzygies among the universal covariants of  $(S_1, S_2, S_3)$ , (Cf. (20)). The following are such syzygies,

$$\begin{aligned} (xy)(yz) + \psi_2 x_2 z_2 + \Omega y_2 &= 0, \\ (xy)(xz) + (xy)x_2 z_2 + \psi_1 y_2 z_2 + \Omega x_2 &= 0, \\ (xz)(yz) + (yz)x_2 z_2 + \psi_3 x_2 y_2 + \Omega z_2 &= 0, \\ (xz)\Omega + (yz)\psi_1 z_2 + (xy)\psi_3 x_2 &= 0, \\ (xy)\Omega + (xz)\psi_2 x_2 + \psi_2 x_2^2 z_2 + (yz)\psi_1 y_2 + \Omega x_2 y_2 &= 0, \\ (yz)\Omega + (xz)\psi_2 z_2 + \psi_2 x_2 z_2^2 + (xy)\psi_3 y_2 + \Omega y_2 z_2 &= 0. \end{aligned}$$

In the known expansion, (wherein  $\beta < \alpha + 1, s > 3$ ),

$$K_s = N^\alpha \eta_s + N^{\beta-1} L \kappa + \dots + NL^{r-1} \psi + L^\delta \omega \pmod{2},$$

$\eta_s$  is always quadratic if  $s$  is even. If  $s$  is odd,  $\eta_s = \{ \overline{K_s} \}$ , and is cubic. We find <sup>(44)</sup>,

$$\eta_4 = C_0 x_1^2 + (C_1 + C_2 + C_3) x_1 x_2 + C_4 x_2^2 = \{ \{ K_4 N, LN \}^4, L \}^2 + N \{ K_4 N, N^3 \}^6,$$

$$\eta_6 = C_0 x_1^2 + (C_1 + C_2 + C_3 + C_4 + C_5) x_1 x_2 + C_6 x_2^2 \\ = \{ \{ K_6, LN \}^4, L \}^2 + N \{ K_6, N^3 \}^6,$$

$$(27) \quad \eta_5 = C_0 x_1^3 + (C_1 + C_2) x_1^2 x_2 + (C_3 + C_4) x_1 x_2^2 + C_5 x_2^3 = \{ K_5, L^2 \}^4 + \{ K_5, N \}^2,$$

$$(28) \quad \eta_7 = C_0 x_1^3 + (C_1 + C_2 + C_4) x_1^2 x_2 + (C_3 + C_5 + C_6) x_1 x_2^2 + C_7 x_2^3 \\ = \{ \{ K_7, L^2 \}^4, L^2 \}^4 + \{ \{ K_7, L^2 \}^4, N \}^2.$$

We have proved that  $\{ \overline{K_s} \}$  is always a simultaneous algebraic covariant of the set  $(L, N, K_s)$ , but the problem of generalization of the transvectant formulary is formidable. We have also,

$$\{ K_s \} = \{ \{ \overline{K_s} \}, L \}^2, \quad [K_s] = \{ \{ \overline{K_s} \}, N \}^2, \quad \langle K_s \rangle = \{ \{ K_s \}, N \}^2.$$

The following are among the algebraic formulae of interest when the modulus is  $p$ ,

$$\{ K_s, L \}^r = C_{p+s-2r+1}, \quad (L = x_1^p x_2 - x_1 x_2^p).$$

In view of the special form of  $L$ , if  $C_{p+s-2r+1}$  is known and  $K_s$  is unknown, this equation can be solved for  $K_s$  by the methods of § IV, Lemma 2, (e). We have also,

$$\{ K_s, N \}^4 = K_s L^{p-2}, \quad \{ K_s, N \}^2 = D_{p+s-1} L^{p-3},$$

$$\{ K_s, N \}^r = E_{(r-1)p+s-r+1} L^{p-r-1},$$

$E$  being of especial note when its order is divisible by  $p-1$ ;  $s = a(p-1)$ .

### Covariants led by prime seminvariants.

A seminvariant with no invariant factor, which is not a product of two seminvariants may be said to be prime. We construct covariants modulo 2 led by each prime case of the typical term of  $f$ . Such a case is  $a$ , and (cf. the syzygies),

$$R(a) = a x_1^2 + A x_1 x_2 + a_1 x_2^2,$$

---

<sup>(44)</sup> The transvectant is always defined as having been freed from the numerical g. c. d. of its terms as given by the transvectant operator.

exists as a covariant. Here  $a_1$  is the conjugate of  $a$  under  $t$ , and  $\Lambda$  is the scale invariant,

$$\begin{aligned} \Lambda = & a_0 b_1 c_0 + a_0 b_2 c_0 + a_1 b_1 c_0 + a_0 b_0 c_1 + a_0 b_0 c_2 + a_0 b_2 c_1 + a_0 b_2 c_3 + a_1 b_1 c_1 + a_0 b_1 c_2 \\ & + a_1 b_0 c_0 + a_1 b_0 c_2 + a_1 b_2 c_1 + a_1 b_2 c_2 + a_0 b_1 c_3 + a_1 b_0 c_3 + a_1 b_1 c_3. \end{aligned}$$

The covariants  $[X_4]$ ,  $Q[X_4]$ , (where  $X_4$  is  $R(a)$ ), are led by  $a + \Lambda$ . If a cubic  $M$ , led by  $\Lambda$ , exists,  $M + Q[X_4]$  will be a cubic led by  $a$ .

LEMMA. - *There exists a cubic covariant which has  $\Lambda$  as its leading coefficient.*

In outline the proof is as follows. If a function primed represents the function's increment with respect to  $(t_1, t_2, t_3)$ , so that  $a + a' = \Lambda$ , and if

$$M = \Lambda x_1^3 + \Lambda_1 x_1^2 x_2 + \Lambda_2 x_1 x_2^2 + \Lambda x_2^3,$$

is a covariant, the relation expressing covariancy gives  $\Lambda_1' = \Lambda_2' = \Lambda$ , because  $\Lambda_1 + \Lambda_2 = \Lambda$  and  $\Lambda' = 0$ . To construct  $M$  we partition  $\Lambda$  into its two parts  $\lambda_1, \lambda_2$  conjugate with respect to  $t$  and try  $\lambda_1$  for  $\Lambda_1$  and  $\lambda_2$  for  $\Lambda_2$ , but it is found that  $\lambda_1' = \lambda_2' \neq \Lambda$ . The  $\lambda_1', \lambda_2'$  have six terms in common with  $\Lambda$  and four other terms. The problem then is to choose by inspection a sum  $\Sigma$  of terms  $a_i b_j c_k$  such that  $\lambda_1' + \Sigma' = \lambda_2' + \Sigma' = \Lambda$ , for then,  $\lambda_1 + \Sigma = \Lambda_1$ ,  $\lambda_2 + \Sigma = \Lambda_2$ . Then a tedious process of inspection gave the self-conjugate  $\Sigma$ ,

$$\Sigma = a_0 b_0 c_0 + a_1 b_0 c_3 + a_0 b_2 c_3 + a_1 b_2 c_2 + a_0 b_1 c_3 + a_0 b_2 c_0 + a_1 b_0 c_0 + a_0 b_0 c_1 + a_1 b_1 c_0 + a_1 b_2 c_3.$$

Hence  $M$  is a covariant, q. e. d., with

$$\begin{aligned} \Lambda_1 = & a_0 b_0 c_0 + a_1 b_0 c_3 + a_0 b_1 c_0 + a_0 b_0 c_2 + a_0 b_2 c_1 + \\ & + a_1 b_1 c_1 + a_1 b_2 c_2 + a_0 b_1 c_3 + a_1 b_0 c_0 + a_1 b_2 c_3, \\ \Lambda_2 = & a_0 b_0 c_0 + a_0 b_2 c_0 + a_1 b_1 c_3 + a_1 b_2 c_1 + a_1 b_0 c_2 + \\ & + a_0 b_1 c_2 + a_0 b_0 c_1 + a_1 b_1 c_0 + a_0 b_2 c_3 + a_1 b_2 c_3. \end{aligned}$$

Let  $K_6 = f_1 f_2 f_3$ ; then  $M$ , as an algebraic covariant is,

$$\begin{aligned} M = & \{(\{K_6, LN\}^4, L)^2 + \{K_6, N^3\}^6 N), L\}^2 N + \{K_6, LN\}^4 \\ & + (\{f_3, L\}^3 L + \{f_3, L\}^2, L\}^2 N + f_3) \{f_1, \{f_2, L\}^2\}^4 + \{(\{K_6, LN\}^4, L)^2 \\ & + (\{f_3, L\}^2, L\}^2 N + f_3) \{f_1, \{f_2, L\}^2\}^4, L\}^2, N\}^2 L. \end{aligned}$$

In view of (28) the formulae (30), (31) below are expressed algebraically. If  $I_j$  is an arbitrary integral polynomial in the invariants of the set (26), now to be augmented by  $\Lambda$ , the typical term of the  $f$  considered above can be written as,

$$(29) \quad r_j = I_j a_0^{m_1} b_0^{m_2} c_0^{m_3} a^{n_1} \lambda^{n_2} \Gamma^{n_3} E^{n_4}, \quad (m_i = 0, 1, \text{ or } 2; n_k = 0 \text{ or } 1).$$

By the two main reduction algorithms described above, a non-linear, (as to  $x_1, x_2$ ), covariant led by  $r_j/I_j$  is reducible in terms of covariants led by prime cases of  $r_j/I_j$ , but this principle will be amplified further later. We now give tables of fundamental covariants. Each covariant is led by the prime seminvariant above it.

TABLE I: *Fundamental covariants of  $(f_1, f_2, f_3)$  under  $G_6$ .*

<i>Cubic order</i>						
$a_0$	$b_0 + b_1$	$c_0$	$\lambda + e^2 + \Delta$	$\Gamma$	$E + b_1 e$	$a$
$Qf_1$	$Q[f_2]$	$f_3$	$Q[R(\lambda)]$	$Q[R(\Gamma)]$	$Q[R(E)]$	$M + Q[X_4]$
<i>Quadratic order</i>						
$a_0$	$b_0$	$c_0$	$\lambda$	$\Gamma$	$E$	$a$
$wf_1$	$f_2$	$\{f_3\}$	$R(\lambda)$	$R(\Gamma)$	$R(E)$	$R(a)$
<i>Linear order</i>						
$a_0$	$b_0 + b_1$	$c_0 + e$	$\lambda + e^2 + \Delta$	$\Gamma$	$E + b_1 e$	$a + \Delta$
$f_1$	$[f_2]$	$[f_3]$	$[R(\lambda)]$	$[R(\Gamma)]$	$[R(E)]$	$[R(a)]$

#### Covariants led by invariants.

All quantities of order  $> 3$  are reducible modulo 2. There exists a cubic covariant led by  $e$ , viz.,  $f_3 + Q[f_3]$ . Hence we have cubics led by  $E$ , and by  $\lambda + \Delta$ . Each of the tabulated linear covariants is led by a binomial, a seminvariant plus an invariant, the latter being zero in two cases. If we change any factor of  $r_j$  into such a binomial, without really altering  $f$ , we merely rearrange the terms of  $f$ , but, if it is done to express  $f$  in terms of leaders of linear covariants of table III (below), it may bring unidentified scale invariants into the absolute term of  $f$ . The latter invariants are centrals of covariants like,

$$(30) \quad \{f_1 f_2^2 f_3^2 R(\Gamma)\} = a_0 b_0^2 c_0^2 \Gamma x_1^2 + \dots$$

But the absolute term, in our linear covariant, must vanish identically or we would have a linear covariant led by an invariant, an impossibility. The principle involved is involved more explicitly in the following discussion.

To lead a covariant cubic, an invariant must not contain any term which is left unaltered by the substitution  $t$ . In general  $f$  is a nonhomogeneous polynomial in  $a_0, b_0, c_0, a, \lambda, \Gamma, E$ . It has an absolute term  $r_0$  free from these

seminvariants being therefore an arbitrary polynomial in forms of the set (26) augmented by  $\Delta$ .

Now,  $r_0 = G + W$ , and  $W$  is the part whose terms contain as factors, one or more invariants of the set,  $\Delta, k, g_1, e, j_1, j_2, j_3, L', K$ . In Table II are given cubic covariants led by these respective invariants. Multiply these cubics by arbitrary polynomials in the fourteen invariants and add, and we have a cubic  $g$  whose first term is  $Wx_1^3$ . The part  $G$  is a polynomial in  $\Delta, q, I, b_1, Q'$  exclusively. Each of these invariants has a term which is left unaltered by  $t$  and these respective terms are different. Hence if  $r_0$  leads a cubic  $h, G=0$ , and,

$$h = g + LJ \pmod{2}, \quad (J \text{ invariant}).$$

The covariant  $\{f_3, f_3\}^2$ , in Table II is a Hessian (with no numerical factor excepting unity), and  $\Delta_2$  is the irreducible covariant,

$$\Delta_2 = ((b_0 + b_2)x_1 + (b_0 + b_1)x_2)((b_1 + b_2)x_1 + (b_0 + b_2)x_2).$$

The last term of the formula opposite  $g_1$  is itself a covariant led by  $e_0^2 \lambda$ , and therefore reducible. Note that  $I_j Q$  is a quadratic led by  $I_j$ .

TABLE II. - Covariants led by invariants.  
Cubic order

Leader	Covariant
$k'$	$[f_2]\Delta_2 \quad (k' = k + b_1q)$
$\Delta$	$M$
$e$	$Q[f_3] + f_3$
$L'$	$Q'Qf_1 + f_1^3$
$j_1$	$Q[f_1 f_2] + f_1 f_2$
$j_2$	$Q[f_1 f_3] + f_1 \{f_3\}$
$j_3$	$Q[f_2 f_3] + b_1 Q[f_3] + b_1 f_3 + \{f_2 f_3\}$
$K$	$Q[ef_3 \{f_3\} + f_3 \{f_3, f_3\}^2] + [e \{f_3\} + \{f_3, f_3\}^2] \{f_3\} + \Delta Q[f_3] + e^2 f_3$
$g_1$	$Q[(e \{f_3\} + \{f_3, f_3\}^2)(ef_3 + [f_3] \{f_3\})] + \{(e \{f_3\} + \{f_3, f_3\}^2)(ef_3 + [f_3] \{f_3\})\}$

**A fundamental system of covariants of  $(f_1, f_2, f_3)$ , (mod. 2).**

A fundamental system is a set of irreducibles which suffice for the reduction of all orders of arbitrary covariants, that is, in the present theory, for the reduction of all covariants of orders 1, 2, 3 whose leading coefficients are in the form  $f$  of which  $r_j$  of (29) is a typical term. If a covariant  $\xi$  is led by  $f$ , we can construct a covariant of like order for each term of  $f$  (led by the term).

The sum  $\eta$  of the latter covariants is a covariant and, since  $\xi - \eta$  has the factor  $x_2$ ,

$$\xi = \eta + XL, \quad (X \text{ an invariant or } 0),$$

and  $\xi$  is reduced. The principle is due to DICKSON. The case of the term  $r_0$  of  $f$  was treated above. The complete story of the reduction of a covariant of order 2 or 3 led by a composite  $r_j$  ( $j \neq 0$ ) involves two steps. First, construct a covariant of the given order led by  $r_j/I_j$ . It will be multiplied by  $I_j$  later. Formula (30) shows the method of construction of a quadratic of led by  $r_j/I_j$  or one of its factors. If we require a cubic, the formula will be of the type,

$$(31) \quad \{ \overline{f_1 f_2^2 f_3^2 R(\Gamma)} \} = a_0 b_0^2 c_0^2 \Gamma x_1^3 + \dots$$

Second, factor  $r_j/I_j$  into two factors  $S, T$  and construct a covariant of the order led by each factor. The covariant whose leader is  $r_j/I_j$  is then reducible by the first two algorithms (italicized) in our Paragraph on reduction techniques. Repeat the process for the covariants led by  $S, T$ . Continue until the two factors required each time become prime seminvariants. The covariant whose leader is  $r_j$  will then have been reduced in terms of covariants of Tables I, II, universal covariants, invariants, and covariants of the first order.

Consider the fundamental covariants of order unity in  $x_1, x_2$ . Every quantic of Table III below is irreducible in view of the Paragraph on syzygies and all such are contained in the Table. If we should adjoin to the seminvariant system (25) a complete system of invariants [9], some new syzygies probably would result which would reduce some forms of Table III.

*The aggregate of irreducibles from the finite number of covariants given in Tables I, II, III, (with  $L, N, A$ , and invariants (26)), is a fundamental system of formal covariants (mod. 2) of the set  $(f_1, f_2, f_3)$ .*

TABLE III. - *Fundamental covariants of order unity.* - With  $R(\lambda) = X_1$ ,  $R(\Gamma) = X_2$ ,  $R(E) = X_3$ ,  $R(a) = X_4$ , the irreducible covariants of the first order in  $x_1, x_2$  are those given by the formula,

$$[f_1^i f_2^j f_3^k \delta],$$

where  $i, j, k$  assume the values 0, 1, or 2, repetitions being permitted, and  $\delta$  runs through the set 1,  $X_1, X_2, X_3, X_4, X_1 X_2, X_1 X_3, X_1 X_4, X_2 X_3, X_2 X_4, X_3 X_4, X_1 X_2 X_3, X_1 X_2 X_4, X_1 X_3 X_4, X_2 X_3 X_4, X_1 X_2 X_3 X_4$ .

\* \* \*

A method for fundamental invariants, not limited to those of (26), can be based upon the theorem stated in the last line of § IV.

Among the generalizations which are possible, beyond the more special situations which we have described in this paper, is a theorem that the invariant

theory of a set of polynomials in any number of variables which are subject to linear transformations, is a correlation between two sets of invariant hyperplanes. On the algebraic side, Hilbert's basis theorem is the leading principle of the correlation. (Cf. § II).

The principles of § IV can be made to apply to the concomitants of any special linear group, used in place of  $G(p)$ , and these methods are the only ones of this generality that have been found.

VII. - Special processes (i, ii, iii).

(i) - Erosion of a solid afloat in a stream.

A suggestion for special inverse invariant processes may be obtainable from some geometrical configuration of a structure definitely described by analytic properties. Some segment, for example, may be given by a formula and when the segment is assumed invariant under alterations of the configuration, some other geometric element may become the dependent variable in a type of equation. Solution of the equation will determine a function from its invariant.

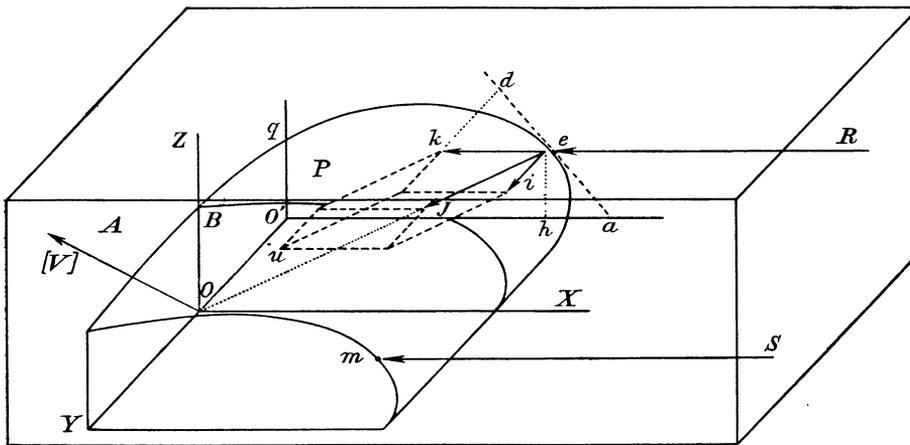


Fig. 2.

Accordingly, consider a liquid in the form of a freely-flowing canal-stream  $A$  in which there is a solid  $B$  of such density that its floating level is at a depth  $\delta$ . Since the upper part of a stream moves faster than the lower part, the stream  $A$  will exert a shear upon the solid  $B$ .

PROBLEM. - To find the equation of the solid's surface considered to be the natural erosion product of the action.

HYPOTHESIS. - *The solid B has a vertical plane of symmetry (the XZ plane), whose horizontal elements are stream-lines of the flow of the liquid. The plane sections of the figure, (as P), are perpendicular to the Y-axis, (OO').*

If stream and solid were at rest the condition for equilibrium is that any pressure vector through the center of mass (taken as O) is counterbalanced by an equal and opposite vector through O. The shearing pressure on B may be described as follows. The pressure R at e, against a small square area of an interposed vertical plane parallel to YOZ, exerted by the cube C of the liquid, of which the square is one face, is proportional to C's momentum. Both mass M and velocity v, of C, are functions of a depth variable  $\theta$  ( $=hO'e$ );  $M=g(\theta)$ ,  $v=f(\theta)$ , and we consider  $\theta$  only between definite limits  $\theta_1 < \theta < \theta_2$ . Hence,

$$R = af(\theta)g(\theta), \quad (=aH(\theta)).$$

The effective pressure on B at e is the projection  $T=eu$ , of the vector R, upon the normal to the surface. Thus T is expressed in terms of R and the angle  $\varphi$  between R and eu. Taking eu as the diagonal of a parallelopiped of forces, we resolve T into components, viz., ek along R produced in P; ei in P along the normal to the curve of section eq; ej along the radial line Oe. The line Oe is r in the polar equation of the surface and O'e is r in the polar equation of eq.

We resolve similarly at m, the point of B symmetrical to e, the parallel vector  $S=R$ . Then the resultant of the pair of components through O is a single vector V which reaches forward in plane XZ, and the resultant of all vectors V, for all pairs (e, m) of B, allowed by  $\theta$ 's limits, is a single vector [V] which pulls B forward without rotation. At e there remain the normal component ei, and the component ek parallel to OX. Both of these vectors have a shearing effect and the total shear due to R is their resultant in P.

With P, (that is, OO'), fixed, and  $\theta_1 < \theta < \theta_2$ , the angle  $\varphi$ , the vector R, and therefore ek, are single-valued functions of  $\theta$ . Hence, to any degree of accuracy, depending upon s,

$$ek = a\theta^{s-1} + b\theta^{s-2} + \dots + l, \quad (=G(\theta)),$$

where a, ..., l are numerical.

INVARIANT PROPERTY. - *We assume (what is in accordance with fact) that, as  $\theta$  varies, the segment dk, parallel to the vector ei, remains invariant.*

We can then determine a differential equation of the curve eq, which involves  $G(\theta)$  among its coefficients. From the shape of eq for different distances OO', the shape of the surface can be determined. With O' as origin we have,

$$\begin{aligned} <aO'e = \theta, \quad O'e = r, \quad \angle O'ea = \psi, \\ \tan \psi = r d\theta / dr, \quad dk = \gamma \quad (\text{constant}). \end{aligned}$$

Hence,

$$dk = ek \cos(\theta + \psi - \pi/2) = G(\theta) \sin(\theta + \psi) = \gamma,$$

or,

$$(32) \quad G(\theta) \left( \frac{dr}{d\theta} \sin \theta + r \cos \theta \right) = \gamma \sqrt{\left[ \left( \frac{dr}{d\theta} \right)^2 + r^2 \right]}.$$

Here  $G(\theta)$  may figure as an arbitrary function and  $\gamma$  as an arbitrary constant. The variables are not separable but a solution  $r = F(\theta)$  exists, and  $F(\theta)$  is an instance of a function determined from one of its invariants.

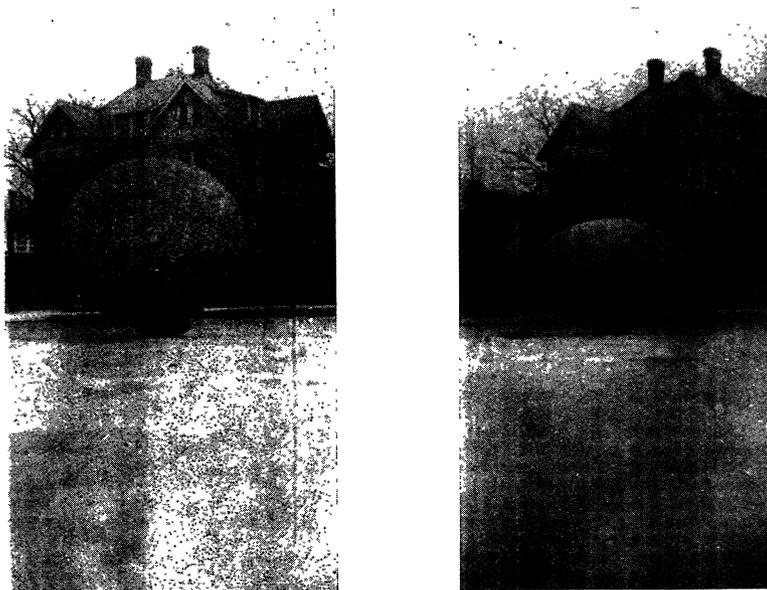


Fig. 3.

GLACIERS. - Our statement of the present problem describes the flow of a deep glacier because the bottom layer of a glacier is under such pressure that its composition is that of semi-crystallized water [10], [11] and is capable of flowing slowly. A boulder afloat within the bottom layer, assumed to contain sand which aids erosion, will be ground to a form which is represented in cross-section by the solution of equation (32). Since the writer has seen a number of such boulders from moraines of the great glacier in Maine, we were able to use their cross-sections as a solution  $r = F(\theta)$  of (32), whence we determine  $G(\theta)$ . These boulders, which are of granite, are true ellipsoids with equation,

$$x^2/a^2 + y^2/c^2 + z^2/b^2 = 1, \quad (a > b > c),$$

The photographs show side view and edge view of one, of dimensions  $a=7$  in.,  $b=6$  in.,  $c=4.4$  in. approximately <sup>(12)</sup>. After  $G(\theta)$  had been determined as above, we generalized it and reversed the equation (32), solving for a generalized form of  $r=F(\theta)$ .

Thus the curve  $eq$  was identified as an ellipse experimentally. We can readily show that the flow of a stream will hold some principal plane of such an ellipsoid in coincidence with the plane  $XZ$ , it being a question of chance as to which principal plane it is. Hence we can choose the least axis ( $c$ ) on  $O'Y$ , whence  $eq$  is,

$$r=I/(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{1/2}=F(\theta), \quad (I \text{ constant}).$$

Substituting this  $F(\theta)$  in (32) we get,

$$G(\theta)=\gamma(\sqrt{[a^4 \sin^2 \theta + b^4 \cos^2 \theta]})/b^2 \cos \theta.$$

Now, the essential symmetries, special cases, and property of being an increasing function of  $\theta$ , of this  $G(\theta)$ , will be preserved if we replace it by,

$$(33) \quad G(\theta)=\gamma(\sqrt{[a^4 \sin^{4p-2} \theta + b^4 \cos^{4p-2} \theta]})/b^2 \cos^{2p-4} \theta, \quad (p > 1/2).$$

Using this form in (32) the integral is,

$$(34) \quad r=J/(b^2 \cos^{2p} \theta \pm a^2 \sin^{2p} \theta)^{1/2p}=F(\theta), \quad (J \text{ constant}).$$

This is an  $(ab)$  section of a surface,

$$(35) \quad E: \quad x^{2p}/a^2 + y^{2p}/c^2 \pm z^{2p}/b^2 = 1, \quad (a \geq b \geq c).$$

In the case of a glacier, the upper sign in (34), ( $p=1$ ), gives the ellipsoid as a product of erosion, and the lower sign gives hyperboloids. The latter are the shapes of the hills which can survive beneath a deep glacier.

*The transformations.* - The form of  $G(\theta)$  in (33) is independent of the size of the solid and depends only on the parameters  $\gamma$ ,  $p$ ,  $a/b$ . Hence the most fundamental transformation leaving  $dk$  invariant is the ternary linear transformation  $\Gamma$  which carries an ellipsoidal  $E$  into an ellipsoidal  $E'$  and leaves the axes of  $E'$  proportional respectively to those of  $E$ . Thus under  $\Gamma$ ,

$$\Gamma: \quad x=\lambda_1 x' + \mu_1 y' + \nu_1 z', \quad y=\lambda_2 x' + \mu_2 y' + \nu_2 z', \quad z=\lambda_3 x' + \mu_3 y' + \nu_3 z',$$

we should have,

$$(36) \quad lx^{2p} + my^{2p} + nz^{2p} = K(lx'^{2p} + my'^{2p} + nz'^{2p}), \\ (l=1/a^2, m=1/c^2, n=\pm 1/b^2, K \text{ constant}).$$

If we assume (36) to be an identity we can solve for some of the  $\lambda_i$ ,  $\mu_i$ ,  $\nu_i$ ,

<sup>(12)</sup> \* Found on Cape Ann, Mass. by MR. W. H. BARKER of Lansdowne, Pa.

(six in the case  $p=1$ ), in terms of the rest as independent parameters. We have carried out the work for  $p=1$ , as shown in what follows.

We note first that the determination of all binary transformations,

$$t: \quad x = \lambda_1 x' + \mu_1 z', \quad z = \lambda_2 x' + \mu_2 z', \quad (\lambda\mu) \neq 0,$$

for which,

$$x^2/a^2 + z^2/b^2 = L(x'^2/a^2 + z'^2/b^2), \quad (L \text{ constant}),$$

is a special solution for transformations which leave  $dk$  invariant, that is, which leave the ellipse  $eq$  relatively invariant. We readily determine  $t$ , as,

$$t: \quad x = \frac{1}{a} [\sqrt{(La^2 - \mu_1^2 b^2)}] x' + \mu_1 z', \quad z = \frac{b^2}{a^2} \mu_1 x' - \frac{1}{a} [\sqrt{(La^2 - \mu_1^2 b^2)}] z'.$$

When we substitute, from  $\Gamma$  in (36), we obtain; ( $p=1$ ),

$$(37) \quad \begin{cases} \lambda_1^2 + m\lambda_2^2 + n\lambda_3^2 = Kl, & l\lambda_1\mu_1 + m\lambda_2\mu_2 + n\lambda_3\mu_3 = 0, \\ \mu_1^2 + m\mu_2^2 + n\mu_3^2 = Km, & l\lambda_1\nu_1 + m\lambda_2\nu_2 + n\lambda_3\nu_3 = 0, \\ \nu_1^2 + m\nu_2^2 + n\nu_3^2 = Kn, & l\mu_1\nu_1 + m\mu_2\nu_2 + n\mu_3\nu_3 = 0, \end{cases}$$

To derive elimination formulas which will be free from extraneous factors the following plan must be pursued. Eliminate  $\lambda_3$  between the first, fourth and fifth equations (37), leaving five equations in eight unknowns, three being free from  $\lambda_2$ . Eliminate  $\lambda_2$  and we then have,

$$\lambda_1 = \sqrt{Kmn}(\mu_2\nu_3 - \mu_3\nu_2) / \{ lm\mu_1^2\nu_2^2 + lm\mu_2^2\nu_1^2 + ln\mu_1^2\nu_3^2 + ln\mu_3^2\nu_1^2 + mn\mu_2^2\nu_3^2 \\ + mn\mu_3^2\nu_2^2 - 2lm\mu_1\mu_2\nu_1\nu_2 - 2ln\mu_1\mu_3\nu_1\nu_3 - 2mn\mu_2\mu_3\nu_2\nu_3 \}^{1/2},$$

together with enough auxiliary equations to eliminate  $\mu_2, \mu_3, \nu_3$  from  $\lambda_1$ , leaving the latter a function of  $\mu_1, \nu_1, \nu_2$ . The solution may then be completed readily and is as follows, when expressed in the form of a transformation  $T$ ;

$$x = \frac{\sqrt{\Delta_2}}{\sqrt{m\Delta_2}} x' + \mu_1 y' + \nu_1 z', \\ y = \frac{l\mu_1\Delta_2\sqrt{mK} - lm\nu_1\nu_2\sqrt{\Delta_1}}{(-lm\nu_1^2 + mnK)\sqrt{m\Delta_2}} x' + \frac{lm\mu_1\nu_1\nu_2 + \sqrt{mK\Delta_1}}{lm\nu_1^2 - mnK} y' + \nu_2 z', \\ z = \frac{(l^2m\nu_1^3 - lmnK\nu_1 + lm^2\nu_1\nu_2^2)\sqrt{\Delta_1} - lm\mu_1\nu_2\Delta_2\sqrt{mK}}{(-lm\nu_1^2 + mnK)\Delta_2\sqrt{m}} x' + \\ + \frac{lmnK\mu_1\nu_1 - l^2m\mu_1\nu_1^3 - lm^2\mu_1\nu_1\nu_2^2 - m\nu_2\sqrt{mK\Delta_1}}{(lm\nu_1^2 - mnK)\sqrt{\Delta_2}} y' + \frac{\sqrt{\Delta_2}}{n} z'.$$

In this result the essential irrationalities, assumed real, are  $\sqrt{m\Delta_1}, \sqrt{m\Delta_2}$  in which

$$\Delta_1 = lmn\mu_1^2\nu_2^2 + l^2n\mu_1^2\nu_1^2 + l^2m\nu_1^4 + lm^2\nu_1^2\nu_2^2 - 2lmnK\nu_1^2 - m^2nK\nu_2^2 - ln^2K\mu_1^2 + mn^2K^2, \\ \Delta_2 = -ln\nu_1^2 - mn\nu_2^2 + n^2K.$$

If  $s$  stands for the alterations under which  $dk$  is invariant,  $dk$  is relatively invariant under  $sT$ .

(ii) - Functional relations between the masses of the planets in a system.

It is known that the integral,

$$r=f(\theta, c_1, c_2),$$

of the equation of central orbits,

$$(38) \quad d^2u/d\theta^2 + u = [F(1/u)]/\gamma^2 u^2, \quad (u=1/r, F=\text{force}),$$

involves, as a parameter, the mass  $m$  of the moving body, here assumed to have the dimensions of a planet. A finite segment of the orbit can therefore be represented by an equation,

$$(39) \quad \theta = \Delta_0(m)r^{n-1} + \Delta_1(m)r^{n-2} + \dots + \Delta_{n-1}(m), \quad (= \Delta(r, m)).$$

The  $\Delta_i(m)$  are rational, integral quantities with real coefficients [12],

$$\Delta_i(m) = e_{0i}m^{s-1} + e_{1i}m^{s-2} + \dots + e_{s-1i},$$

in which  $m$  has a range of continuous variation but so small that the orbital perturbations caused by the variation are not sufficient to render the orbit unstable. The parameter  $m$  cannot be varied *ab initio* over the masses of the sequence of planets in a central system.

For any value  $\theta$  in (39) the distances from the center of force  $o$  to the respective planets are given by a generalized form of BODE'S formula, viz.,

$$q_{j+2} = g(q_2/q_1)^j + h, \quad (j=0, \dots, k),$$

$g = q_1(q_2 - q_3)/(q_1 - q_2)$ ,  $h = (q_1q_3 - q_2^2)/(q_1 - q_2)$ ,  $q_i$  being the distance from  $o$  to planet number  $i$ , and  $q_1, q_2$ , necessarily given by observational measurements. It is obviously an algebraic simplification to subtract  $h$  from the roots of  $\Delta(r, m)$ , and multiply the roots of the result by  $1/g$ . These processes lead to equations of fictitious or auxiliary orbits,

$$(40) \quad \theta = \gamma_0(m)r^{n-1} + \gamma_1(m)r^{n-2} + \dots + \gamma_{n-1}(m), \quad (= \Gamma(r, m)),$$

each of which is moved outward by one space if we multiply the roots of  $\Gamma(r, m)$  by  $z = q_2/q_1$ . Here  $\theta$  should be allowed only a small variation which, by a choice of axes, may be held near to zero. With  $z$  known we will therefore have the following as an orbital system;

$$(41) \quad z^{(n-1)e}\theta = \gamma_0(m)r^{n-1} + z^e\gamma_1(m)r^{n-2} + \dots + z^{(n-1)e}\gamma_{n-1}(m), \quad (e=0, \dots, k).$$

The independence of these equations can be emphasized by using a different letter  $m$  in each; also a different  $r$ . Relations between the different masses can be shown to exist by means of (41), but more simply as follows. If we form an equation like (39) for the orbit  $q_{e+2}$ , its auxiliary will have an equation,

$$(42) \quad \theta = \lambda_0(m')r^{n-1} + \lambda_1(m')r^{n-2} + \dots + \lambda_{n-1}(m'), \quad (=A(r, m')).$$

Since the equation for the mass  $m'$  is unique and  $\lambda_v$  may be assumed to be of the same order as  $\gamma_v$  we have,

$$(43) \quad \lambda_v(m') = z^{(v+1-n)e} \gamma_v(m), \quad (v=0, \dots, n-1),$$

whence the functional relation between  $m'$  and  $m$  must be linear, that is, homographic in general,

$$H(m, m') : m = \frac{\alpha_1 m' + \alpha_2}{\beta_0 m' + \beta_1}.$$

If  $\lambda_v$  is the same function as  $\gamma_v$ , (43) is a relation of covariancy of  $\gamma_v$  under direct transformation by  $H$ . Therefore a homographic relation between the masses of any two planets, correlating their respective orbital equations, will exist if an expansion (41), of generality sufficient to represent the system of orbits, can be formed, having covariants of  $H$  as the  $\gamma_v$ .

The key to this construction is the periodic homography. The period must be greater than the number of planets in the system. The homography  $H$  will be periodic, and will have the period  $w$  mentioned concurrently, if the parameters satisfy one of the following relations. Each  $H(m, m')$  generates a formal cyclic group.

$$\begin{aligned} \beta_1 &= -\alpha_1, & w &= 2, \\ \alpha_2 &= -(\alpha_1^2 + \alpha_1 \beta_1 + \beta_1^2) / \beta_0, & w &= 3, \\ \alpha_2 &= -(\alpha_1^2 + \beta_1^2) / 2\beta_0, & w &= 4, \\ \alpha_2 &= -[3\alpha_1^2 + 4\alpha_1 \beta_1 + 3\beta_1^2 - (\alpha_1 + \beta_1)^2 \sqrt{5}] / 2\beta_0, & w &= 5, \\ \alpha_2 &= -(\alpha_1^2 - \alpha_1 \beta_1 + \beta_1^2) / 3\beta_0, & w &= 6. \end{aligned}$$

The following propositions are important, though, for the sake of brevity, proofs are omitted.

A homography of period three combined with one of period five is a homography of period fifteen, thus some higher orders, also, are represented in the list.

If a set of real numbers  $m_1, m_2, \dots$ , are permuted in a simple cycle by a homography in which the parameters are complex imaginaries, then  $m_1, m_2, \dots$  can also be permuted cyclically by a homography  $H$  whose parameters are real. Hence only real homographies need be considered here.

If a homography is periodic with period  $> 2$ , its poles are imaginary.

*Covariants of periodic homographies.* - Iteration of an  $H(p', p)$  of period

$w$  gives a set  $S$  of linear fractional functions which are permuted cyclically by the transformation. When  $w=3$  the set is,

$$S: p, \quad [a_1 p - (\alpha_1^2 + a_1 \beta_1 + \beta_1^2)/\beta_0]/(\beta_0 p + \beta_1), \quad [-\beta_1 p - (\alpha_1^2 + a_1 \beta_1 + \beta_1^2)/\beta_0]/(\beta_0 p - a_1).$$

When the cycle of  $w$  functions is known, the quantic in homogeneous indeterminates  $x_1, x_2$  which has the functions as roots, is a covariant under direct transformation by  $H(x_1'/x_2', x_1/x_2)$ , since the transformation on  $x_1/x_2$  induces a cogredient transformation on the roots, which are therefore merely permuted. With  $w=3$ , the covariant is,

$$\begin{aligned} P = & (\beta_0 p - a_1)(\beta_0 p + \beta_1)x_1^3 + [-\beta_0^2 p^3 + 3(\alpha_1^2 + a_1 \beta_1 + \beta_1^2)p - (\alpha_1^3 - \beta_1^3)/\beta_0] x_1^2 x_2 \\ & + [\beta_0(a_1 - \beta_1)p^3 - 3(\alpha_1^2 + a_1 \beta_1 + \beta_1^2)p^2 + (\alpha_1^2 + a_1 \beta_1 + \beta_1^2)^2/\beta_0^2] x_1 x_2^2 \\ & + [a_1 \beta_1 p^3 + (\alpha_1^3 - \beta_1^3)p^2/\beta_0 - (\alpha_1^2 + a_1 \beta_1 + \beta_1^2)^2 p/\beta_0^2] x_2^3, \end{aligned}$$

and the transformations are,

$$R: \quad x_1 = [a_1 x_1' - (\alpha_1^2 + a_1 \beta_1 + \beta_1^2)x_2'/\beta_0]\zeta, \quad x_2 = (\beta_0 x_1' + \beta_1 x_2')\zeta.$$

where  $\zeta$  is arbitrary. From  $RP=P'$  we find,

$$P = -(\alpha_1 + \beta_1)D\zeta^3 P',$$

$D\zeta^3$  being the determinant of  $R$ . We can assume  $-(\alpha_1 + \beta_1)D\zeta^3 = z^{-1}$ , (Cf. (41)), i. e. choose the modulus of  $P$  for  $z$ . We also require an absolute covariant. Let  $P_1, P_2$  be two covariants of type  $P$ , order  $w$ , and modulus  $M$ , of  $H(x_1'/x_2', x_1/x_2)$  of period  $w$ . Then, by polynomial approximation,

$$Q(x) = P_1(x)/P_2(x), \quad (x = x_1/x_2),$$

can be expressed in the form,

$$x_2^v Q(x) = a_0 x_1^v + a_1 x_1^{v-1} x_2 + \dots + a_w x_2^v.$$

In the finite region delimited by the determinations  $(Q_i, x_{(i)})$  used in calculating  $Q(x)$ , the latter is an absolute  $\varepsilon$ -covariant of  $H$ .

We have now determined (40) so the  $\gamma_v$  will have the required invariant properties (43) under a periodic  $H(x, x')$ . When (42) is the same as  $F$ , ( $c_i$  arbitrary constants),

$$F: \quad \theta = c_0 Q(x)^{n-1} r^{n-1} + c_1 Q(x)^{n-2} P(x) r^{n-2} + \dots + c_{n-1} P(x)^{n-1}.$$

then (41) is the same as  $H^e F$ , except for  $\varepsilon$ -terms,  $H^e$  being the  $e$ -th iteration of  $H$ , q. e. d.

All masses of the planetary system can be obtained from the second mass by the iteration of a properly calculated  $H(m, m')$ , but the first, obtainable from the inverse of  $H$ , should not be used as a basis for the calculation of  $H$ . The main practical object is the determination of  $H$ .

### The system of moons (planets) of Saturn.

We chose the system of moons of Saturn as the planetary system most likely to give accurate results as a numerical instance of the above theory. That is, the generalized Bodeian law for the distances of these moons (from Saturn), gives, with the value  $z=1.3$  as the Bodeian ratio-number, accurate values of the distances. [12] This system is the best known illustration of the Bodeian principle.

We substituted successively in the equation,

$$\beta_0 mm' + \beta_1 m' - a_1 m - a_2 = 0,$$

for  $(m, m')$ , ( $xa$  Enceladus,  $ya$  Tethys), ( $ya$  Tethys,  $.187a$  Dione), ( $.187a$  Dione,  $.4a$  Rhea), and solved for the ratios  $a_1/\beta_0$ ,  $a_2/\beta_0$ ,  $\beta_1/\beta_0$ . This gives the homography,

$$H(m', m): \quad m' = \frac{Ama + Ba^2}{Cm + Da},$$

$$A = -.187y^2 - .213xy + .1496y - .0139876,$$

$$B = .4xy^2 - .039831y^2 - .109769xy + .0139876x,$$

$$C = -y^2 + .587y - .213x - .034969,$$

$$D = xy^2 - .374xy + .0748x - .039831y.$$

Beginning with  $m = .4a$  (Rhea) we iterated this  $H$  four times, that is, across four Bodeian intervals, from Rhea to Titan and equated the result to 21 (Titan). The unit of mass here is  $a/100000$ ,  $a$  being the mass of Saturn. Numbers such as  $.187a$  (Dione) were treated as if exact but, as this computation progresses, no decimal number should be cut off, as sufficiently accurate, short of fourteen decimal places. We thus obtain an equation,  $J=0$ , of order 12 in  $(x, y)$ . Knowing that the mass of Enceladus is near  $.025a$ , we construct a series of solutions  $(x, y)$  of  $J=0$  with  $x$  in the vicinity of  $.025a$ . One such solution, substituted in  $H(m', m)$ , would give to  $H$  its desired numerical form, except that  $H$  itself imposes certain arithmetical restrictions. All results of its iteration, in numerical form, should be positive numbers. Hence  $C$  and  $D$  should be opposite, and  $C$  and  $A$  the same in sign. Also the zero  $-aB/A$  and the singularity  $-aD/C$  should be near together in value, for if any  $m'$  produced by iteration falls within the interval  $I$  between the zero and the singularity, the next  $m'$  will be negative. It is soon seen that the value  $.025a$ , which astronomers have published for  $xa$  (Enceladus), is the most favorable  $x$ -value in our series of solutions  $(x, y)$  of  $J=0$ . The latter equation then becomes an octic in  $y$ , the pertinent solution of which is  $ya = .137241a$  (Tethys). The uniqueness of the resulting  $H$  in the vicinity  $(x, y)$  considered on  $J=0$ ,

appears to be a case of an important theorem yet unproved. Our final result for  $H$  is,

$$H_1(m', m): \quad m' = \frac{.22906831ma - .05887998a^2}{2.14313749m - .44087723a}.$$

The poles of  $H_1$  are imaginary as the theory requires.

Substituting  $m' = .025a$  (Enceladus) in  $H_1$  and solving for  $m$  we get,  $m = .2727112a$ , (Mimas). The results obtained by iterating  $H_1$  across the seventeen Bodeian intervals from Mimas to Phoebe are shown in Table IV wherein the numbers in parentheses are the mass-values given previously in astronomical tables.

TABLE IV. - *The mass-values of the moons of Saturn.*

Mimas	Enceladus	Tethys	Dione	Rhea	Lucretia *
.2727112a (.007a)	.025a (.025a)	.137241a (.11a)	.187a (.187a)	.4a (.4a)	.0786482a
Clio *	Minerva *	Titan	Hyperion	Terpsichore *	Faustina *
.1500575a	.2054483a	21.a (21.277a)	.1066207a	.1622444a	.23308a Revised to .264a
Japetus	Clymene *	Kala *	Theresa *	Tyche *	Phoebe
.0127615a	.1353155a	.1848087a	.369278a	.0733445a	.148328a

The asterisks designate moons which no longer exist or have not been discovered.

The small revision of the mass of Faustina should be explained. The mass given by  $H_1^{14}$  is  $m' = .23308a$  and this value is on the interval  $I$  between the singularity and the zero of  $H_1$ ;  $I(.205716a, .25704a)$ . Trial calculations show that this impasse cannot be corrected by small alterations alone of the mass of Enceladus (.025a). It is a result of the several possibilities for small errors which we have mentioned but especially of the fact that we have treated the mass of Dione and that of Rhea as if they were exact. Accordingly we arbitrarily replaced the value .23308a by .264a thus moving the mass-value for Faustina a little way out of the interval  $I$ . It is perhaps fortunate, since we cannot expect perfect accuracy, that, after eleven iterations, we have a

method of introducing a corrective alteration that is both definite and much restricted. Wishing to avoid conjectures, we do not account for the fact that the mass-value obtained for Mimas is too large.

The fact that a unique homography so accurately produces the mass-values of this planetary system appears to be a contradiction of the idea that chance has played any essential part in the cosmic creative processes, as far as the sizes of the planets of a central system are concerned [15].

### (iii) - Invariants in the theory of orbits.

Further variety in the program of inverse processes in invariant theory may be obtained and this will be evident from our third special problem. The question next considered is that of the determination of the equation of the orbit of a celestial body from its invariants of stability. We derive the equation of the arbitrary stable arc of finite length. Most of our illustrations were drawn from the theory of the comets. The problem of the invariance of periods of revolution is considered.

### The stabilizing invariants.

A curve is said to be a stable orbit of a celestial mass if the curve's position and form are such that the central forces will constrain the mass to move on the curve. Since least action is a vital principle in central motion we regard the form of the orbital curve, and not the form of the force-function  $F$  in (38), as the element expected to have essential simplicity [13]. In a previous paper [12], I have derived the most general form of  $F$  in order that the orbit should be stable. It is,

$$(44) \quad F = G_n(r) = \gamma^2 \lambda^2 [2p(r)^2 - rp(r)p'(r) + r^2/\lambda^2]/r^5,$$

( $p' = \partial p/\partial r$ ,  $p(r) = ar^{n-1} + br^{n-2} + \dots + k$ ),  $r$  being the distance of the astral body from the center of force. Another definition of a stable arc would be, any integral curve of (38) when  $F$  has the value (44). For increasing integral values of  $n$ ,  $G_n$  is an increasingly accurate gravitational formula. The number  $a$  decreases rapidly as  $n$  increases. We can say that  $G_5$  is « practically accurate » for astrophysical theory, and,

$$(45) \quad G_5(r) = \gamma^2 \lambda^2 [-\{ 2a^2r^8 + 3abr^7 + (b^2 + 2ac)r^6 + (ad + bc)r^5 \} \\ + (be + cd)r^3 + (d^2 + 2ce + 1/\lambda^2)r^2 + 3der + 2e^2]/r^5 = \gamma^2 \lambda^2 A_5(r).$$

It is important that the potential  $rG_5(r)$  is nearly reproduced, with sign reversed, by the reciprocation  $r = 1/r'$ .

The segment of the orbit, for which (44) is valid as the force, has the equation,

$$(46) \quad \int dr/p(r) = \lambda\theta + \lambda_1, \quad (\lambda, \lambda_1 \text{ constant}),$$

but (39), which we express more simply as,

$$C: \quad \theta = \nu(m)r^{n-1} + \xi(m)r^{n-2} + \dots + \pi(m)r + \varrho(m),$$

can also be the equation of this segment. If we make (46) and  $C$  identical, we obtain  $n-2$  rational expressions in  $\nu, \dots, \pi$ , the vanishing of which are necessary and sufficient conditions that  $C$  should be stable. These expressions,

$$C_{in}, \quad (i=1, \dots, n-2), \quad (\text{Cf. } R_4, \text{ p. 303}),$$

are invariants and may be called invariants of stability. When  $n=5$ ,

$$C_{15} = 4[16\nu O^4 - 36\nu\xi O^2\pi + 16\nu^2 O\pi^2 + 9\nu\xi^2\pi^2]/\pi^5\lambda$$

$$C_{25} = [48\xi O^4 - 108\xi^2 O^2\pi + 96\nu\xi O\pi^2 + 27\xi^3\pi^2 - 32\nu O^3\pi - 16\nu^2\pi^3]/\pi^5\lambda$$

$$C_{35} = [32O^5 - 96\xi O^3\pi + 48\nu O^2\pi^2 + 54\xi^2 O\pi^2 - 24\nu\xi\pi^3]/\pi^5\lambda.$$

**THEOREM.** - *The equation of any stable finite arc, ( $n=5$ ), can be expressed in the form,*

$$(47) \quad \theta = y \left[ \frac{O(m)^3}{\pi(m)^2} \right] r^4 + x \left[ \frac{O(m)^2}{\pi(m)} \right] r^3 + O(m)r^2 + \pi(m)r + \varrho(m),$$

where  $(x, y)$  are the (cartesian) coordinates of a common intersection of the three curves,

$$(48) \quad \begin{cases} (9x^2 - 36x + 16y + 16)y = 0, \\ 27x^3 - 108x^2 + 48x + 96xy - 16y^2 - 32y = 0, \\ 54x^2 - 96x - 24xy + 48y + 32 = 0. \end{cases}$$

In proof of this theorem, the elimination of  $\nu$  between  $C_{25}=0$ , and  $C_{35}=0$ , gives,

$$243\xi^5 - 729\xi^4 \left( \frac{O^2}{\pi} \right) - 108\xi^3 \left( \frac{O^2}{\pi} \right)^2 + 1728\xi^2 \left( \frac{O^2}{\pi} \right)^3 - 1728\xi \left( \frac{O^2}{\pi} \right)^4 + 512 \left( \frac{O^2}{\pi} \right)^5 = 0.$$

Hence,  $\xi = xO^2/\pi$  and,

$$(49) \quad 243x^5 - 729x^4 - 108x^3 + 1728x^2 - 1728x + 512 = 0.$$

Then  $\nu = yO^3/\pi^2$ , in which,

$$y = (27x^2 - 48x + 16)/(12x - 24).$$

Substitution of these values  $\nu, \xi$  in  $C_{i5}$ , ( $i=1, 2, 3$ ), gives (48). The equations from (46) onward hold within the field ( $Y$ ) of approximation which is assumed when  $C$  is derived by polynomial approximation. The powers of  $O/\pi$ , which are involved as factors against the expressions (48) were cancelled. They are small for some units of distance and large for other units, while the set (48) is unaltered by a change of units. That is, the equations hold regardless of the factors, q. e. d.

A solution of (48) is  $x = .5934$ ,  $y = .1763$ , this  $x$  being a root of (49). If we use  $n=6$ , the equation of  $C$  will be (47) with the term  $r^2 z_1 O(m)^4 / \pi(m)^3$  added to the right hand side. Then  $(x, y, z_1)$  is the common intersection of four algebraic surfaces, and the new  $(x, y)$  will be a little more accurate than the  $(x, y)$  of (48).

To find the substitutions for which the  $C_{in}$  are invariant, we may use the generators of the algebraic  $Q$ , (Cf. (1), § I), as given by DICKSON

$$\begin{aligned} V: \quad r_1 &= -r_2', \quad r_2 = r_1'; & T_h: \quad r_1 &= r_1' + hr_2', \quad r_2 = r_2'; \\ S_k: \quad r_1 &= r_1', \quad r_2 &= kr_2', & (k \neq 0). \end{aligned}$$

When these in succession are tried on the homogeneous form of  $C$ , it is found that only  $S_k$  leaves  $C_{in}$  invariant, q. e. d.

The arbitrary phase of the ground-function (47) depends upon the arbitrariness of  $O, \pi, \rho$ . There is but one intersection of (48).

Equation  $C$  is in the rational or parabolic form. We shall be able to compare it with the general functional form in certain cases.

### Can an ellipse which is very eccentric be the permanent orbit of a comet?

HALLEY'S is a typical comet, and its orbit has been considered to be an ellipse of large eccentricity, which is described to a close approximation by the following data. (The ellipse of the data is a little longer than that of the comet). We refer to the ellipse of the data as  $Y$ , and we can proceed as if the orbit had been transferred to the ecliptic.

$$(50) \left\{ \begin{array}{l} \text{Unit of distance } 22046875 \text{ miles; semi-axes } a=80, b=20, \\ \text{Perihelion distance } D=a(1-\sigma)=2.5403, (=56006000 \text{ miles}), \\ \text{Aphelion distance } r_1'=157.459667, (=3471493600 \text{ miles}), \\ \text{Eccentricity } \sigma = [\sqrt{(a^2 - b^2)}] / a = \frac{1}{4} \sqrt{15} = .9682458, \\ \text{Equation } r = 5 / (1 - .9682458 \cos \theta). \end{array} \right.$$

The aphelion is on the positive polar axis and has the coordinates  $(r_1', \theta_1')$ , viz.,  $(80 + 20\sqrt{15}, 0)$ . The intersection of  $Y$  with the mean orbit of Jupiter; (Mean distance from the Sun 483000000 miles), is the point  $Q_2: (r_2', \theta_2')$ , viz.,  $(21.90786, .648351)$ . We shall prove the following,

LEMMA 1. - *An arc of the ellipse  $Y$ , chosen in the vicinity of the intersection of  $Y$  with the orbit of Venus, is not a stable arc.*

Five points on  $Y$  in the vicinity noted are  $P_i: (r_i, \theta_i^{\text{rad.}})$ , as follows; (The mean distance from the Sun to Venus being taken as 67000000 miles),

$$(51) \quad \left\{ \begin{array}{l} P_0: r_0 = 2.9023382, \quad \theta_0 = 2.4135092 = 138^\circ 17' 2'', \\ P_1: r_1 = 2.9690874, \quad \theta_1 = 2.3552685 = 134^\circ 56' 49.1'', \\ P_2: r_2 = 3.0389794, \quad \theta_2 = 2.3002372 = 131^\circ 47' 37.9'', \\ P_3: r_3 = 3.1122409, \quad \theta_3 = 2.2477901 = 128^\circ 47' 19.9'', \\ P_4: r_4 = 3.1891221, \quad \theta_4 = 2.1974667 = 125^\circ 54' 20.2''. \end{array} \right.$$

The equation,

$$\theta = \nu r^4 + \xi r^3 + O r^2 + \pi r + \rho,$$

determined by these points is,

$$(52) \quad \theta = .44686r^4 - 5.93300r^3 + 29.83118r^2 - 67.84843r + 61.39025.$$

It is the equation of the arc  $P_0P_4$ . In order to determine whether it is a case of the type-form (47) we calculate the results,

$$yO^3/\pi^2 = 1.01667, \quad xO^2/\pi = -7.78302.$$

Comparing these numbers with the respective coefficients of  $r^4$  and  $r^3$  in (52) we conclude that the discordance is too great and that it cannot be corrected. No ellipse of large eccentricity will satisfy this test for stability.

If a comet travelling on an ellipse  $Y$  is considerably perturbed in the region corresponding to its intersection with the orbit of Venus, it will not reestablish  $Y$  as its orbit. The question therefore arises, what closed curve, possible as the orbit of a comet, possesses the property of stability throughout its length? [14].

#### The central force which engages a comet.

We obtain the answer to this question from the standpoint of formula (44) which includes the central force for the case of any stable central motion of a single mass.

Whether a comet is an inert remnant of the primordial nebula (SIMON NEWCOMB) or a composition within which some chemical action, involving an

effect of illumination, is taking place (W. W. CAMPBELL), or a cloud which has integrated from the substance of a « planetary nebula » surrounding, distantly, the Sun, it is a reasonable conjecture that its motion is mechanically equivalent to a mere flight of particles which influence each other gravitationally only in a measure insufficient to destroy the stability of central motion of any constituent particle.

Let the origin  $O$  (Fig. 1) be the center of the force (Sun). Project all of the aforesaid particles of the comet by dropping a perpendicular from each to the plane of the orbit of the nucleus. The feet of these perpendiculars will trace central orbits, each having the property of stability, and a chosen group of these orbits will be properly contained in a region  $\varrho_0$ . The equation of a chosen orbit  $C_1$  in  $\varrho_0$  is given by the formula (13), that is,

$$(53) \quad \theta = \varphi(\zeta_{i_1 i_2}; \mu; r).$$

Here  $r$  is continuous over a range  $\varrho > l > 0$ . and  $\mu$  is a number of a point-set on a segment  $(\mu^{(1)}, \dots, \mu^{(q)})$ .

In the perturbational field  $\tau$  immediately surrounding  $C_1$ ,  $\mu$  may be regarded as the mass  $m$  of a particle of which  $C_1$  is the orbit Cf. (40). We choose the lower bounding radius, of  $\varrho_0$ , as the polar axis, and assume

POSTULATE 1. - *The angular width  $\theta_0$  of  $\varrho_0$  is of such smallness that  $s \theta_0 = 0$  within  $(Y)$ ,  $s$  being taken from the transformation  $T_0$  which returns a perturbed orbit of  $\tau$  to its primary  $C_1$ , (Cf.  $R_1$ , p, 301),*

$$T_0: \quad \theta' = \theta, \quad r' = r + sp(r), \quad (s \doteq 0, \quad p(r) = ar^{n-1} + br^{n-2} + \dots + k).$$

We note that  $G(r)$  of (44) was derived by means of  $T_0$ , and secondly that, with  $\varrho$  conveniently extensive, the effect of angular narrowing of  $\varrho_0$  is to make the orbital arcs in  $\varrho_0$  like eccentric ellipses focal at  $O$  rather than approximate circles centered at  $O$ .

The formal inverse of (53) is now,  $m = \psi(r, \theta)$ . Expressing  $\psi$  on  $\varrho$  by polynomial approximation,  $(\theta \doteq 0, \theta^2 = 0$  in  $\varrho_0)$ ,

$$(54) \quad \begin{aligned} m &= (A + \theta A_1)r^{n-1} + (B + \theta B_1)r^{n-2} + \dots + (K + \theta K_1) \\ &= h(r) + \theta h_1(r), \end{aligned} \quad (A, \dots, K_1 \text{ constant}).$$

In the central force-function for a stable orbit  $C_1$ , viz.,

$$G(r) = \gamma^2 \lambda^2 [2p(r)^2 - rp(r)p'(r) + r^2/\lambda^2]/r^5 = \gamma^2 \lambda^2 A_n(r), \quad (p' = \partial p / \partial r),$$

we may now use  $h(r) + \theta h_1(r)$  for  $p(r)$ , that is,  $h(r)$  for  $p(r)$ , since  $s\theta_0 = 0$ , and  $\theta < \theta_0$  in  $\varrho_0$ . We then find,

$$(55) \quad G(r) = \gamma^2 \lambda^2 [m \{ 2m - 4\theta h_1(r) + \theta r h_1'(r) \}]/r^5 + \gamma^2/r^3.$$

POSTULATE 2. - *The mass  $m$  is small, such that the bracket expression in (55) is zero, in (Y)*

In point of fact the mass of a whole comet may be small enough to satisfy this postulate, but we can assume this only for the mass  $m$  of one of its particles. Moreover, when advantage results from the broader interpretation, we can replace  $r$  by  $r_1 + w$  in  $T_0$ , and in (54), giving a new  $p$ , a new set  $A, \dots, K_1$ , and therefore a new  $h$  and  $h_1$ . The first  $G$  is reproduced with  $r_1$  in place of  $r$  and (55) becomes  $G(r_1)$ . Stability of (46) is preserved by this transformation if the positive  $w$  is properly limited above. Therefore,

THEOREM. - *The central force, as it acts upon the diffuse mass called a comet, is equal in (Y), to the formula of the inverse cube,  $J(r) = \gamma^2 / (r - w)^3$ .*

**The transcendental equation of the orbit.**

When  $V/r_1^3$  is substituted for the arbitrary  $F(1/u)$  in (38), i. e., in

$$D(r_1, \theta, F) = d^2u/d\theta^2 + u - F(1/u)/\gamma^2 u^2 = 0, \quad (u = 1/r_1),$$

the equation can be integrated and the result is,

$$(56) \quad r = 1/(xe^{a\theta} + \eta e^{-a\theta}) + w, \quad [= 1/\Gamma(\theta, \alpha, x, \eta) + w, \quad e = 2.71828\dots].$$

Arbitrary constants are  $x, \eta, \alpha$ , the latter being a function of  $V/\gamma^2$ . This orbit is a double spiral ( $R_2$ , p. 244). In the vicinity of the origin it consists of two oppositely directed whirls, each approaching asymptotically the small circle  $r = w$ , this circle being a species of ROCHE limit. In the outer part of the plane the curve is a single oval. Its aphelion point may be, in fact, at infinity but we give attention only to cases where the aphelion  $\Omega_1$  is at a finite distance from  $O$ . We shall assume  $O\Omega_1$  as the positively directed polar axis. By definition the outer node  $\delta$  of the curve (in the negative direction from the Sun) is the perinodal point and the next consecutive node  $v$ , on the positive side of the Sun is the anodal point.

LEMMA 2. - *The polar axis can always be chosen so as to be a unique line of symmetry of the curve (56).*

If we rotate the polar axis through an angle  $-\beta$ , the equation (56) is transformed into,

$$r = 1/[(xe^{a\beta})e^{a\theta} + (\eta e^{-a\beta})e^{-a\theta}] + w.$$

Write,

$$\zeta = xe^{a\beta} = \eta e^{-a\beta},$$

then,

$$\beta = [\log_e (\eta/x)]/2\alpha,$$

a unique result, assuming that  $\eta, x, a$  are different from zero. The equation now becomes,

$$(57) \quad r = 1/\zeta(e^{a\theta} + e^{-a\theta}) + w,$$

and it is evident that the positive coordinate axis is a line of symmetry; (If  $(r', \theta')$  is on the curve,  $(r', -\theta')$  is also), q. e. d.

The conclusion is forced that (57), for properly chosen  $a, \zeta, w$  must be an approximation of an ellipse which is focal at the Sun. A special case is where we would make (57) coincide as near as possible with the orbit of HALLEY'S, comet that is, with  $Y$  of (50). We would substitute the numerical coordinates of three points for  $(r, \theta)$  in (57) and solve for  $a, \zeta, w$ . These points are the coordinates  $(r_1', \theta_1')$  of the aphelion  $\Omega_1$ , the coordinates  $(r_2', \theta_2')$  of the point where Jupiter's orbit intersects  $Y$ , and  $(r_3', \theta_3') = (24.99251, .613444)$ . We would then plot (57) and  $Y$  together making them coincide as nearly as their respective equations permit.

This work can be simplified some by means of a preliminary theorem about  $a$ . Let  $(r_2, \theta_2), (r_2, -\theta_2)$ , be any two points symmetrically situated on an orbit (57), and  $(r_1, 0)$  the aphelion. Then,

$$2\zeta = 1/(r_1 - w), \quad \zeta(e^{a\theta_2} + e^{-a\theta_2}) = 1/(r_2 - w),$$

whence,

$$e^{2a\theta_2} - 2\left(\frac{r_1 - w}{r_2 - w}\right)e^{a\theta_2} + 1 = 0,$$

and we obtain a general formula for  $a$ ,

$$(58) \quad a = \frac{1}{\theta_2} \log_e \left\{ [r_1 - w + \sqrt{(r_1 - w)^2 - (r_2 - w)^2}] / (r_2 - w) \right\}.$$

The function  $a$  remains invariant when the point  $(r_2, \theta_2)$  is moved along the curve.

In the determination, described above, of a special equation (57), where,

$$r_1' = 157.459667, \quad r_2' = 21.90786, \quad \theta_2' = .648351, \quad \text{etc.},$$

we find,

$$a = 4.267975, \quad \zeta = .00322685, \quad w = 2.51.$$

The equation of the orbit is, therefore,

$$(59) \quad r = 1/\{ .00322685(e^{4.267975\theta} + e^{-4.267975\theta}) \} + 2.51.$$

Figure 4 shows the curve (59) and the ellipse  $Y$ . Any arc of (59) satisfies the conditions for stability given by (47).

The equation (57) can be made to represent, closely within the solar system, an ellipse, a parabola, or a hyperbola, by proper choices of the set  $(\alpha, \zeta, w)$  [14]. More than two hundred of the observed orbits of comets could not be distin-

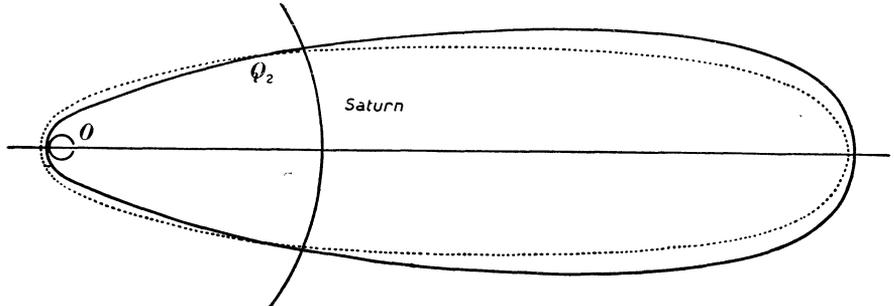


Fig. 4.

guished from parabolas. Related to the properties of the transcendental curves is the fact that the great comet of 1811, which was perturbed from its normal course by Jupiter's attraction, at the beginning of its recessive motion, approximated  $\delta v$  as its path. A comet must have such momentum that it will span the perinodal region as a projectile. The jump required will be of almost zero duration if the radius  $w$  is but little less than the comet's perihelial distance. The node  $\delta$  is an « almost minimal » point.

**The period of Halley's comet. Invariant periods.**

We give a simple determination of the period of HALLEY'S comet. An equation from the known theory of central motion, (center of force at  $O$ ), which involves the time variable  $t$ , is

$$dt = r^2 d\theta / \gamma.$$

The time required for a planetary body  $N$  to complete a revolution around  $O$  on a simple closed circuit  $r = F(\theta)$  may therefore be written,

$$T = \frac{1}{\gamma} \int_0^{2\pi} [F(\theta)]^2 d\theta.$$

If  $r = F(\theta)$  is the equation (57), the indefinite integral of  $r^2$  is,

$$W(\theta) = \int r^2 d\theta = -\frac{1}{2\alpha\zeta^2} \left( \frac{1}{e^{2\alpha\theta} + 1} \right) + \frac{2w}{\alpha\zeta} \tan^{-1} e^{\alpha\theta} + w^2\theta + H, \quad (H \text{ constant}).$$

Particularizing (57) to (59),  $W(\theta)$  becomes a numerical function of  $\theta$ . Accepted Astronomical data give this comet (HALLEY'S) a speed of « about one mile per second » at aphelion. We shall use this speed as .6914 miles per second, or

$$l = 4181587.2$$

miles in 70 days on a circular arc  $\Omega$ , (center of circle at  $O$ ), which is bisected by the aphelion point. Hence  $\Omega$  is an arc of radius,

$$r_1' = 3471493600$$

miles. The angle subtended at  $O$  by  $\Omega$  is,

$$E = .00120455 \text{ radians, } (= l/r_1').$$

With  $\alpha = 4.267975$ ,  $\zeta = .003227$ ,  $w = 2.51$ , time-unit = one day, we have

$$70\gamma = 2W(\theta) \Big|_0^{E/2} = 29.87884, \quad \gamma = .42684.$$

The period of this comet is, therefore,

$$T = \frac{2}{.42684} W(\theta) \Big|_0^\pi = 76.08 \text{ years.}$$

The next return to perihelion of this comet will be in 1986.

\* \* \*

LAPLACE (1749-1827) emphasized the importance of the invariability of the period of the Earth's revolution around the Sun, and referred to this invariance as « one of the most remarkable phenomena in the system of the universe » [15]. Thus he was among the first to recognize an invariant; was second only to LAGRANGE who had noticed, in 1773, the invariance of the discriminant of a binary quadratic quantic under a linear transformation. No complete analytic proof of LAPLACE'S invariance has ever been given. We here prove a theorem which includes, as a special case, the invariance of the Earth's period.

We have stated that (46) is the equation of only a segment of the orbit which corresponds to  $G(r)$  as the force, but, if the segment is of length above some definite minimum there will be enough discrete points upon it to determine the orbit completely. This being assumed,  $G(r)$  may be regarded as the force at any point on the stable orbit. Its general equation can therefore be determined. We have only to integrate.

$$D(r, \theta, G(r)) = 0.$$

With,

$$G(r) = \gamma^2 \lambda^2 [2p(r)^2 - rp(r)p'(r) + r^2/\lambda^2]/r^5 \\ = \gamma^2 \lambda^2 [-\{(n-3) a^2 r^{2n-2} + A_1 r^{2n-3} + \dots + A_{2n-7} r^5\} + Lr^3 + Mr^2 + Ur + V]/r^5,$$

and,

$$f(r) = 2\lambda^2 \left[ \frac{a^2}{2} r^{2n-2} + \frac{A_1}{2n-7} r^{2n-3} + \frac{A_2}{2n-8} r^{2n-4} + \dots + \frac{A_{2n-7}}{1} r^5 + Lr^3 + \frac{1}{2} (M - \lambda^{-2}) r^2 + \frac{1}{3} Ur + \frac{1}{4} V + C_1 r^4 \right],$$

the integral is,

$$(60) \quad \theta + C_2 = \int dr / \sqrt{f(r)}, \quad (C_1, C_2 \text{ arbitrary constants}).$$

We can now identify,

$$(61) \quad \theta + \lambda_1/\lambda = \int dr / \lambda p(r),$$

that is (46), with any arc of the orbit, assuming the latter to be a closed curve of one circuit, for, since (60), (61) are the same functional forms on the arc, we have,

$$\sqrt{f(r)} = \lambda p(r).$$

Note also that the hyperelliptic integrand in the case of any orbit, becomes rational when the orbit is stable.

We shall refer to a maximum (of  $r$ ) on the segment (61) as an aphelion and to the next consecutive extreme, a minimum, if existent on the segment, as a perihelion.

LEMMA 3. - *The aphelion and perihelion distances are roots of  $p(r) = 0$ .*

In fact, from (61),  $dr/d\theta = \lambda p(r)$ , and, since we have assumed the existence of the extremes, the Lemma follows from the elementary theory of such extremes.

THEOREM. - *The time of rotation from aphelion, ( $r = \sigma$ ), to perihelion, ( $r = \nu$ ), on a stable orbit as represented by (61), is an  $\varepsilon$ -invariant under the arbitrary perturbations taken as equivalent to the transformation,*

$$T_0 : \quad \theta' = \theta, \quad r' = r + sp(r), \quad (s (= \varepsilon) \doteq 0).$$

Our proof is as follows. The equation used above to introduce the time-variable  $t$ , can be written,

$$dt = r^2 dr / \gamma (dr/d\theta) = r^2 dr / \gamma \lambda p(r).$$

Therefore the time of rotation from aphelion to perihelion is,

$$\tau = \frac{1}{\gamma \lambda} \int_{\nu}^{\sigma} \frac{r^2}{p(r)} dr.$$

When we transform  $\tau$  by  $T_0$ , since  $\sigma, \nu$  are roots of  $p(r)$ , the limits of the integral remain unaltered. We consequently obtain,

$$\frac{1}{\gamma\lambda} \int_{\nu}^{\sigma} \frac{r'^2}{p(r')} dr' = \frac{1}{\gamma\lambda} \int_{\nu}^{\sigma} \frac{r^2}{p(r)} dr + \frac{\varepsilon}{\gamma\lambda} (\sigma^2 - \nu^2), \quad \text{q. e. d.}$$

If the orbit is that of the Earth,  $\nu$  is so near in value to  $\sigma$  that the  $\varepsilon$ -term is of the second order of magnitude and negligible. If the orbit is that of HALLEY'S comet there is an aphelion distance but no proper perihelion. The arc upon which the comet makes its circuit about the Sun may be said to provide a weak perihelion, ( $r=\nu$ ). There will remain, however, an  $\varepsilon$ -term which can be made small but not zero by decreasing  $\varepsilon$ . It is known from observations that the period of the comet is not exactly invariant but depends upon the relative positions of the planets when the comet passes through their field.

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