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ON CONTINUOUS TRANSFORMATION
OF SOME FUNCTIONS INTO AN ORDINARY DERIVATIVE

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1. - A function $f(x)$ is called an ordinary derivative in $[0, 1] = [0 \leq x \leq 1]$ if there exists a continuous function $\mathfrak{B}(x)$ such that $\frac{d}{dx} [\mathfrak{B}(x)] = f(x)$.

This paper is intended to solve the following *problem D* which was putted to me by NICOLAS LUSIN: Let $f(x)$ be a finite function of class 1 having the property of DARBOUX in $[0, 1]$ ⁽¹⁾. We must find a continuous and essentially increasing function $x = \varphi(t)$ [$\varphi(0) = 0, \varphi(1) = 1$] such that $f[\varphi(t)]$ is an ordinary derivative in $[0 \leq t \leq 1]$.

Let $f(x)$ be a finite function and let

$$(y) \qquad y_1, y_2, y_3, \dots$$

be the sequence of all rational numbers y_n such that there are two points a_n and b_n belonging to $[0, 1]$ and satisfying the condition $f(a_n) < y_n < f(b_n)$. Denote by $E_{y_n} \{E^{y_n}\}$ the set of all points x of $[0, 1]$ satisfying the condition $f(x) < y_n \{f(x) > y_n\}$. If E is a measurable set of points, then mE will denote the measure of E .

DEFINITION 1. - Let (ϱ, Δ) be any pair of positive numbers ϱ, Δ , and let E be any measurable set. We shall say that a point x_0 of E is a *density* (ϱ, Δ) *point of E* if for each interval $i_{x_0} = (x_0 - \delta', x_0 + \delta'')$ where $\delta = \delta' + \delta'' \leq \Delta$ there exists the inequality

$$(1) \qquad m[i_{x_0} E] \geq \delta - \frac{1}{\varrho^3} \cdot \delta^2.$$

In our Note ⁽²⁾ we have introduced the following definition.

DEFINITION 2. - Let

$$\Delta_1, \varrho_1, \Delta_2, \varrho_2, \Delta_3, \varrho_3, \dots$$

⁽¹⁾ G. DARBOUX: *Sur les fonctions discontinues*. [Ann. Ec. Norm. Sup. (2) 4, pp. 109-110].

⁽²⁾ *Sur les fonctions dérivées*. [Bulletin des Sciences Mathématiques, (2) 64, pp. 116-121 (1940)].

be a sequence of positive numbers such that :

$$(i) \quad \varrho_n \geq |y_i| \quad \text{for all } i \leq n;$$

$$(ii) \quad \varrho_1 < \varrho_2 < \varrho_3 < \varrho_4 < \dots; \quad \Delta_1 \leq \Delta_2 \leq \Delta_3 \leq \Delta_4 \leq \dots$$

A finite function $f(x)$ will be called approximately (ϱ_n, Δ_n) continuous in $[0, 1]$ if there exists a system \bar{P} of perfect sets,

$$(\bar{P}) \quad \bar{P}_{y_r}^{r+s}, \bar{P}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; \quad r=1, 2, 3, \dots)$$

such that :

$$(i) \quad \bar{P}_{y_r}^{r+s} \subset [0, 1], \quad \bar{P}_{r+s}^{y_r} \subset [0, 1],$$

$$\bar{P}_{y_r}^{r+s} \subset \bar{P}_{y_r}^{r+s+1} \subset E_{y_r}, \quad \bar{P}_{r+s+1}^{y_r} \subset \bar{P}_{r+s}^{y_r} \subset E^{y_r};$$

$$(ii) \quad E_{y_r} = \lim_{s \rightarrow \infty} \bar{P}_{y_r}^{r+s}, \quad E^{y_r} = \lim_{s \rightarrow \infty} \bar{P}_{r+s}^{y_r};$$

(iii) if $y_r < y_t$ and if M is the greater of the integers r, t , every point of the set

$$\bar{P}_{y_r}^{M+s} \{ \bar{P}_{M+s}^{y_t} \}, \quad (s=0, 1, 2, 3, 4, \dots)$$

is a density $(\varrho_{M+s}, \Delta_{M+s})$ point of the set

$$\bar{P}_{y_t}^{M+s} \{ \bar{P}_{M+s}^{y_r} \}.$$

In the same Note we have proved the following

THEOREM I. - Every approximately (ϱ_n, Δ_n) continuous in $[0, 1]$ function is an ordinary (exact) derivative in $[0, 1]$.

The proof of this theorem will rest on the following

LEMMA 1. - We suppose that for a finite summable in $[0 \leq x \leq 1]$ function $f(x)$ and for any interval $(x_0 - \delta_1', x_0 + \delta_1'')$ contained in $[0, 1]$, there exists a sequence of integers,

$$(n) \quad n_1, n_2, n_3, n_4, \dots$$

and a sequence of measurable sets,

$$(\mathcal{E}) \quad \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \dots$$

such that :

$$(i) \quad \mathcal{E}_1 < \mathcal{E}_2 < \mathcal{E}_3 < \dots < \mathcal{E}_n < \dots, \quad \lim_{n \rightarrow \infty} \mathcal{E}_n = (x_0 - \delta_1', x_0 + \delta_1'');$$

$$(ii) \quad m(C\mathcal{E}_k) \leq \frac{2\delta_1}{3}, \quad \delta_1 \left[1 - \frac{2\delta_1}{3} \right] \leq m\mathcal{E}_1 \leq \delta_1,$$

where

$$\delta_1 = \delta_1' + \delta_1'';$$

(iii) if x_i is any point of \mathcal{E}_1 , then $|f(x_i) - f(x_0)| < \eta$, and if x_k is any point of \mathcal{E}_k , then $|f(x_k)| \leq \varrho_{n_{k-1}}$.

2. - In order to solve our problem D we proceed in the following manner.

In the first place we construct for a given finite function $f(x)$ of class 1 having the property of DARBOUX a characteristic system P of perfect sets

$$(P) \quad P_{y_r}^{r+s}, \quad P_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, 4, \dots) \quad (3).$$

In the second place we pass from the characteristic system P to a perfect system of sets:

$$(Q) \quad Q_{y_r}^{r+s}, \quad Q_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots) \quad (3).$$

In the third place we construct a system \bar{Q} of perfect sets

$$(\bar{Q}) \quad \bar{Q}_{y_r}^{r+s}, \quad \bar{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots)$$

which enjoys the following properties:

(i) there exists a correspondence CS of the similitude

$$(CS) \quad \dot{Q}_{y_r}^{r+s} \leftrightarrow \bar{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r} \leftrightarrow \bar{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots)$$

between the sets of the system \bar{Q} and the sets of a perfect system \dot{Q} of the sets which is obtained by making the complete corrections of the initial perfect system Q (3);

(ii) the set

$$\lim_{n \rightarrow \infty} \sum_{s=1}^{s=n} \bar{Q}_{y_s}^n \left\{ \lim_{n \rightarrow \infty} \sum_{s=1}^{s=n} \bar{Q}_n^{y_s} \right\}$$

is dense in the segment $[0 \leq t \leq 1]$;

(iii) if $y_r < y_t$ and if M is the greater of the integers r and t , every point of the set

$$\bar{Q}_{y_r}^{M+s} \left\{ \bar{Q}_{M+s}^{y_t} \right\}$$

is a density $(\varrho_{M+s}, \Delta_{M+s})$ point of the set

$$\bar{Q}_{y_t}^{M+s} \left\{ \bar{Q}_{M+s}^{y_r} \right\}$$

for all $s=0, 1, 2, 3, \dots$;

(3) I. MAXIMOFF: *Some Theorems on the Functions of class 1 having the property of Darboux*. [Rendiconti del Circolo Matematico di Palermo].

I. MAXIMOFF: *On Continuous Transformation of Some Functions into Approximately Continuous*. [Annals of Mathematics, 1941].

$$(iii) \quad m \{ C[\bar{Q}_Y^n \cdot \bar{Q}_n^y] \} \leq a_n,$$

where a_n is a positive number such that the series $a_1 + a_2 + a_3 + a_4 + \dots$ is convergent and $Y\{y\}$ denote the greater {the least} of the numbers $y_1, y_2, y_3, \dots, y_n$.

At last, we construct the continuous and essentially increasing function $x = \psi(t)$ [$\psi(0) = 0, \psi(1) = 1$] which transforms the sets

$$\dot{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, 4, \dots; r=1, 2, 3, 4, \dots)$$

respectively into the sets

$$\bar{Q}_{y_r}^{r+s}, \quad \bar{Q}_{r+s}^{y_r}.$$

If these conditions are fulfilled we shall say that

(i) the system \dot{Q} of perfect sets

$$(\dot{Q}) \quad \dot{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots)$$

is the *final perfect system* of sets for $f(x)$, and will be denoted par DQ ;

(ii) the system \bar{Q} of perfect sets

$$(\bar{Q}) \quad \bar{Q}_{y_r}^{r+s}, \quad \bar{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots)$$

is the system of our problem D for $f(x)$.

In order to construct the sets of the system DQ and of the system \bar{Q} we shall need the following lemmas.

LEMME 2. - If ϱ, Δ are positive numbers and if p is any perfect everywhere non dense set, then there exists a perfect everywhere non dense set P such that every point of p is a density (ϱ, Δ) point of P .

Proof. - We construct, first of all, a perfect everywhere non dense set P' such that every point of p is a density point of P' . Let (a, b) be any contiguous interval of P' and let

$$(\delta) \quad \delta_1, \delta_2, \delta_3, \dots$$

be any sequence of intervals such that

(i) the extremities of these intervals are rational numbers;

(ii) the length of each of these intervals does not surpass Δ ;

(iii) each of these intervals contains at least one of the points of p

and $p \subset \sum_k \delta_k$.

Now we will consider a transformation T of the set P' into a new perfect set P , $P' = TP$. To determine this transformation T we fix one of the extremities of any contiguous interval (a, b) of the set P and we denote this extremity with $f(a, b)$. Let F be the set consisting 1) of all points $f(a, b)$, 2) of all the extre-

mities of all the intervals $\delta_1, \delta_2, \delta_3, \dots$ and 3) of all points of the set p . The transformation T must shorten the contiguous intervals of P without changing the points of the set F .

First of all, we shorten each contiguous interval (a, b) of P' having points in common with δ_1 in such a manner that

$$m[\delta_1 CP'] \leq \frac{\delta_1^2}{\varrho^3}.$$

This is the first shortening which transforms the set P' into a new perfect set P'' .

Without changing the points of the set F we shorten each contiguous interval of P'' having points in common with δ_2 in such a manner that

$$m[\delta_2 CP''] \leq \frac{\delta_2^2}{\varrho^3}.$$

This is the second shortening which transforms the set P'' into a new perfect set P''' .

Without changing the points of the set F we shorten each contiguous interval of P''' having points in common with δ_3 in such a manner that

$$m[\delta_3 CP'''] \leq \frac{\delta_3^2}{\varrho^3}$$

and so on.

This process may be continued illimitedly and as a result of it we shall obtain the perfect set P satisfying all the conditions of the lemma 2.

LEMMA 3. - If p, q are any two perfect everywhere non dense sets such that $p \cdot q = 0$, then there exists a pair of perfect everywhere non dense sets P, Q such that $P \cdot Q = 0$ and every point of the set $p\{q\}$ is a density (ϱ, Δ) point of the set $P\{Q\}$.

Proof. - First of all, we find the perfect everywhere non dense sets P', Q' such that every point of the set $p\{q\}$ is a density (ϱ, Δ) point of the set $P'\{Q'\}$. The sets p and q have no points in common, therefore the set q is contained in the sum $S = (a_1, b_1) + (a_2, b_2) + \dots + (a_k, b_k)$ of the contiguous intervals of p . Let (a_r, β_r) be any interval such what the set $(a_r, b_r)q$ will be contained in (a_r, β_r) and $a_r < a_r < \beta_r < b_r$. We form the perfect set

$$P = P' - \sum_r (a_r, \beta_r)P'$$

and a perfect everywhere non dense set P_r contained in (a_r, β_r) and such that every point of the set $(a_r, \beta_r)q$ is a density (ϱ, Δ) point of P_r . Evidently, the sum $Q = \sum_r P_r$ is a perfect everywhere non dense set. It is easily seen that P

and Q are the sets satisfying all the conditions of our lemma.

3. - We now turn back to the construction of the sets of the system \bar{Q} . The procedure \bar{Q} of this construction is based on the precedent lemmas. To construct the sets

$$\bar{Q}_{y_r}^{r+s}, \quad \bar{Q}_{r+s}^y, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots)$$

we shall consider these sets in pairs

$$[\bar{Q}_{z_r}^n, \quad \bar{Q}_n^{z_{n+1-r}}]$$

where $z_1^n, z_2^n, z_3^n, \dots, z_n^n$ is the increasing sequence of numbers $y_1, y_2, y_3, \dots, y_n$. The pair $[\bar{Q}_{z_r}^n, \bar{Q}_n^{z_{n+1-r}}]$ will be called a pair of order n and of index r , $\bar{Q}_{z_r}^n$ is the first set and $\bar{Q}_n^{z_{n+1-r}}$ is the second set of this pair.

If $r=1$, this pair is said to be of class 1.

If $1 < r \leq n+1-r$, this pair is said to be of class 2.

If $r > n+1-r$, this pair is said to be of class 3.

Let c_n^s denote the number of all the pair s of class $s=1, 2, 3$. We define the order of the construction of the sets \bar{Q} by the following rule R :

(i) firstly we construct the sets of the pair of index $r=1$, next the sets of the pair of index $r=2$, thereupon the sets of the pair of index $r=3$ and so on;

(ii) then we construct the set $\bar{Q}_{z_r}^n$, and after the set $\bar{Q}_n^{z_{n+1-r}}$ of the same pair

$$[\bar{Q}_{z_r}^n, \quad \bar{Q}_n^{z_{n+1-r}}]$$

of class 3;

(iii) we construct the sets of each pair of class <3 simultaneously and conjointly.

Denote by K_n^r the operation of the construction of the sets of a pair of order n and of index r if $1 \leq r \leq n+1-r$, i. e. if this pair is of class <3 .

Denote by D_n^s ($s=1, 2, 3, \dots, 2c_n^3$) the operation of the construction of the set belonging to the class 3 where s indicate the order of this operation defined by the rule R .

Let G_k be one of the operations K_n^r, D_n^s

$$(n=1, 2, 3, \dots; r=1, 2, 3, \dots; s=1, 2, 3, \dots)$$

where k indicate the place of the operation G_k in the system of all these operations defined by the rule R .

Each operation Q_k must be accompanied by the passage from one perfect system Q of the sets to another perfect system which will be denoted by $G_k Q$. After each operation G_k it must be established some *correspondence of the similitude* $\pi_k: x \leftrightarrow t$ between the points t of the already formed sets \bar{Q} and the points x of the corresponding sets Q of the system $G_k Q$.

Thus, each step of the procedure \bar{Q} consists

1) of the operation G_k , 2) of the operation construing the sets of the perfect system $G_k Q$ and 3) of the operation establishing the correspondence π_k .

The general scheme of the step S_k of the procedure \bar{Q} will be described in the following lines. We suppose that the correspondence $\pi_{k-1} : x \leftrightarrow t$ is given. Denote by $X_{k-1} \{T_{k-1}\}$ the set of all points $x \{t\}$ partaking of the correspondence π_{k-1} .

We now consider the following cases.

First case: the operation G_k brings to the simultaneous construction of the sets \bar{Q}_1, \bar{Q}_2 of one pair. Let $Q_1 \{Q_2\}$ be the set of the perfect system $G_{k-1} Q$ having the same indices as those of the sets $\bar{Q}_1 \{\bar{Q}_2\}$ ⁽⁴⁾. Let $q_1 = X_{k-1} Q_1, q_2 = X_{k-1} Q_2$. Evidently, we can attach to the set $q_1 \{q_2\}$ a set $\bar{q}_1 \{\bar{q}_2\}$ corresponding in virtue of π_{k-1} to the set $q_1 \{q_2\}$. Let $(a, b) \{(A, B)\}$ be any contiguous interval of the set $q_1 \{q_2\}$ and let $(\bar{a}, \bar{b}) \{(\bar{A}, \bar{B})\}$ be the contiguous interval of the set $\bar{q}_1 \{\bar{q}_2\}$ corresponding in virtue of π_{k-1} to the interval $(a, b) \{(A, B)\}$. We will construct the perfect everywhere non dense sets \bar{Q}_1, \bar{Q}_2 in such a manner that $[\bar{a}, \bar{b}] \bar{Q}_1, [\bar{A}, \bar{B}] \bar{Q}_2$ are the perfect everywhere non dense sets. But it may happen that the sets $[a, b] Q_1, [A, B] Q_2$ are not perfect. In order to transform the sets $[a, b] Q_1, [A, B] Q_2$ into the perfect sets it is necessary to make the original correction. We must transform this original correction into the complete correction. This gives us the perfect system $G_k Q$. We now establish the correspondence of similitude $\varphi(a, b) \{\varphi(A, B)\} : [a, b] Q_1 \leftrightarrow [\bar{a}, \bar{b}] \bar{Q}_1 \{[A, B] Q_2 \leftrightarrow [\bar{A}, \bar{B}] \bar{Q}_2\}$ where $Q_1 \{Q_2\}$ is the set of the perfect system $G_k Q$ having the same indices as those of the set $\bar{Q}_1 \{\bar{Q}_2\}$. We adjoin to the correspondence π_{k-1} the correspondences $\varphi(a, b) \{\varphi(A, B)\}$ for all the intervals $(a, b) \{(A, B)\}$, then we obtain the new correspondence of similitude π_k .

Second case: the operation G_k brings to the construction of the first set of a pair of class 3. In this case, we repeat the precedent reasoning, but leaving out the letters Q_2, \bar{Q}_2, q_2 and the intervals (A, B) .

Third case: the operation G_k brings to the construction of the second set of a pair of class 3. In this case we repeat the precedent reasoning, but leaving out the letters Q_1, \bar{Q}_1, q_1 and the intervals (a, b) .

It is obvious that the passage from the system $G_{k-1} Q$ to the system $G_k Q$ runs without changing the sets Q of order $\leq h_k - 1$ if h_k is the order of the sets Q_1, Q_2 . [Note. We shall say that the set $Q_{y_r}^{r+s} \{Q_{r+s}^{y_r}\}$ is the set of order $r+s$]. This means that

$$G_{k-1} Q_{y_r}^{r+s} = G_k Q_{y_r}^{r+s}, \quad G_{k-1} Q_{r+s}^{y_r} = G_k Q_{r+s}^{y_r}$$

(4) We shall say that the sets Q_q^p and \bar{Q}_q^p have the same indices p, q .

for $r+s \leq h_k-1$ where

$$G_t Q_{y_r}^{r+s}, \quad G_t Q_{r+s}^{y_r}$$

are the sets of the system $G_t Q$. In pursuance of this by continuation of the procedure \bar{Q} we obtain the *final perfect system* \dot{Q}

$$(\dot{Q}) \quad \dot{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots).$$

Describing above the general scheme of the step S_k of the procedure \bar{Q} we have omitted the details of the construction of the sets \bar{Q} . We will now complete this void space.

Here are the details.

First case: the considered pair $[\bar{Q}_{z_1}^n, \bar{Q}_n^{z_n}]$ is of class 1. Denote by ψ_{n-1} the correspondence by similitude: $x \leftrightarrow t$ between the points t of the formed sets \bar{Q} of order $n-1$ and the points x of the corresponding sets Q of the perfect system which is obtained after the precedent corrections and which will be denoted by $G_{k-1} Q$ so that $\psi_{n-1} = \pi_{k-1}$, $Q_1 = Q_{z_1}^n$, $Q_2 = Q_n^{z_n}$ where $Q_{z_1}^n$, $Q_n^{z_n}$ are the sets of the perfect system $G_{k-1} Q$. By applying the general scheme of the step S_k we determine the sets q_1, q_2 and the contiguous intervals $(a, b) \{(A, B)\}$ of the set $q_1 \{q_2\}$. Let $\bar{q}_1 \{\bar{q}_2\}$ be the set corresponding to the set $q_1 \{q_2\}$ in virtue of π_{k-1} and let $(\bar{a}, \bar{b}) \{(\bar{A}, \bar{B})\}$ be the contiguous interval of $\bar{q}_1 \{\bar{q}_2\}$ corresponding to the interval $(a, b) \{(A, B)\}$ in virtue of π_{k-1} . Now we construct the perfect everywhere non dense set $P[\bar{a}, \bar{b}] \{P[\bar{A}, \bar{B}]\}$ contained in the segment $[\bar{a}, \bar{b}] \{[\bar{A}, \bar{B}]\}$ and containing the points $\bar{a}, \bar{b} \{\bar{A}, \bar{B}\}$. After that we adjoin to the set $\bar{q}_1 \{\bar{q}_2\}$ the sets $P[\bar{a}, \bar{b}] \{P[\bar{A}, \bar{B}]\}$ for all the intervals $[\bar{a}, \bar{b}] \{[\bar{A}, \bar{B}]\}$, we obtain then the set

$$\bar{Q}_1 = \bar{Q}_{z_1}^n \{\bar{Q}_2 = \bar{Q}_n^{z_n}\}.$$

Second case: the considered pair

$$[\bar{Q}_{z_1}^n, \bar{Q}_n^{z_{n+1-r}}]$$

is of class 2. We assume that the sets of this pair are constructed by the operation G_k . In order to make use of the general scheme we shall put

$$Q_1 = Q_{z_1}^n, \quad Q_2 = Q_n^{z_{n+1-r}}$$

where

$$Q_{z_1}^n, \quad Q_n^{z_{n+1-r}}$$

are the sets of the perfect system $G_{k-1}Q$ and we define the sets $q_1 \{q_2\}$ as it was shown in the general scheme of the step S_k . Let $\bar{q}_1 \{\bar{q}_2\}$ be the set corresponding to the set $q_1 \{q_2\}$ in virtue of π_{k-1} .

We form the sets

$$\bar{R}_{z_r^n} = \bar{q}_1 + \bar{Q}_{z_{r-1}^n}, \quad \bar{R}_{n^{n+1-r}} = \bar{q}_2 + \bar{Q}_n^{z_{n+1-r}}$$

It is easily seen that

$$\bar{R}_{z_r^n} \cdot \bar{R}_{n^{n+1-r}} = 0.$$

We construct with the lemma 3 the perfect everywhere non dense set $Q_{z_r^n} \{Q_n^{z_{n+1-r}}\}$ such

(i) every point of the set $\bar{R}_{z_r^n} \{\bar{R}_{n^{n+1-r}}\}$ is a density (ϱ_n, Δ_n) point of the set

$$\bar{Q}_{z_r^n} \{\bar{Q}_n^{z_{n+1-r}}\};$$

(ii) $\bar{Q}_{z_r^n} \cdot \bar{Q}_n^{z_{n+1-r}} = 0.$

Third case : the considered pair $[\bar{Q}_{z_1^n}, \bar{Q}_n^{z_{n+1-r}}]$ is of class 3. In order to make use of the general scheme we shall put

$$Q_1 = Q_{z_r^n} \{Q_2 = Q_n^{z_{n+1-r}}\}.$$

Next we determine the sets \bar{q}_1, \bar{q}_2 so as it was shown in the general scheme of the step S_k . After that we form the set

$$\bar{R}_{z_r^n} = \bar{q}_1 + \bar{Q}_{z_{r-1}^n} \{\bar{R}_{n^{n+1-r}} = \bar{q}_2 + \bar{Q}_n^{z_{n+1-r+1}}\}$$

and we construct the perfect everywhere non dense set

$$\bar{Q}_{z_r^n} \{\bar{Q}_n^{z_{n+1-r}}\}$$

such that every point of the set $\bar{R}_{z_r^n} \{\bar{R}_{n^{n+1-r}}\}$ is a density (ϱ_n, Δ_n) point of the set $\bar{Q}_{z_r^n} \{\bar{Q}_n^{z_{n+1-r}}\}$.

We shall construct all the sets \bar{Q} of class 3 keeping to this model except for the set $\bar{Q}_n^{z_1^n}$ which must satisfy the following complementary condition :

the length of the greater of the contiguous intervals of the set $\bar{Q}_{z_n^n} + \bar{Q}_n^{z_1^n}$ contained in $[0, 1]$ is less than $\frac{1}{n^2}$.

Thus, the procedure \bar{Q} of the construction of the sets

$$(\bar{Q}) \quad \bar{Q}_{y_r}^{r+s}, \quad \bar{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots)$$

is entirely determined. The principal result of the precedent discussion may be stated in the form of the following:

THEOREM 2. - If $f(x)$ is a finite function of class 1 having the property of DARBOUX in $[0 \leq x \leq 1]$, then there exists a *perfect system* \bar{Q} of sets

$$(\dot{Q}) \quad \dot{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots)$$

contained in $[0 \leq x \leq 1]$ and a system \bar{Q} of perfect everywhere non dense sets

$$(\bar{Q}) \quad \bar{Q}_{y_r}^{r+s}, \quad \bar{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots)$$

contained in $[0 \leq t \leq 1]$ which enjoys the following properties:

(i) there exists a correspondence *CS* of similitude

$$\dot{Q}_{y_r}^{r+s} \leftrightarrow \bar{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r} \leftrightarrow \bar{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots)$$

between the sets of the system \bar{Q} and the sets of the system \dot{Q} ;

(ii) the set

$$\lim_{n \rightarrow \infty} \sum_{s=1}^{s=n} \bar{Q}_{y_s}^n \left\{ \lim_{n \rightarrow \infty} \sum_{s=1}^{s=n} \bar{Q}_n^{y_s} \right\}$$

is dense in the segment $[0 \leq t \leq 1]$;

(iii) if $y_r < y_t$ and if M is the greater of the integers r and t , then every point of the set

$$\bar{Q}_{y_r}^{M+s} \left\{ \bar{Q}_{M+s}^{y_t} \right\}$$

is a density (ρ_{M+s}, Δ_{M+s}) point of the set

$$\bar{Q}_{y_t}^{M+s} \left\{ \bar{Q}_{M+s}^{y_r} \right\}$$

for all $s=1, 2, 3, \dots$;

$$(iiii) \quad m[C(\bar{Q}_Y^n, \bar{Q}_n^y)] \leq \alpha_n$$

where α_n is a positive number such that the series

$$\alpha_1 + \alpha_2 + \alpha_3 + \dots$$

is convergent and $Y\{y\}$ is the greatest {the least} of the numbers $y_1, y_2, y_3, \dots, y_n$.

4. - Properties of the functions $\psi_n(t)$. Let

$$x = \psi_n(t), \quad [\psi_n(0) = 0, \psi_n(1) = 1]$$

be the continuous and essentially increasing function which transforms the sets

$$\dot{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots)$$

respectively into the sets

$$\bar{Q}_{y_r}^{r+s}, \quad \bar{Q}_{r+s}^{y_r}, \quad (s=0, 1, 2, 3, \dots; r=1, 2, 3, \dots).$$

It is obvious that this function $x=\psi_n(t)$ express the correspondence of the similitude ψ_n . This function $\psi_n(t)$ enjoys the following properties.

First property: this function transforms the sets

$$\bar{Q}_n^r, \quad \bar{Q}_n^{y_r}, \quad (r=1, 2, 3, \dots, n; n=1, 2, 3, \dots)$$

respectively into the sets

$$\dot{Q}_n^r, \quad \dot{Q}_n^{y_r}, \quad (r=1, 2, 3, \dots; n=1, 2, 3, \dots).$$

Second property. Let

$$Q_m = \sum_{s=1}^{s=m} (Q_{y_s}^m + Q_m^{y_s}), \quad \bar{Q}_m = \sum_{s=1}^{s=m} (\bar{Q}_{y_s}^m + \bar{Q}_m^{y_s}).$$

Then, if t_0 is any point of Q_p and if $p < q$, we have

$$\psi_q(t_0) = \psi_p(t_0).$$

Third property: the sequence of the functions

$$\psi_1(t), \quad \psi_2(t), \quad \psi_3(t), \dots$$

is uniformly convergent. In reality, let t be any point which do not belong to the set \bar{Q}_p , consequently, t will belong to a contiguous interval $i_t = (J_1, J_2)$ of the set \bar{Q}_p . The functions $\psi_p(t)$ and $\psi_q(t)$ are continuous and essentially increasing, therefore

$$\psi_p(J_1) < \psi_p(t) < \psi_p(J_2), \quad \psi_q(J_1) < \psi_q(t) < \psi_q(J_2).$$

Since $\psi_p(J_1) = \psi_q(J_1)$, $\psi_p(J_2) = \psi_q(J_2)$, if $p < q$, we have

$$|\psi_q(t) - \psi_p(t)| < |\psi_p(J_2) - \psi_p(J_1)|.$$

But $(\psi_p(J_1), \psi_p(J_2))$ is a contiguous interval of the set Q_p corresponding in virtue of $x=\psi_p(t)$ to the contiguous interval (J_1, J_2) of the set \bar{Q}_p . Let l_p be the length of the greatest of the contiguous intervals of Q_p , then we have

$$(l) \quad |\psi_q(t) - \psi_p(t)| < l_p$$

for every point t of the segment $[0 \leq t \leq 1]$, for we have $|\psi_q(t) - \psi_p(t)| = 0$

for every point t of \bar{Q}_p . The set $\lim_{p \rightarrow \infty} Q_p$ is dense in $[0 \leq x \leq 1]$, therefore $\lim_{p \rightarrow \infty} l_p = 0$.

Let ε be an arbitrary, positive, small number. Then we can find a positive integer ν_ε such that $l_p < \varepsilon$ for $p > \nu_\varepsilon$. Thus,

$$|\psi_q(t) - \psi_p(t)| < \varepsilon \quad (l)$$

in all cases when $q > p > \nu_\varepsilon$, consequently, the sequence $\psi_1(t), \psi_2(t), \psi_3(t), \dots$ is convergent. Let $\psi(t) = \lim_{n \rightarrow \infty} \psi_n(t)$. From the inequality (l) we deduce

$$|\psi(t) - \psi_p(t)| < l_p.$$

Since $\lim_{p \rightarrow \infty} l_p = 0$ we conclude that the sequence

$$\psi_1(t), \psi_2(t), \psi_3(t), \dots$$

is uniformly convergent, consequently, the function $\psi(t)$ is continuous in $[0 \leq t \leq 1]$. Since $\psi_n(0) = 0, \psi_n(1) = 1$, we have $\psi(0) = 0, \psi(1) = 1$.

Properties of the function $\psi(t)$.

The first property: if a point t_0 belong to the set \bar{Q}_ν , then $\psi_\nu(t_0) = \psi(t_0)$. In reality,

$$\psi_\nu(t_0) = \psi_{\nu+1}(t_0) = \psi_{\nu+2}(t_0) = \dots = \psi(t_0).$$

The second property: $\psi(t)$ is an essentially increasing function in $[0 \leq t \leq 1]$.

Proof. - In fact, let us suppose $t_1 < t_2$. Then $\psi_n(t_1) < \psi_n(t_2)$, no matter what n is, consequently, $\lim_{n \rightarrow \infty} \psi_n(t_1) \leq \lim_{n \rightarrow \infty} \psi_n(t_2)$, or $\psi(t_1) \leq \psi(t_2)$. Since the set \bar{Q}_n is dense in $[0 \leq t \leq 1]$, we can find an integer ν such that the interval (t_1, t_2) contains at least two points J_1 and $J_2, J_1 < J_2$, of the set \bar{Q}_ν , consequently, $t_1 < J_1 < J_2 < t_2$. Thence we deduce $\psi_\nu(t_1) < \psi_\nu(J_1) < \psi_\nu(J_2) < \psi_\nu(t_2)$. But $\psi_\nu(J_1) = \psi(J_1), \psi_\nu(J_2) = \psi(J_2)$, therefore $\psi_\nu(t_1) < \psi(J_1) < \psi(J_2) < \psi_\nu(t_2)$.

Passing to the limit we obtain

$$\psi(t_1) \leq \psi(J_1) < \psi(J_2) \leq \psi(t_2)$$

from which it results $\psi(t_1) < \psi(t_2)$.

Third property: the function $f[\psi(t)]$ is an ordinary derivative in $[0 \leq t \leq 1]$.

Proof. - At first we construct for $f(x)$ any characteristic system P of perfect sets

$$(P) \quad P_{y_r}^n, \quad P_n^{y_r}, \quad [r=1, 2, 3, \dots, n; n=1, 2, 3, \dots].$$

In the second place, we pass from this system P to a perfect system Q of sets

$$(Q) \quad Q_{y_r}^n, \quad Q_n^{y_r}, \quad [r=1, 2, 3, \dots, n; n=1, 2, 3, \dots].$$

In the third place, we construct using the theorem 2 a final perfect system \dot{Q} of the sets

$$(\dot{Q}) \quad \dot{Q}_{y_r}^n, \quad \dot{Q}_n^{y_r}, \quad [r=1, 2, 3, \dots, n; n=1, 2, 3, \dots]$$

and simultaneously the system \bar{Q} of perfect everywhere non dense sets

$$(\bar{Q}) \quad \bar{Q}_{y_r}^n, \quad \bar{Q}_n^{y_r}$$

satisfying all the conditions of the theorem 2.

In the fourth place, we form the system \dot{P} of the sets

$$(\dot{P}) \quad \dot{P}_{y_r}^n, \quad \dot{P}_n^{y_r}, \quad [r=1, 2, 3, \dots, n; n=1, 2, 3, \dots],$$

where

$$\dot{P}_{y_r}^n = P_{y_r}^n + \dot{Q}_{y_r}^n, \quad \dot{P}_n^{y_r} = P_n^{y_r} + \dot{Q}_n^{y_r}.$$

It is easily seen that the system \dot{P} is also a characteristic system of the sets for $f(x)$.

We now assume that the function $x=\psi(t)$ transforms the sets

$$(\dot{P}) \quad \dot{P}_{y_r}^n, \quad \dot{P}_n^{y_r}, \quad [r=1, 2, 3, \dots, n; n=1, 2, 3, 4, \dots]$$

respectively into the sets

$$(\bar{P}) \quad \bar{P}_{y_r}^n, \quad \bar{P}_n^{y_r}.$$

Let \bar{a} be any point of the set $\bar{P}_{y_r}^n \{ \bar{P}_n^{y_t} \}$ ($y_r < y_t; n \geq r, n \geq t$) and let $a = \psi(\bar{a})$. Now we consider the following cases.

First case: the point a belong to the set

$$\dot{Q}_{y_r}^n \{ \dot{Q}_n^{y_t} \}.$$

In this case the point \bar{a} belongs to the set

$$\bar{Q}_{y_r}^n \{ \bar{Q}_n^{y_t} \}.$$

The set $\dot{Q}_{y_t}^n \{ \dot{Q}_n^{y_r} \}$ is contained in $\dot{P}_{y_t}^n \{ \dot{P}_n^{y_r} \}$, therefore the set $\bar{Q}_{y_t}^n \{ \bar{Q}_n^{y_r} \}$ is contained in the set $\bar{P}_{y_t}^n \{ \bar{P}_n^{y_r} \}$, consequently, the point \bar{a} being a density (ϱ_n, Δ_n) point of the set $\bar{Q}_{y_t}^n \{ \bar{Q}_n^{y_r} \}$ is a density (ϱ_n, Δ_n) point of the set $\bar{P}_{y_t}^n \{ \bar{P}_n^{y_r} \}$.

Second case: the point a does not belong to the set $\dot{Q}_{y_r}^n \{ \dot{Q}_n^{y_t} \}$, consequently, the point a belongs to a contiguous interval i_a of this set. On the other hand, the point a belong to the set $\dot{P}_{y_r}^n \{ \dot{P}_n^{y_t} \}$ and does not belong to the set $\dot{Q}_{y_r}^n \{ \dot{Q}_n^{y_t} \}$, consequently, the point a belong to the set $P_{y_r}^n \{ P_n^{y_t} \}$, for

$$\dot{P}_{y_r}^n = P_{y_r}^n + \dot{Q}_{y_r}^n \{ \dot{P}_n^{y_t} = P_n^{y_t} + \dot{Q}_n^{y_t} \}.$$

Finally, the point a does not belong to the set

$$Q_{y_r}^n \{ Q_n^{y_t} \},$$

