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CONCERNING THE SEMI-CONTINUITY OF ORDINARY INTEGRALS OF THE CALCULUS OF VARIATIONS

by EDWARD JAMES MCSHANE (Princeton).

The present note is an addendum to my paper entitled: *Existence Theorems for Ordinary Problems of the Calculus of Variations* (1), and the notations and definitions of that paper will be used here. We are concerned with an ordinary integrand $F(x, y, y')$ and its associated parametric integrand, defined for $x' > 0$ by the equation $G(x, y, x', y') = x' F(x, y, y'/x')$ and for $x' = 0$ by a passage to the limit $x' \to 0$. As before, we define

$$I[y] = \int_{x_0}^{x_1} F(x, y, y')dx, \qquad J[C] = \int_C G(x, y, x', y')ds.$$

One of the most important theorems of E. T. is theorem 6.1, which is concerned with the semi-continuity of $J[C]$. The proof of this theorem involved a rather intricate construction. The purpose of the present note is to give another proof which is both shorter and simpler. We therefore assume as in E. T. that $F(x, y, y')$ is continuous together with its partial derivatives as to $y^{1'}$ and $y^{2'}$ for all (x, y) on a closed set A and all y'. Also, we define $\bar{K}(M)$ to be the class of all curves C: $x = x(s)$, $y = y(s)$ lying in A, having length $\leqq M$, and such that $x'(s) \geqq 0$ wherever it is defined. We can then state

THEOREM 6.1. - *If $I[y]$ is positive quasi-regular on A, then $J[C]$ is lower semi-continuous on the class $\bar{K}(M)$.*

Let us first observe that by use of lemma 4.2 of E. T. we find exactly as in the proof of theorem 4.1 that it is sufficient to prove the following: If the curves $C_0, C_1, \ldots,$ of the class $\bar{K}(M)$ have the respective representations (2) $z^i = z_0^i(t)$, $z^i = z_1^i(t), \ldots,$ $0 \leqq t \leqq 1$, where all the functions $z_n^i(t)$ satisfy the same LIPSCHITZ condition (say of constant Q) and $z_n^i(t)$ converge uniformly to $z_0^i(t)$, then the inequality

$$\liminf \int_0^1 G(z, z_n')dt \geqq \int_0^1 G(z_0, z_0')dt \tag{1}$$

(1) Annali di Pisa, Vol. III (1934), pp. 183-211. This paper we henceforth cite as E. T.

(2) We recall that the triple $z = (z^0, z^1, z^2)$ is another symbol for the triple (x, y^1, y^2).

holds. Letting h be any positive number, we write the identity

$$
\begin{aligned}
(2)\qquad \int_0^1 G(x_n, y_n, x_n'+h, y_n')dt - \int_0^1 G(x_0, y_0, x_0'+h, y_0')dt = \\
= \int_0^1 [G(x_n, y_n, x_n'+h, y_n') - G(x_0, y_0, x_n'+h, y_n')]dt + \\
+ \int_0^1 E(x_0, y_0, x_0'+h, y_0', x_n'+h, y_n')dt + \\
+ \int_0^1 (x_n'-x_0')G_0(x_0, y_0, x_0'+h, y_0')dt + \\
+ \int_0^1 (y_n^{\alpha'}-y_0^{\alpha'})G_\alpha(x_0, y_0, x_0'+h, y_0')dt.
\end{aligned}
$$

The arguments of the functions G, G_i are contained in the bounded (3) closed set $[(x, y)$ in A, $h \leqq x' \leqq Q+h$, $|y^{i'}| \leqq Q]$, and on this set the functions are continuous. Hence as in the proof of Lemma (4.1), the first, third and fourth integrals on the right tend to 0; and since $E \geqq 0$, we find that for every $h > 0$ we have

$$(3)\qquad \liminf \int_0^1 G(x_n, y_n, x_n'+h, y_n')dt \geqq \int_0^1 G(x_0, y_0, x_0'+h, y_0')dt.$$

Suppose now that inequality (1) does not hold: there then exist numbers $\delta > 0$, H such that

$$(4)\qquad \liminf \int_0^1 G(z_n, z_n')dt + \delta < H < \int_0^1 G(z_0, z_0')dt.$$

By lemma 2.2, G_0 is upper semi-continuous for z on A and $z^{0'} \geqq 0$; hence it is bounded above, say $G_0 \leqq N$, where $N > 0$. Consequently

$$
\begin{aligned}
G(x_n, y_n, x_n'+h, y_n') = G(x_n, y_n, x_n', y_n') + \\
+ hG_0(x_n, y_n, x_n'+\theta h, y_n') \leqq G(x_n, y_n, x_n', y_n') + Nh.
\end{aligned}
$$

Hence for every $h < \delta/N$ we have

$$(5)\qquad \int_0^1 G(x_n, y_n, x_n'+h, y_n')dt < \int_0^1 G(x_n, y_n, x_n', y_n')dt + \delta.$$

On the other hand, by lemma 2.3 of E. T., G is bounded below for the arguments $0 \leqq x' \leqq Q+1$, $|y^{i'}| \leqq Q$, (x, y) on A; and for every x, y, y' we have

$$\lim_{h \to 0} G(x, y, x'+h, y') = G(x, y, x', y').$$

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(3) There is no loss of generality in assuming A bounded, for all the curves C_0, $C_1, \ldots,$ lie in a bounded portion of A.

(If $x'>0$, this follows from the continuity of G; if $x'=0$, it is the equation which defines $G(x, y, 0, y')$). Hence, using (4), we find that there exists an $h<\delta/N$ such that

(6) $$\int_0^1 G(x_0, y_0, x_0'+h, y_0')dt>H.$$

Combining (4), (5) and (6) we have

$$\liminf \int_0^1 G(x_n, y_n, x_n'+h, y_n')dt<H<\int_0^1 G(x_0, y_0, x_0'+h, y_0')dt.$$

This contradicts inequality (3), and the theorem is established.

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