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ON FUNCTIONS OF RECTANGLES AND THEIR APPLICATION TO ANALYTIC FUNCTIONS

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1. - BESICOVITCH has recently proved the following generalizations of the OSGOOD and the RIEMANN theorems [1].

A) If a function $f(z)$ of a complex variable, defined in an open simply connected domain D, is known to be continuous at all points of D and to be differentiable at all points, except, possibly, at the points of a set E of finite or of enumerably infinite linear measure [2], then $f(z)$ is also differentiable at the points of E, and thus is holomorphic in the domain D.

B) If a function $f(z)$ of a complex variable, defined in an open simply connected domain D is known to be bounded in the domain and to be differentiable (i. e. to have a finite derivative) at all points of the domain, except, possibly, at the points of a set E of linear measure zero, then, for every point a of E, the limit of $f(z)$, as z tends to a through values of $D-E$, exists, and the function $f(z)$, defined at the points of E by the values of these limits, is also differentiable at the points of E and thus is holomorphic in the domain D.

In this paper we intend to give theorems *A*) and *B*) in a more abstract form, viz. in a form of theorems on additive functions of rectangles. In the case $\alpha=0$, where α denotes the order of length (see below), the exceptional sets considered in Theorems 5.1 and 5.2 become enumerable and we refind the well known theorems of LEBESGUE and DE LA VALLÉE POUSSIN. In the case $\alpha=1$ we obtain the theorems of BESICOVITCH in a slightly more general form.

2. - The *Lebesgue measure* and the *diameter* of a point set E will be denoted respectively by $|E|$ and $\delta(E)$. Given an enumerable family of sets $\mathbf{E}=\{E_i\}$ and a number $\alpha \geqslant 0$, we shall put

$$\delta_\alpha(\mathbf{E})=\delta_\alpha(\{E_i\})=\underset{i=1,2,\ldots}{\text{upper bound}}\,[\delta(E_i)]^\alpha$$

$$\sigma_\alpha(\mathbf{E})=\sigma_\alpha(\{E_i\})=\sum_i [\delta(E_i)]^\alpha.$$

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(1) BESICOVITCH [1], Theorems 2, 1. Cf. also TONELLI [1].

(2) A set E is said to be of enumerably infinite linear measure if it can be split into an enumerable set of sets of finite linear measure.

Let E be a point set. By $\Lambda_a^{(n)}(E)$ we denote the lower bound of all numbers $\sigma_a(\mathfrak{C})$, where $\mathfrak{C}$ is an arbitrary family of circles covering E and satisfying the condition $\delta_a(\mathfrak{C}) < n^{-1}$. The limit $\Lambda_a(E) = \lim_{n\to\infty} \Lambda_a^{(n)}(E)$ will be called the *length of order* a of the set E (3). In the sequel it will be generally assumed that $0 \leqslant a \leqslant 2$. It will be readily seen that $\Lambda_2(E) = 4\pi^{-1}|E|$.

A set E that is the sum of a sequence of sets of finite length of order a is said to be of *enumerably infinite length of order* a (4). For the sake of brievity we shall term such sets the B_a-sets. B_0-sets coincide, obviously, with the enumerable ones. In the case $a=1$ the expression « of order a » and the index a in the above notation will be usually omitted.

3. - We shall only consider the rectangles and squares with sides parallel to the axis. A function $F(I)$ of rectangles is said to be additive if $F(I_1+I_2) = F(I_1)+F(I_2)$ for any pair of adjacent rectangles. It is said to be continuous if $F(I)\to 0$ whenever $\delta(I)\to 0$.

Let now $F(I)$ be a function of rectangles and x an arbitrary point. Consider the four expressions

$$\overline{F}(x) = \overline{\lim_{\delta(S)\to 0}} \frac{F(S)}{|S|}, \qquad \underline{F}(x) = \underline{\lim_{\delta(S)\to 0}} \frac{F(S)}{|S|}$$

$$\overline{F}_a(x) = \overline{\lim_{\delta(S)\to 0}} \frac{F(S)}{[\delta(S)]^a}, \qquad \underline{F}_a = \underline{\lim_{\delta(S)\to 0}} \frac{F(S)}{[\delta(S)]^a},$$

S denoting an arbitrary square containing x. The numbers $\overline{F}(x)$ and $\underline{F}(x)$ are called respectively the *upper* and the *lower derivatives* of $F(x)$ at the point x. When $\overline{F}(x) = \underline{F}(x)$ we shall call this common value the *differential coefficient* of $F(x)$ at the point x and shall denote it by $F'(x)$.

We shall say $F(I)$ has the property (L_a^+) in a rectangle I_0 if $\underline{F}_a(x) > -\infty$ everywhere in I_0; if, moreover, $\underline{F}_a(x) \geqslant 0$ everywhere in I_0, we shall say that $F(I)$ has the property (l_a^+). The analogous properties (L_a^-) and (l_a^-) correspond respectively to the inequalities $\overline{F}_a(x) < +\infty$, $\overline{F}_a(x) \leqslant 0$. Finally, if a function has both the properties (l_a^+) and (l_a^-) (respectively (L_a^+) and (L_a^-)) it will be said to have the property (l_a) (respectively (L_a)).

4. - $\mathfrak{P}_n$ will denote the n-th *net* on the plan, i. e. the enumerable set of squares into which the plan is divided by the two systems of parallel lines

$$x = k\cdot 2^{-n}, \qquad y = k\cdot 2^{-n} \qquad (k=0, \pm 1, \pm 2, \ldots).$$

The squares belonging to $\mathfrak{P}_n$ will be called *meshes* of order n.

(3) See HAUSDORFF [1]; HAHN [1], pp. 459-461.

(4) See footnote (1).

LEMMA 4.1. - *Given a set E and non negative numbers N, $\varepsilon>0$, $\alpha\leqslant 2$, there exists a sequence $\mathfrak{S}=\{S_n\}$ of meshes of order $>N$, which satisfy the following conditions:*

(i) $$\sigma_\alpha(\mathfrak{S})\leqslant 32[\Lambda_\alpha(E)+\varepsilon],$$

(ii) *to any point x of E there corresponds an integer $n>0$, such that any mesh of order n that contains* ([5]) *x belongs to $\mathfrak{S}$.*

Proof. - Let $\mathfrak{C}=\{C_i\}$ be a sequence of circles such that

$$(4.1)\qquad E\subset\sum_{i=1}^{\infty}C_i,\qquad \delta(\mathfrak{C})<2^{-N-1}\quad\text{and}\quad \sigma_\alpha(\mathfrak{C})\leqslant\Lambda_\alpha(E)+\varepsilon.$$

Let, for every i, N_i denote the posive integer such that

$$(4.2)\qquad 2^{-N_i}>\delta(C_i)\geqslant 2^{-N_i-1}.$$

It is easily seen that there exist at most four meshes of order N_i that have points in common with C_i. Let $\mathfrak{S}$ be the set of all meshes of orders $N_1, N_2,...., N_i,....$ that have points in common respectively with the circles $C_1, C_2,...., C_i,....$ The set $\mathfrak{S}$ obviously satisfies the condition (ii). Next, it follows from (4.1) and (4.2) that

$$\sigma_\alpha(\mathfrak{S})\leqslant 4\cdot 2^{\frac{\alpha}{2}}\sum_{i=1}^{\infty}2^{-\alpha N_i}\leqslant 4\cdot 2^{\frac{3\alpha}{2}}\sum_{i=1}^{\infty}[\delta(C_i)]^\alpha\leqslant 32\sigma_\alpha(\mathfrak{C})\leqslant 32[\Lambda_\alpha(E)+\varepsilon]$$

and so the condition (i) is also satisfied.

5. - LEMMA 5.1. - *If an additive and continuous function $F(I)$ has the property (l_α^+), where $0\leqslant\alpha\leqslant 2$, in a rectangle I_0, and if $\underline{F}(x)\geqslant 0$ everywhere in I_0, except, perhaps, for x belonging to a B_α*-set *$D\subset I_0$, then $F(I_0)\geqslant 0$.*

Proof. - On account of the continuity of $F(I)$ we may assume that I_0 is a mesh, say of order N_0.

Let ε be an arbitrary positive number and let $G(I)=F(I)+\varepsilon\cdot|I|$. Put

$$D=\sum_i D_i,\quad\text{where}\quad \Lambda_\alpha(D_i)<+\infty\quad\text{for } i=1, 2,....$$

Let $R_{i,n}$ denote the set of points x in I_0 such that

$$(5.1)\qquad G(S)>-\varepsilon\cdot 2^{-i}[\Lambda_\alpha(D_i)+1]^{-1}\cdot[\delta(S)]^\alpha$$

for any mesh S of order $\geqslant n$ containing x. Since $F(I)$ possesses the property (l_α^+), we have

$$I_0=\sum_{n=1}^{\infty}R_{i,n}\quad\text{for any } i=1, 2,....$$

([5]) There exist at most four meshes that have this property.

and therefore, since the sets $R_{i,n}$ are measurable (B) (more exactly, any of them is the product of a sequence of open sets)

$$(5.2) \qquad \Lambda_a(D_i) = \sum_{n=1}^{\infty} \Lambda_a(D_{i,n}),$$

where $D_{i,n} = D_i \cdot (R_{i,n} - R_{i,n-1})$ for $n > 1$, and $D_{i,1} = D_i \cdot R_{i,1}$ $(i = 1, 2, \ldots)$.

Now, by Lemma 4.1, there exists for any pair of positive integers n, i, an enumerable set $\mathfrak{S}_{i,n}$ of meshes of order $> n$ that are contained in I_0 and satisfy the following conditions:

$$(5.3) \qquad \sigma_a(\mathfrak{S}_{i,n}) \leqslant 32\,[\Lambda_a(D_{i,n}) + 2^{-n}],$$

(5.4) for each point x in $D_{i,n}$ there exists an integer $n \geqslant N_0$, such that any mesh of order n containing x belongs to $\mathfrak{S}_{i,n}$,

(5.5) each mesh S that belongs to $\mathfrak{S}_{i,n}$ has common points with $D_{i,n}$, and, consequently, satisfies the inequality (5.1).

Let us put $\mathfrak{S} = \sum_{i,n=1}^{\infty} \mathfrak{S}_{i,n}$. It easily follows from (5.5), (5.3) and (5.2) that for any sequence $\{S_k\}$ of (different) meshes of $\mathfrak{S}$ we have the inequality

$$(5.6) \qquad \sum_{k=1}^{\infty} G(S_k) \geqslant -\varepsilon \cdot \sum_{i=1}^{\infty} \left\{ 2^{-i} [\Lambda_a(D_i) + 1]^{-1} \cdot \sum_{n=1}^{\infty} \sigma_a(\mathfrak{S}_{i,n}) \right\} \geqslant -32\varepsilon.$$

We shall say that a rectangle has the property (A) if it is the sum of a finite number of meshes S, each of which either belongs to $\mathfrak{S}$ or satisfies the inequality $G(S) > 0$. If follows from (5.6) that

$$(5.7) \qquad G(I) \geqslant -32\varepsilon$$

for any rectangle I having the property (A).

We are now going to prove that I_0 has the property (A). In fact, suppose that it does not possess this property. Then, by the well known argument, a decreasing sequence $\{I_0 = S_1, S_2, \ldots, S_k, \ldots\}$ of meshes can be found, so that no S_k has the property (A). Hence, each S_k neither belongs to $\mathfrak{S}$ nor satisfies the inequality $G(S_k) > 0$. However this is impossible, for, if the limiting point x_0 of the sequence $\{S_k\}$ belonged to D, it would follow from (5.4) that at least one S_k belonged to $\mathfrak{S}$; if, on the contrary, $x_0 \varepsilon I_0 - D$, then $\underline{G}(x_0) = \underline{F}(x_0) + \varepsilon > 0$ and, consequently, $G(S_k) > 0$ for all k sufficiently large. Hence I_0 has the property (A) and therefore the inequality (5.7) holds for $I = I_0$. Thus

$$F(I_0) = G(I_0) - \varepsilon\,|\,I_0\,| \geqslant -(32 + |\,I_0\,|)\varepsilon,$$

and, since ε may be chosen arbitrarily small, $F(I_0) \geqslant 0$.

THEOREM 5.1. - *If an additive and continuous function $F(I)$ has the property $(\underline{U}_a^+)$ $(0 \leqslant a < 2)$ in a rectangle I_0, and if the inequality* $-\infty \neq \underline{F}(x) \geqslant \psi(x)$

where $\psi(x)$ is a summable function, holds everywhere in I_0, except, perhaps, on a B_α-set, then, for any rectangle $I \subset I_0$, we have

$$F(I) \geqslant \int_I \psi(x)dx.$$

Proof. - Let $\Psi(I)$ be a minorant ([6]) of $\psi(x)$, i. e. an additive and continuous function of rectangles such that $+\infty \neq \overline{\Psi}(x) \leqslant \psi(x)$ for every x in I_0. From the inequality $\overline{\Psi}(x) \neq +\infty$ it follows that $\Psi(I)$ has the property (l_α^-) and, therefore, the difference $\Delta(I) = F(I) - \Psi(I)$ has the property (l_α^+). Furthermore, everywhere in I_0, except, perhaps, on a B_α-set, we have the inequality

$$\underline{\Delta}(x) \geqslant \underline{F}(x) - \overline{\Psi}(x) \geqslant \underline{F}(x) - \psi(x) \geqslant 0.$$

Hence, by the preceding lemma, $\Delta(I) \geqslant 0$, i. e. $F(I) \geqslant \Psi(I)$. Since the last inequality holds for any minorant $\Psi(I)$ of $\psi(x)$, we have

$$F(I) \geqslant \int_I \psi(x)dx$$

and the theorem is established.

From theorem 5.1 we obtain at once

THEOREM 5.2. - *If an additive and continuous function $F(I)$ has the property (l_α) $(0 \leqslant \alpha < 2)$ in a rectangle I_0 and if both derivatives $\overline{F}(x)$ and $\underline{F}(x)$ are summable over I_0 and finite everywhere in I_0, with the exception at most of a B_α-set, then $F(I)$ is an absolutely continuous function in I_0, and, therefore*

$$F(I) = \int_I F'(x)dx$$

for any rectangle $I \subset I_0$.

6. - Now let $f(z)$ be a (complex) continuous function of a complex variable. For any rectangle I consider the complex integral

$$\int_{(I)} f(z)dz = U(I) + iV(I) \tag{6.1}$$

taken along the boundary (I) of I in the positive sense. The real and imaginary parts, $U(I)$ and $V(I)$, of this integral are both additive and continuous functions of I, and $f(z)$ being continuous, they satisfy the condition (l_1). Next, il is easily seen that $U'(z) = V'(z) = 0$ at any point z at which $f(z)$ has a differential coefficient. Moreover, the derivatives $\overline{U}(z)$, $\underline{U}(z)$, $\overline{V}(z)$, $\underline{V}(z)$ are finite, whenever

$$\overline{\lim_{h \to 0}} \left| \frac{f(z+h) - f(z)}{h} \right| < +\infty.$$

([6]) See DE LA VALLÉE-POUSSIN [1], pp. 74-76.

Hence, using the MORERA theorem, we deduce from Theorem 5.2 that:

If a complex continuous function $f(z)$ is differentiable almost everywhere in an open region R and if $\overline{\lim}_{h\to 0}\left|\frac{f(z+h)-f(z)}{h}\right| < \infty$ *everywhere, except, perhaps, on a set of enumerably infinite length, then $f(z)$ is holomorphic in R.*

7. - Lemma 5.1 as well as Theorems 5.2 and 5.3 hold true if the conditions (l_a^+) and (l_a) are replaced respectively by (L_a^+) and (L_a), provided that the exceptional B_a-sets are simultaneously replaced by *sets of length zero of order a*. The proofs become even simpler. Consequently by the argument similar to that used in § 6, we get the second theorem of BESICOVITCH generalized as follows:

If a complex function $f(z)$, bounded in an open region R, is differentiable almost everywhere in R, and if $\overline{\lim}_{h\to 0}\left|\frac{f(z+h)-f(z)}{h}\right| < \infty$ *everywhere in R, with the possible exception of a set of length zero, then $f(z)$ is equivalent* (i. e. *almost everywhere equal*) *to a function holomorphic in R* (7).

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(7) It should be noticed that from the hypothesis of the theorem it follows that $f(z)$ is measurable on any straight line in R, and consequently the complex integral (6.2) may be considered in the LEBESGUE sense. We also need the following analogue of the MORERA theorem: *if the complex integral* (6.1) *of a bounded and measurable function vanishes for any rectangle I, then $f(z)$ is equivalent to a holomorphic function.* This follows at once from the MORERA theorem by the well known argument of integral means.

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