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The mechanics of the stability of a central orbit

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THE MECHANICS OF THE STABILITY OF A CENTRAL ORBIT

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I.

The astronomical type of orbit and the trajectory of an electron in the atomical realm, may both be studied, in relation to stability, through the circumstance of their perturbations. Especially significant is the automatism of self-restitution under small perturbations, which is a property of these trajectories. The orbital arrangements which form the atom of any chemical element, under typical conditions, undergo small disturbances, with the result that matter can pass through a great variety of modifications while the atoms themselves retain their essential identity through this power of restitution. A. BERTHOUD of Neuchâtel discourses (1) as follows upon this property: « The principle of conservation of the elements is thus nothing more than the expression of the fact that, in chemical phenomena, these modifications of the atom are never sufficiently radical to cause a complete upset in the balance of the (orbital) system which always returns to its original state as soon as the primary conditions are re-established ».

Postulates of the property of self-restitution under perturbations, of an astronomical orbit, form the foundation of this paper. Its problem involves the question of the determination of all potential functions for which central orbits are stable, and applies to electronical orbits which satisfy BOHR'S laws.

Definition: A continuous central orbit, (represented by one or by two polar equations), is stable if the rotating body is maintained upon it permanently, by the potential.

II.

Consecutive Curves and Interpolation.

Let C be any branch of a continuous space curve, $\{r=f_1(\varphi),\ \theta=g_1(\varphi)\}$, which is an orbit of a material particle and consider any curve C' consecutive to C according to the following definition (2). Instead of the two curves joining A

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(1) A. BERTHOUD: *The New Theories of Matter and the Atom* (Translated); HERBERT SPENCER: *First Principles* (4 ed.), p. 501.

(2) DIENGER: *Grundriss der Variationsrechnung*, p. 1.

and B take two broken lines, one being a system of chords, a_1a_2, a_2a_3,...., of C, and the other a system of segments b_1b_2, b_2b_3,...., choosing a_i, b_i, as corresponding vertices. Assume all distances a_ib_i less than δ with δ arbitrarily small. Suppose that all segments a_ia_{i+1} approach zero, their number in the meantime increasing indefinitely so that there is transition to the curve C. Simultaneously then b_1b_2, b_2b_3,.... can be made to approach as a limiting position a curve C' such that the distance between any two corresponding points on the two respective curves is less than δ.

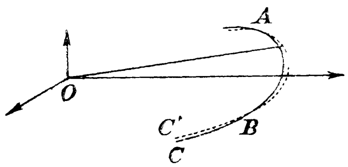

Fig. 1.

III.

The Perturbations of an Astral Orbit.

A planet in solitary rotation about a sun whose gravitational attraction is newtonian would describe a plane elliptical orbit. Astronomical suns, however, are in fact attended by planets in groups, some being moons. The mutual attractions of the planets within a group perturb the orbits of all so that the actual trajectories, constantly shifting in all three dimensions by small amounts represented by δ of § II, exhibit perturbations from truly elliptical forms. In addition account must sometimes be taken of orbital alterations due to relativity and inequalities due to small changes in the central potential.

The circumstances relating to electronical orbits are analogous.

Definition: Consider the problem of two bodies where a heavenly body subject to small perturbations, is otherwise in stable rotation around another, that is, around a center of force. The totality of finite orbital segments which pass near a given segment S of the totality in a chosen time interval, is the field F of perturbed orbits in the vicinity of S: (By definition).

Whether the stable orbit is a closed contour about the center of force, or not, we can waive the question of the actual mechanical description of all of the curves of F and imagine as drawn near a segment of the stable orbit a field of perturbed orbits any one of which could be traversed by the body if it were properly perturbed as it passes through the field.

Except where further stipulations are added the following are our hypotheses.

Postulate 1. - We assume that the potential due to the center of force O is a definite function of the radius r, constant over a small sphere whose center is O.

Postulate 2. - We assume it to be always possible to draw through the field F a segment C, (which may or may not belong to F), consecutive to the segments S in the sense of § II, such that the segments of F would all return to coincidence [3] with C if the perturbative causes were removed.

The postulates 1 and 2 assign to the orbit of reference C the power of self-restitution within the field during the operation of the perturbing forces. The existence of C is to be proved by the determination of its equation.

The manner in which the perturbing influences are assumed to be removed is of importance. In the astronomical problem we suppose the release to be sufficiently gradual that the equivalent variation of the central perturbing function described in § VI is not discontinuous or even sudden, as this would be equivalent to an impact upon the rotating body. This restriction will be found to be inherent in the analysis, (cf. § IX).

If the rotating planet is in a librating plane or if its perihelion is advancing it may be necessary to establish an upper bound to the time interval which delimits the field.

IV.

Restitution by Transformation.

The following construction relates to a three-dimensional field about an orbit of reference C.

Let C' be an arbitrary curve of the field. The coordinates of P, which is any point of C, are $OP=r$, $<XOS=\varphi$, $<SOP=\theta$.

The coordinates of Q are $\{r+\sigma, \varphi, \theta+\tau\}$. We have $RQ=\sigma$, $<QOP=-\tau$, and σ and τ are small. The center of the force is O.

Let the equations of C be $\{\varphi=f(r), \varphi=g(\theta)\}$ and assume as observationally known the coordinates of n points Q, distributed with some regularity along C':

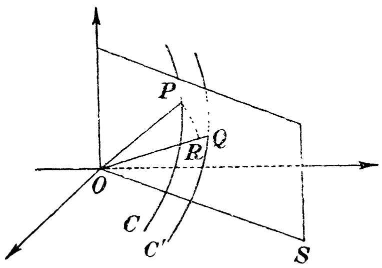

Fig. 2.

$$Q_1;\ (r_1', \varphi_1, \theta_1'),\dots,\quad Q_n;\ (r_n', \varphi_n, \theta_n').$$

A plane SOQ_i drawn through such a point intersects C in a point P_i: $(r_i, \varphi_i, \theta_i)$, where $r_i'=r_i+\sigma_i$, $\theta_i'=\theta_i+\tau_i$.

If (r, σ) are variable coordinates they may be considered to be functionally connected through the n determinations, (r_i, σ_i), $(i=1,\dots, n)$, and the coefficients $\alpha,\dots, \varkappa$, in the function

$$(1)\qquad \sigma=\alpha r^{n-1}+\beta r^{n-2}+\dots+\varkappa=tp(r), \qquad (t\doteq 0),$$

may be assumed to have been calculated by methods current in the theory of curve-fitting, from (r_i, σ_i), $(i=1,\dots, n)$.

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[3] In the sense of analytic approximation.

Likewise, we may write

$$\tau = a'\theta^{n-1} + \beta'\theta^{n-2} + \dots + \varkappa' = sq(\theta), \qquad (s \doteq 0),$$

since (θ, τ) are functionally connected through n determinations (θ_i, τ_i), $(i=1,\dots,n)$, furnished by the observations. The accuracy of the representations (1) evidently depends upon the magnitude of n. The length of the field F is determined by that of the segment C' which contains the n observed points Q_i. The curve C' traverses F as $a,\dots,\varkappa'$, are allowed to vary. If the equations of C' are assumed as $\{\varphi'=\eta(r'),\ \varphi'=\zeta(\theta')\}$, there follows:

THEOREM 1. - *Any curve C' of the field F can be returned to approximative coincidence with the orbit of reference C by a transformation of the type,*

$$(2) \qquad T; \quad \varphi'=\varphi, \quad r'=r+tp(r), \quad \theta'=\theta+sq(\theta), \quad (s, t \doteq 0).$$

The operation TC' therefore represents the quality of self-restitution which is inherent in the orbit of reference as a result of the assumed central potential. We revert to this theme in section VII. The property of self-restitution is a necessary condition for stable astral motion.

V.

The Equations of the Perturbed Orbit.

The assigned equations of C' and the conditions of the problem may be combined to give,

$$\eta(r') = \eta(r+\sigma) = \eta(r) + tp(r)\eta'(r) = f(r), \qquad (\eta' = d\eta/dr),$$
$$\zeta(\theta') = \zeta(\theta+\tau) = \zeta(\theta) + sq(\theta)\zeta'(\theta) = g(\theta), \qquad (\zeta' = d\zeta/d\theta).$$

Hence, by solving the two linear differential equations we determine the equations of C', from those of C assumed as arbitrary. The solutions are,

$$(3) \qquad \eta(r) = e^{-\int \frac{dr}{\pi(r)}} \left[\int e^{\int \frac{dr}{\pi(r)}} f(r)dr/\pi(r) + A \right], \qquad (\pi(r) = tp(r)),$$

$$(4) \qquad \zeta(\theta) = e^{-\int \frac{d\theta}{\varrho(\theta)}} \left[\int e^{\int \frac{d\theta}{\varrho(\theta)}} g(\theta)d\theta/\varrho(\theta) + B \right], \qquad (\varrho(\theta) = sq(\theta)).$$

From these at once the equations of C', in the form, $\varphi'=\eta(r')$, $\varphi'=\zeta(\theta')$, may be written.

If the small numbers s, t are negligible beyond the second powers only, the two differential equations are

$$(5) \qquad \begin{cases} \frac{1}{2} t^2 p(r)^2 \eta'' + tp(r)\eta' + \eta = f(r), & (\eta'' = d^2\eta/dr^2), \\ \frac{1}{2} s^2 q(\theta)^2 \zeta'' + sq(\theta)\zeta' + \zeta = g(\theta), & (\zeta'' = d^2\zeta/d\theta^2). \end{cases}$$

Thus whatever the forces may be which, under the postulates, actuate the motion, the following is the orientation of the figure of the orbit. The locus of $\varphi=\eta(r)$ is an arbitrary surface of which every section by a plane through the z axis is a circle with its center at the origin O. We refer to it as a pseudo-sphere. The surface $\varphi=\zeta(\theta)$ is a cone A with its vertex at O. The intersection of these two surfaces is the orbit C' and C is consecutive to the latter.

The coordinates assumed for points of C are of the heliocentric type.

VI.

The Perturbing Function.

In this section a more special case is assumed in which the cone A is the (r, φ) plane, both orbits C, C' being curves in this plane. The point O being the polar origin, the equation of C may be derived in the well-known form:

$$\frac{\gamma^2}{r^4}\frac{d^2r}{d\varphi^2}-\frac{2\gamma^2}{r^5}\left(\frac{dr}{d\varphi}\right)^2-\frac{\gamma^2}{r^3}=-P(r), \tag{6}$$

P being the force function, and by interchange of the dependent and independent variables this becomes,

$$\frac{\gamma^2}{r^4}\frac{d^2\varphi}{dr^2}\Big/\left(\frac{d\varphi}{dr}\right)^3+\frac{2\gamma^2}{r^5}\left(1\Big/\frac{d\varphi}{dr}\right)^2+\frac{\gamma^2}{r^3}=P. \tag{7}$$

The transformation is now

$$T_0; \qquad \varphi'=\varphi, \qquad r'=r+tp(r), \qquad (t\doteq 0),$$

and the equation (4) does not exist. In the equation of C', viz: $\varphi'=\eta(r')$, it is convenient to drop the primes. The force function P_1 whose trajectory is the arbitrary C' is then obtained by substituting $\eta(r)$ for φ in (7).

We abbreviate as follows:

$$R=e^{-\int dr/\pi}\int e^{\int dr/\pi}f(r)dr/\pi, \qquad S=Ae^{-\int dr/\pi}-f(r). \tag{8}$$

Then from (3),

$$\frac{d\eta}{dr}=-\frac{1}{\pi}(R+S), \tag{9}$$

$$\frac{d^2\eta}{dr^2}=\frac{\pi'+1}{\pi^2}(R+S)+f'(r)/\pi. \tag{10}$$

Hence,

$$M=P_1(r)=\frac{\gamma^2[-(\pi'-2\pi/r+1)\pi(R+S)-f'\pi^2]}{r^4(R+S)^3}+\frac{\gamma^2}{r^3}.$$

Similarly the force P corresponding to C is obtained by replacing φ by f in (7);

$$N=P(r)=\frac{\gamma^2 f''}{r^4 f'^3}+\frac{2\gamma^2}{r^5 f'^2}+\frac{\gamma^2}{r^3}.$$

Hence the perturbing function whose effect, when added to N is to shift C into C' is $L=M-N$ and is equal to

$$\gamma^2 \left\{ \frac{-(\pi' - 2\pi/r + 1)\pi(R+S) - f'\pi^2}{r^4(R+S)^3} - \frac{f''}{r^4 f'^3} - \frac{2}{r^5 f'^2} \right\}. \tag{11}$$

The use of L as a practical formula requires the knowledge of the equation of the orbit of reference in coordinates of the heliocentric type. Note that when our formulary is adapted to planetary perturbations in the solar system it is not assumed that newtonian gravitation is invariably the law of the central force.

Every curve of the field F is the free trajectory of a force function M corresponding to a definite choice of the set $\alpha, \ldots, \varkappa$, and M is obtained by adding the perturbing function L to the given force function N.

VII.

The Orbit of Reference.

The mechanical property of self-restitution will be as well fulfilled if, instead of the curve C' being returned to coincidence with C by the removal of perturbations, it returns to some other curve C_0 which could be brought into coincidence with C by a small translation which does not also bend the curve, (cf. Postulate 2).

Both C and C_0, when plane, are integral curves of (6). In three dimensions they are integral curves of the simultaneous equations ([4]),

$$\left\{ \begin{aligned} & \frac{d\theta}{d\varphi} \sec\theta \cos\varphi + \frac{h}{g} \cos\theta + \sin\theta \sin\varphi = 0, \qquad (g, h, \text{ constant}), \\ & \frac{d}{d\varphi} [r^{-4} \cos^{-4}\theta] \left[\left(\frac{dr}{d\varphi}\right)^2 + r^2 \left(\frac{d\theta}{d\varphi}\right)^2 + r^2 \cos^2\theta \right] = \frac{2}{g^2} P(r, \theta) \frac{dr}{d\varphi}, \end{aligned} \right. \tag{12}$$

where $P(r, \theta)$ is the force function.

We refer to the case where C' returns to C_0 instead of to C as that of an advancing orbit of reference. A planetary orbit whose perihelion advances is an example.

Consider the formal equations obtained by transforming C' by the corresponding T:

$$\left\{ \begin{aligned} \varphi' &= E[f(r'), \pi(r'), dr'] = e^{-\int dr'/\pi(r')} \left[\int e^{\int dr'/\pi(r')} f(r') dr'/\pi(r') + A \right] \\ &= \eta(r') = E[f(r) + \pi(r) f'(r), \pi(r), dr] \equiv f(r) + (A + A') e^{-\int dr/\pi(r)}, \\ \varphi' &= E[g(\theta'), \varrho(\theta'), d\theta'] = E[g(\theta) + \varrho(\theta) g'(\theta), \varrho(\theta), d\theta] \\ &= g(\theta) + (B + B') e^{-\int d\theta/\varrho(\theta)}, \end{aligned} \right. \tag{13}$$

([4]) These were derived by the author. They are probably new.

where A, A', B, B' are arbitrary constants. It follows that C' is returned to C as a particular case, but in general it is transformed into a neighboring curve, which can be C_0, viz:

$$(14)\qquad \begin{cases} \varphi = f(r) + (A+A')e^{-\int dr/\pi(r)}, \\ \varphi = g(\theta) + (B+B')e^{-\int d\theta/\varrho(\theta)}. \end{cases}$$

The rest of this paper treats exclusively of plane motion, the second equation (14) being omitted. Since the determination of the potential may be said to be the principal objective, this is particularization only in appearance, if we assume that the force is uniform with respect to all directions outward from the center.

By writing $f(r)$ as $f(r, a, b)$ where a, b are the constants of integration of (6), the hypothesis that C_0 is (14) gives

$$f(r, a, b) + (A+A')e^{-\int dr/\pi(r)} = f(r, a+\delta, b+\varepsilon), \qquad \delta \doteq 0, \qquad \varepsilon \doteq 0.$$

Therefore a special case of the equation, $\varphi = f(r)$, of C_0, is

$$(15)\qquad \varphi = \lambda e^{-\int dr/\pi(r)} + \mu.$$

Substituting from (15) in (7) we obtain the corresponding special formula for the force:

$$(16)\qquad P(r) = \frac{\gamma^2}{\lambda^2 r^5}\left[\lambda^2 r^2 + (2\pi^2 - \pi(\pi'+1)r)e^{2\int dr/\pi(r)}\right].$$

Proposition: To determine the general formula for $f(r)$. The object in constructing the field F was to study the question of mechanical stability. Let T_0, (cf. § VI), as an expression involving t and p, be written as $T(t,p)$. Suppose K to be a curve C' of F and let t_1 be the value of t for which $T(t_1,p)K = C_0$. If t_2, t_3 are any admissible values of t such that $t_2 + t_3 = t_1$, the curve $T(t_2,p)K = L$, is a curve of the field. For,

$$T(t_3,p)L = T(t_3,p)T(t_2,p)K = T(t_1,p)K = C_0.$$

Hence L is the curve of F, which is returned to C_0 by $T(t_3,p)$.

Thus K belongs to a single parameter system S, containing C_0 and forming a subset of F. The curves of the system are obtainable from K by the transformations $T(t_i, p)$ of the set $[T(t,p)]$ obtained by varying t in $T(t,p)$. The parameter is t, with a limited, though continuous, range of variation and C_0 corresponds to the value $t = t_1$. If S is the system of integral curves of a (first order) differential equation, the latter must be a universal invariant of $[T(t,p)]$: But we can readily determine all of the first order universal invariants. Thus we shall determine a differential equation which C_0 satisfies, identify it with (7) and determine $P(r)$ in its most general form.

VIII.

Universal Invariants.

Keeping in mind the essential fact of the mechanics, that any T_0 operates to return C' to a C_0 whenever perturbative conditions permit T_0 to act, let us determine the universal invariants of the first oder of the set $[T(t,p)]$. We assume the general form, in the primed variables, as $Q(r', \varphi', dr', d\varphi') = 0$. Transforming Q by the typical T_0 of $[T(t,p)]$,

$$(17) \quad Q(r', \varphi', dr', d\varphi') = Q(r + tp, \varphi, dr + tp'dr, d\varphi) = Q(r, \varphi, dr, d\varphi) + tp\frac{\partial Q}{\partial r} + tp'dr\frac{\partial Q}{\partial(dr)} = Q(r, \varphi, dr, d\varphi), \quad (p' = dp/dr).$$

Hence Q is the solution of the linear partial differential equation for which LA GRANGE'S system is

$$(18) \quad \frac{dr}{p(r)} = \frac{d(dr)}{p'(r)dr} = \frac{dQ}{O}.$$

The general solution, from (18), is readily found to be

$$Q = \psi(dr/p(r), \varphi, d\varphi), \quad (\psi \text{ arbitrary}).$$

Hence the general form for the equation of C_0, universally invariant under $[T(t,p)]$, is

$$(19) \quad dr/p(r) = \lambda(\varphi)d\varphi, \quad (\lambda \text{ arbitrary}).$$

The integral is

$$(20) \quad \sigma = \mu(\varphi), \ \text{i. e.} \ \varphi = \Delta(\sigma), \quad \left[\mu(\varphi) = \int \lambda(\varphi)d\varphi,\ \sigma = \int dr/p(r)\right],$$

(Δ arbitrary). Hence, $\quad f(r) = \Delta(\sigma) - (A + A')e^{-\sigma/t}, \quad$ (q. e. f.),

and by substitution of φ from (20) in (7),

$$(21) \quad P(r) = \frac{\gamma^2}{r^3}\left[\frac{2p^2}{r^2(\Delta'(\sigma))^2} + \frac{p\Delta''(\sigma) - pp'\Delta'(\sigma)}{r(\Delta'(\sigma))^3} + 1\right], \quad (\Delta' = d\Delta/d\sigma).$$

In case the rotating body describes a closed contour around the center of force, which is of period $2n\pi$, then, since C_0 coincides with it in F, C_0 is of period $2n\pi$. Since T_0 is identity in φ, all curves of S are periodic and (19) is, — that is, $\lambda(\varphi)$ is a periodic function if the whole orbit is periodic.

LEMMA 1. - *Excluding the use of any field curve C_0 for which the following limit is non-existent or very large:*

$$\lim_{a\to\infty}\left[\frac{d^a\mu}{d\varphi^a}\Big/\frac{d^{a-1}\mu}{d\varphi^{a-1}}\,a\right], \text{ (5)}$$

we can take $\mu(\varphi)$ to be a binomial, linear in φ.

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(5) CAUCHY'S convergence ratio for MACLAURIN'S expansion of $\mu(\varphi)$.

Proof: The field F may be short like the arc used in computations of the elements of a planetary orbit from three observations. Assuming this, let the polar axis be chosen to intersect C_0 in the central region of F. The entire arc C_0 will be traversed by the end of the radius vector (of C_0), as φ varies between narrow limits near to zero. The MACLAURIN expansion of $\mu(\varphi)$ will converge rapidly and $\mu(\varphi)$ may, with accuracy, be replaced by its first two or first three terms. Thus

$$\mu(\varphi)=\mu(0)+[d\mu(\varphi)/d\varphi]_0\varphi\,; \tag{22}$$

also the value zero for $[d\mu/d\varphi]_0$ can be avoided by a small alteration of position of the polar axis. If, for accuracy, $\mu(\varphi)$ is taken to be quadratic in φ the force $P(r)$ in (21) involves $\sqrt{(a\sigma+\text{const.})}$ and is not single-valued, (cf. Postulate 1).

The question, affecting generality, whether we do not pass below an admissible minimum of length for F when requiring the expansion of $\mu(\varphi)$ to reduce to its initial terms, is avoided by assuming the distance from the center of force to the field to be sufficiently large in comparison with the length of F.

Using (20) in the form $\varphi\lambda+\mu=\sigma$, ($\lambda$, μ constant), we obtain, under the assumptions and postulates, (from (7)).

THEOREM 2. - *The orbit of reference of Postulate* 2 *is proved to exist by the determination of its equation. It passes through the field F as an arc of*

$$\int dr/p(r)=\varphi\lambda+\mu, \qquad (\lambda,\ \mu\ \text{constant}). \tag{23}$$

A necessary condition that a central orbit should remain stable after small perturbations is that the central force should be

$$P(r)=\frac{\gamma^2}{r^3}\left[\frac{2\lambda^2p(r)^2}{r^2}-\frac{\lambda^2p(r)p'(r)}{r}+1\right]. \tag{24}$$

The force was determined from a segment of the orbit (within the field). The whole orbit is then obtained by integrating (6). *The formulas* (21), (24) *are equivalent to a generalized law of gravitation.*

The proof when $\lambda(\varphi)$ is periodic excludes, as an orbit of reference, the formula (15). This is the orbit (19) for which, under the present hypotheses,

$$\lambda(\varphi)=-1/(\varphi-\mu).$$

If the stated periodicity of the whole orbit is not assumed, (15) is an admissible C_0.

IX.

Newtonian gravitational force as a special case.

The forms of $P(r)$, for the successive values of n in (24), with

$$p(r)=ar^{n-1}+br^{n-2}+\ldots.+k,$$

are as follows:

$$(25)\quad \begin{cases} n=2; & P=\gamma^2\lambda^2[(a^2+\lambda^{-2})/r^3+3ab/r^4+2b^2/r^5], \\ n=3; & P=\gamma^2\lambda^2[ab/r^2+(b^2+2ac+\lambda^{-2})/r^3+3bc/r^4+2c^2/r^5], \\ n=4; & P=\gamma^2\lambda^2[-a(ar+b)+(ad+bc)/r^2 \\ & \qquad +(c^2+2bd+\lambda^{-2})/r^3+3cd/r^4+2d^2/r^5],\ldots \end{cases}$$

Several new conclusions relating to astronomy have been derived from these results, among them the proof that the mass of a stable asteroid is necessarily greater than a fixed lower limit ([6]). We conclude with a computation of an example of such a minimum mass.

The analysis associated with this problem also shows in detail how the formula (25), ($n=3$), becomes, in a special instance, the newtonian gravitational law of inverse squares.

We write the equation of C_0 in the form,

$$(26)\qquad \varphi=\tau\int dr/p(r)+\delta, \qquad (\tau,\ \delta \text{ constant}),$$

with,

$$p(r)=ar^2+br+c, \qquad b^2-4ac<0.$$

The integrated form of (26) is,

$$(27)\qquad r=v\tan(e\varphi+\beta)-u,$$

where,

$$e=m/2\tau, \qquad m=\sqrt{4ac-b^2}, \qquad u=b/2a, \qquad v=m/2a, \qquad \beta=-e\delta.$$

The formula for central force analogous to (25), $n=3$, may now be reduced to,

$$P=2\gamma^2e^2\left[\frac{u/v^2}{r^2}+\frac{3u^2/v^2+1/2e^2+1}{r^3}+\frac{3u^3/v^2+3u}{r^4}+\frac{(u^2+v^2)^2/v^2}{r^5}\right].$$

Note that when u is large and v small evidently the orbit of reference (27), ($\varphi=\tau\sigma+\delta$), can pass through the field F as *an ellipse with small eccentricity.* Then the above formula may be written, ($\varepsilon\doteq 0$, $\varepsilon^2=0$),

$$(28)\qquad P=16\gamma^2e^2\left[\frac{\left(1+\frac{3}{2}\varepsilon\right)u/v^2}{r^2}+\frac{1/2e^2+1}{8r^3}+\frac{3u}{8r^4}+\frac{2u^2+v^2}{8r^5}\right],$$

and, when r is large enough to make the last two terms negligible within the approximations being employed, this is the newtonian force plus a perturbative correction. The latter may be negligible or considerable according to choice of the constant e. We select the arbitrary γ so $16\gamma^2e^2$ is the gravitational constant. Note that the set (u, v) can remain fixed while the set (a, b, c) enjoys one degree of freedom.

([6]) Proc. Indiana (U. S. A.) Acad. Science, Vol. 40 (1931), p. 265.

The product of the masses M, M'' of the two attracting bodies is now

$$MM''=\left(1+\frac{3}{2}\varepsilon\right)u/v^2,$$

and since u is large, (the aphelion distance when $\beta=0$), and v small, this product is seen to be necessarily large. Hence, if M is the mass of the sun, the other planetary body can not have a mass M'' so small that MM'' would cease to be large. The limiting value of M'' will now be computed for the case of an asteroid in solitary rotation upon the earth's orbit.

The procedure consists, first, in the choice of a unit of mass, appropriate for use by an observer situated upon the revolving asteroid Σ. We assume the mean density of Σ to be that of the earth.

Secondly, we determine the minimum mass of Σ, for which the formulas allow the gravitational attraction to remain approximately newtonian.

With the linear unit chosen as 100,000 miles, the equation of the path of the stable earth is

$$(r_c),\qquad r=h/[1-a\cos\varphi],\qquad (r_c\equiv r,\ h/(1-a)=930).$$

By placing an instance of (27), viz. [7],

$$(r_d),\qquad r=-V\tan\varphi+u,\qquad (u=930),$$

in coincidence with (r_c) at a chosen point, (taken as $\varphi=10^0$), the eccentricity a being assigned, the quantity V is determined. It may be computed from the equation $r_c=r_d$. We then solve for the mass M of the sun and the mass M' of the earth from,

$$MM'=u/V^2,\qquad M=332000M'.$$

The unit will be the fractional $1/M$ part of the sun's mass.

Next, we substitute Σ for the earth on the path and place a field curve

$$(r_e),\qquad r=-v\tan e\varphi+u,\qquad (u=930),$$

in coincidence with (r_c) at the same point, (where $\varphi=10^0$). Writing M'/X for the mass of Σ, v is determined from $MM'/X=u/v^2$, and e from $r_c=r_e$, (X being assigned).

Thus, u, v, e being known, the first term S and the second T, of the bracketed expression in (28), are determined. The last two terms are negligible. But T must be small in comparison with S, else the newtonian law is vitiated. This comparison gives the test for largeness and smallness in the values of M''.

When $a=.01$ and $X=10{,}000$ the values at the point $\varphi=10^0$ are,

$$S=.1420/r^2,\qquad T=.6620/r^2,$$

and these are evidently too near to equality.

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[7] Cartesian equation: $x^2(x^2+y^2-u^2)+2uVxy-V^2y^2=0$.

There are, then, two and only two ways to secure divergence between the computed values of S and T. First we can diminish the eccentricity. However, it is assumed that an asteroid will not maintain forever an orbit whose eccentricity remains less than .00001. Secondly we can decrease X, that is, increase the diameter Y of Σ. The following table shows the typical results.

a	M'	X	Y	Sr^2	Tr^2	S/T
.01	.0651	10000	370 mi.	.1420	.6620	.2
.001	.6596	1000	790 mi.	144.4444	.06617	2183.
.001	.6596	10000	370 mi.	14.4444	.6615	22.
.001	.6596	100000	170 mi.	1.4444	6.614	.2
.0001	6.6075	1000	790 mi.	14494.99	.0659	219955.
.0001	6.6075	10000	370 mi.	1449.499	.6615	2191.
.0001	6.6075	100000	170 mi.	144.95	6.614	22.

Thus with a at its minimum (.0001), as Y ranges from 790 miles to 170 miles, the ratio S/T varies from near 220000, which is sufficiently large with some to spare, to approximately 22 which is too small. Hence the least diameter is near to 370 miles. With $a=.00001$, the least diameter is approximately 170 miles. Therefore:

LEMMA 2. - *The minimum diameter of an asteroid whose mean distance from the Sun, and mean density are respectively those of the earth, exists, and is not less than 170 miles. With a smaller diameter the asteroid is either unstable or else the potential between it and the Sun is not newtonian* (*Cf. the hypotheses of lemma* 1).

The analogous computation for a least asteroid on the orbit of Mars, gives a minimum which is smaller by about 100 miles, the mean density being assumed equal to that of Mars.

A possible verification of the result is obtained if we consider the indentations (« Craters ») on the surface of the moon to have been caused by asteroids falling throughout the ages. The diameters of these indentations range from small values up to near 130 miles, but not larger.

The zone of asteroids is limited in position by the gravitational laws.