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MARIUSZ LEMANCZYK

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THE CENTRALIZER OF MORSE SHIFTS

Mariusz LEMANCZYK

Abstract We examine the centralizer of Morse shifts.
1/ Let $x=b^0 \times b^1 \times \ldots$ be a regular Morse sequence and $|b^i| \leq r$. Then
a/ $C(T)=\{T^i\sigma^j : i\in Z,\ j=0,1\}$ where T is the shift and σ is the mirror map.
b/ There are no roots of T.
2/ There are Morse shifts with uncountable centralizer.
Let $\mathcal{T}^{\{n_t\}}$ be the class of all ergodic automorphisms τ with $\exp(2\pi i/n_t)$ in the point spectrum of τ. We introduce some number $d^{\{n_t\}}(\tau)$ for $\tau\in\mathcal{T}^{\{n_t\}}$ and prove that if $d^{\{n_t\}}(\tau)<\infty$ then τ is coalescent.

Introduction Let $(X,\mathcal{B},\mu)$ be a Lebesgue space and T an invertible transformation of $(X,\mathcal{B},\mu)$. By $C(T)$ we mean the centralizer of T i.e. the group of all automorphisms S of (X,μ) with TS=ST. The centralizer is an important invariant in ergodic theory. It can state some ergodic properties of T. In particular knowing $C(T)$ we can usually answer whether T has roots or T is embeddable in measurable flows. Moreover, if P is a finite generator of T then SP, $S\in C(T)$ are the only generators with the same finite distributions as P.

In the present paper we investigate centralizers of Morse shifts. These shifts play an important role in ergodic theory in providing concrete examples of dynamical systems with required properties /[7], [12], [15] /.

There are some direct reasons to compute the centralizers of Morse shifts. As we shall see in Section 5 the property to have an uncountable centralizer is a typical one in the class of all automorphisms acting in a fixed Lebesgue space. On the other hand examples of automorphisms with the trivial centralizer i.e. $C(T)=\{T^i, i\in Z\}$, are well-known / mixing rank one, minimal self-joining automorphisms [8] [20] /. Our main theorem / Theorem 1 / provides a large class of automorphisms with countable but not trivial centralizer.

Consider Morse dynamical systems as examples in topological dynamic / [16] /. We see that their topological properties are usually common for all Morse sequences / [3], [16] /. In particular the group of all homeomorphisms of $\mathcal{O}_x$ commuting with the shift $C^{top}(x)$ is equal to $\{T^i\sigma^j : i\in Z, j=0,1\}$. It is interesting to know whether $C^{top}(x)=C(x)$ or not. Surprisingly it turns out that in our class the answer can be negative as well as positive.

2. Notations Now, we introduce a bit of terminology. Each element $B=(b_0,\ldots,b_{k-1})\in\{0,1\}^k$ will be called a block, k is called the length of B and we denote it by $|B|$. Denote $B[i,j]=(b_i,b_{i+1},\ldots,b_j)$, $B[i,i]=B[i]$. The block $\tilde{B}=(\tilde{b}_0,\ldots,\tilde{b}_{k-1})$ is defined by setting $\tilde{b}_i=0$ if $b_i=1$ and $\tilde{b}_i=1$ if $b_i=0$. Let $C=(c_0,\ldots,c_{m-1})$ be another block. Then the product $B\times C$ is defined by $B\times C=B^{c_0}B^{c_1}\ldots B^{c_{m-1}}$ where $B^0=B$, $B^1=\tilde{B}$. Let $|B|=|C|=k$. Then $\bar{d}(B,C)=\frac{1}{k}\operatorname{card}\{i: 0\le i\le k-1\ B[i]\ne C[i]\}$. If $|B|\le|C|$ then $fr(B,C)=\operatorname{card}\{i: 0\le i\le |C|-|B|\ C[i,i+|B|-1]=B\}$. We will say B appears in C at i within δ if $\bar{d}(B,C[i,i+|B|-1])<\delta$. If $\bar{d}(B,C[i,i+|B|-1])=0$ we say simply B appears in C at i.

Now, let $b^0, b^1, b^2, \ldots$ be finite blocks of lengths at least two beginning with zero and put

/1/ $\quad x = b^0 \times b^1 \times b^2 \times \ldots$

We set $\lambda_i = |b^i|$, $r_i = \min\left\{\frac{1}{\lambda_i} fr(0, b^i),\ \frac{1}{\lambda_i} fr(1, b^i)\right\}$, $i = 0, 1, \ldots$

The sequence x defined in /1/ is said to be a Morse sequence if

/i/ infinitely many of the b^i's are different from $0\ldots0$,

/ii/ infinitely many of the b^i's are different from $01\ldots010$ and

/iii/ $\sum_{i=0}^{\infty} r_i = \infty$

Obviously /i/ follows from /iii/.

If x is a Morse sequence then one can find an almost periodic point $w \in X = \{0,1\}^Z$ such that $w[k] = x[k]$, $k \geqslant 0$ /[12]/.

Put $\mathcal{O}_x = \overline{\{T^i w: i \in Z\}}$ where T is the shift on X.

It is known $(\mathcal{O}_x, T)$ is strictly ergodic /[12]/. The unique T-invariant measure / ergodic / we shall denote by μ_x and the system $\Theta(x) = (\mathcal{O}_x, T, \mu_x)$ will said to be a Morse dynamical system / Morse shift /.

Denote by σ the mirror map on $\mathcal{O}_x$, i.e. $\sigma(y) = \tilde{y}$, $\tilde{y}[i] = \widetilde{y[i]}$, $i \in Z$. Then $T\sigma = \sigma T$ and by strictly ergodicity of $\mathcal{O}_x$ σ preserves μ_x.

Kwiatkowski in [13] has found out the structure of $\mathcal{O}_x$ / the set X(x) described there is contained in $\mathcal{O}_x$ and the set $\mathcal{O}_x \setminus X(x)$ is countable /. Namely, let

$$D_i^{n_t}(j) = \{y \in \mathcal{O}_x: y[-i+kn_t, -i+(k+1)n_t - 1] = c_t^j, k = 0, \pm 1, \pm 2, \ldots\}$$

$i = 0, \ldots, n_t - 1$, $t \geqslant 0$, $j = 0, 1$; $c_t = b^0 \times b^1 \times \ldots \times b^t$ and put $D_i^{n_t} = D_i^{n_t}(0) \cup D_i^{n_t}(1)$

Then $D^{n_t} = (D_0^{n_t}, \ldots, D_{n_t - 1}^{n_t})$ is a partition of $\mathcal{O}_x$ into open and closed subsets. Moreover, for every $y \in X(x)$ and every $t \in N$ there is only one i, $0 \leqslant i \leqslant n_t - 1$ such that $y[-i+kn_t, -i+(k+1)n_t - 1] = c_t$ or $\tilde{c}_t$

Denote $n_m^t = \lambda_t \cdot \ldots \cdot \lambda_{t+m}$, $c_m^t = b^t \times \ldots \times b^{t+m}$ $t, m \geqslant 0$, $c_m^0 = c_m$, $n_m^0 = n_m$, $x_t = b^t \times b^{t+1} \times \ldots$, $\mu_{x_t} = \mu_t$. c_t or $\tilde{c}_t$ will be called t-symbols.

In what follows we will say about properties of x instead of T on $\mathcal{O}_X$ and for example if no confusion becomes we shall write $C(x)$ instead of $C(T)$.

3. Coalescence Let $(X,\mathcal{B},\mu)$ be a Lebesgue space. We say an automorphism $\tau : X \circlearrowleft$ is coalescent if every endomorphism of (X,μ) commuting with τ is necessarily invertible. Consider the class of all ergodic automorphisms τ of $(X,\mathcal{B},\mu)$ for which $Sp(\tau) \supset G\{n_t : t \geq 0\}$ where $Sp(\tau)$ is the group of all eigenvalues of unitary operator U_τ defined in the following way $U_\tau(f) = f \circ \tau$. Here $n_t = \lambda_0 \cdot \ldots \cdot \lambda_t$, $t \geq 0$ and $\lambda_t \geq 2$. $G\{n_t : t \geq 0\}$ denotes the group generated by $\{\exp 2\pi i/n_t\}$ Let us notice that $\exp(2\pi i/n_t) \in Sp(\tau)$ iff there is a n_t-stack for τ i.e. a partition $(A, \tau A, \ldots, \tau^{n_t-1}A)$ of X /[2]/. Moreover it is not difficult to verify that ergodicity of τ implies that there is only one / reordering if necessary elements of another n_t-stack / n_t-stack for τ, so we denote it by $D^{n_t} = (D_0^{n_t}, \ldots, D_{n_t-1}^{n_t})$. In addition if $n_t \mid n_{t+1}$ then $D^{n_t} \leq D^{n_{t+1}}$. If $\tau \in \mathcal{T}^{\{n_t\}}$ then we get a sequence of T-invariant partitions $D^{n_0} \leq D^{n_1} \leq \ldots$. Let $D = (D_i)_{i \in I}$ be the limit partition. We assert card D_i, is a constant number for all $i \in I$ / i.e. either card $D_i = \infty$ $i \in I$ or card $D_i = m$ for some natural m / a.e. μ. Indeed D is τ-invariant and measurable partition, so our claim easily follows from [1].

Put $d^{\{n_t\}}(\tau) =$ card D_i, $i \in I$. Let us observe that $d^{\{n_t\}}(\tau)$ is an invariant of isomorphy.

Proposition 1 If $d^{\{n_t\}}(\tau)$ is finite then τ is coalescent.

Proof Let $\tau : (X,\mathcal{B},\mu) \circlearrowleft$ and ζ be any τ-invariant and measurable partition of X and let $f : X \to X/\zeta$ be canonical map. It is sufficient to show /[9]/ that if $(\tau, X, \mathcal{B}, \mu)$ and $(\tau/\zeta, X/\zeta, \mathcal{B}/\zeta, \mu/\zeta)$ are isomorphic then ξ is equal to the partition into points.

So, let us suppose it. Thus there is the sequence $\{\bar{D}^{n_t}\} \nearrow \bar{D}$ of n_t-τ/ξ-stacks and $d^{\{n_t\}}(\tau) = d^{\{n_t\}}(\tau/\xi)$. Let $\bar{D}_i$ be any "typical" atom from $\bar{D}$. Therefore $\bar{D}_i = \bigcap_{t \geq 0} \bar{D}^{n_t}_{i_t}$ so $f^{-1}(\bar{D}_i) = \bigcap_{t \geq 0} f^{-1}(\bar{D}^{n_t}_{i_t}) = \bigcap_{t \geq 0} D^{n_t}_{j_t} = D_j \in D$ because the preimage carries n_t-stacks into n_t-stacks. Hence f cannot stick together points as soon as they belong to the same atom D_j /because of card $\bar{D}_i$ = card D_j /.

Finally, $f^{-1}(B/\xi)$ contains σ-algebra generated by n_t-τ-stacks $t \geq 0$, so $\xi \geq D$. Therefore ξ must be equal to the partition into points.

<u>Remark 1</u> /[13]/ For every Morse sequence x, $d^{\{n_t\}}(x) = 2$.

<u>Remark 2</u> If $d^{\{n_t\}}(\tau) = \infty$ then τ need not be coalescent. For instance if τ is a Morse shift and τ' any Bernoulli automorphism then $\tau \times \tau'$ cannot be coalescent / [9], [18] /.

<u>4. Centralizer and simple spectrum</u> In this section we formulate and prove some characterization of automorphism having simple spectra that we need in the following.

<u>Proposition 2</u> Let $\tau: (X, \mu) \circlearrowleft$ be an automorphism of a Lebesgue space. Then U_τ has a simple spectrum iff the unitary centralizer of τ, $C^{unit}(\tau) = \{V: L^2(X,\mu) \circlearrowleft,\ V \text{ is unitary},\ VU_\tau = U_\tau V\}$ is abelian.

<u>Proof</u> If U has a simple spectrum then every unitary operator V, $VU_\tau = U_\tau V$ is a function of τ i.e. there exists a bounded function f such that $V = f(\tau) = \int_{U_\tau} f dE$, where E is the spectral measure of U_τ. Let $V' \in C^{unit}(\tau)$ then $V' = f'(\tau)$. Hence $VV' = \int_{U_\tau} f dE \circ \int_{U_\tau} f' dE = \int_{U_\tau} f f' dE = V'V$ /[5]/.

Now, suppose τ does not have simple spectrum. Then there are $f_1, f_2 \in L^2(X,\mu)$ such that $L^2(X,\mu) = B_1 \oplus B_2 \oplus C$, where B_i is the cyclic space generated by f_i, i.e. $B_i = \mathrm{span}\,(U_\tau^j f_i,\ j \in Z)$, $i=1,2$, C is U_τ-invariant and there exists $U_1 : B_1 \longrightarrow B_2$ which is unitary and $U_1 \circ U_\tau|_{B_1} = U_\tau|_{B_1} \circ U_1$ /[5]/. We define two unitary operators V, V' on $L^2(X,\mu)$ setting

$$\begin{array}{lll} V(b_1) = U_1(b_1) & V'(b_1) = U_\tau(b_1) & b_1 \in B_1, \\ V(b_2) = U_1^{-1}(b_2) & V'(b_2) = b_2 & b_2 \in B_2, \\ V(c) = c & V'(c) = c & c \in C. \end{array}$$

It is easy to see that $V, V' \in C^{unit}(\tau)$ but $VV' \neq V'V$. Indeed, if $VV' = V'V$, $U_\tau|_{B_1}$ and $U_\tau|_{B_2}$ are identity and a contradiction to ergodicity of τ.

It is known /[13]/ that every Morse sequence x has a simple spectrum. Combining this with Proposition 2 we have obtained

<u>Corollary 1</u> For every Morse sequence x, $C(x)$ is abelian.

<u>5. A class of Morse sequences with uncountable centralizer</u>

In this section we give a class of Morse sequences with uncountable centralizer. We also provide some arguments that the property to have an uncountable centralizer is a typical one.

Let $(X,\mathcal{B},\mu)$ be a Lebesgue space and τ be an ergodic automorphism of (X,μ). Let us consider the group $\mathcal{S}$ of all automorphisms $S: (X,\mu)\circlearrowleft$ with the weak topology $\mathcal{W}$ /[6]/ defined in the following way

$$S_n \xrightarrow[W]{} S \text{ iff } \mu\{S_n E \triangle SE\} \xrightarrow[n]{} 0 \text{ for every } E \in \mathcal{B}.$$

Now, we recall some known results on the weak topology.

/2/ $(\mathcal{S}, \mathcal{W}, \circ)$ is a topological group /[6]/,

/3/ $(\mathcal{S}, \mathcal{W})$ is completely metrizable /[6]/,

/4/ $S_n \xrightarrow[W]{} S$ iff $U_{S_n} \Rightarrow U_S$ i.e. $\|U_{S_n} f - U_S f\| \xrightarrow[n]{} 0$ /[6]/,

/5/ $C(\tau)$ is a close set in $\mathcal{W}$,

/6/ If $\tau^{i_t} \longrightarrow S$, $i_t \nearrow \infty$ then $S \in C(\tau)$, $\tau^{i_t - i_{t-1}} \longrightarrow \mathrm{id}$ and $C(\tau)$ is a perfect set, so from the Bair's property $C(\tau)$ is uncountable /[11]/.

We let $\mathfrak{S}^1$ denote the class of all $S\in\mathfrak{S}$ with $S^{i_t}\longrightarrow id$ for some sequence $\{i_t\}\nearrow\infty$. Then $\mathfrak{S}^1$ contains a dense G_δ set of automorphisms of (X,μ). Indeed, if S admits a cyclic approximation with speed $o(1/n)$ then $U^{i_t}\Longrightarrow id$ for some sequence $\{i_t\}$ and moreover the class of all automorphisms admitting a cyclic approximation with a fixed speed contains a dense G_δ set /[10]/. So, we have proved the property to have an uncountable centralizer is a typical one in $\mathcal{W}$. Let us observe that the class $\mathfrak{S}^1$ is closed under taking factors, so if $S\in\mathfrak{S}^1$ then S does not have mixing factors, in particular $h(S)=0$. But a stronger fact is true. If $S\in\mathfrak{S}^1$ then S is disjoint from all mixing transformations /[21]/.

Now, we are able to show there are Morse shifts with uncountable centralizer.

Given a Morse sequence $x=b^0\times b^1\times\ldots$ we denote

$$p_t=\mu_t(00)+\mu_t(11),\quad q_t=\mu_t(01)+\mu_t(10)$$

<u>Proposition 3</u> Let $x=b^0\times b^1\times\ldots$ be a Morse sequence. If $\lim_{t\to\infty}p_t=0$ then $C(x)$ is uncountable.

<u>Proof</u> We will prove that x admits a cyclic approximation with speed $o(1/n)$.

We have $\xi_t=\{D_i^{n_t}(j):\ i=0,\ldots,n_t-1,\ j=0,1\},t\geq 0$ / see Section2 /

From it follows that $\xi_t\nearrow\varepsilon$

We define a cyclic approximation S_t putting

$$S_tD_i^{n_t}(j)=D_{i+1}^{n_t}(j)\qquad i=0,\ldots,n_t-2,\ j=0,1$$

/7/

$$S_tD_{n_t-1}^{n_t}(j)=D_0^{n_t}(1-j)$$

Now, we wish to estimate $A_t=\sum_{j=0}^{1}\sum_{i=0}^{n_t-1}\mu_x(TD_i^{n_t}(j)\vartriangle S_tD_i^{n_t}(j))$

We then get $A_t=2\mu_x(TD_{n_t-1}^{n_t}(0)\vartriangle S_tD_{n_t-1}^{n_t}(0))\leq\frac{2}{n_{t+1}}(fr(00,b^{t+1})+fr(11,b^{t+1}))=$

$=2\,\dfrac{fr(00,b^{t+1})+fr(11,b^{t+1})}{\lambda_{t+1}}\dfrac{1}{n_t}\leq 2p_{t+1}\dfrac{1}{n_t}$

so $A = o(1/2n_t)$.

Therefore x admits desired cyclic approximation.

6. The measure-theoretic centralizer of regular Morse sequences. This section is devoted to prove the main result of the paper

Theorem 1 Let $x=b^0\times b^1\times\ldots$ be a regular Morse sequence satisfying /11/ and let $S\in C(x)$. Then $S=T^i\sigma^j$ for some $i\in Z$, $j=0,1$.

We start with presenting our main techniques / Proposition 4,5/ needed in proving of Theorem 1.

Let $x=b^0\times b^1\times\ldots$ be a Morse sequence.

A measurable function $\varphi: X=\{0,1\}^Z$ is said to be a code of length k if

/i/ $\varphi T = T\varphi$,

/ii/ $\varphi(y)[0]$ depends only on $y[-k,k]$, i.e. if $y[-k,k]=y'[-k,k]$ then $\varphi(y)[0]=\varphi(y')[0]$,

/iii/ k is the smallest natural number satisfying /ii/ and we denote it by $|\varphi|$.

The following Proposition establishes a list of properties of finite codes that we will need.

Proposition 4 /a/ Let φ be finite code. Then for a.e. $y,y'\in\mathcal{O}_x$ if $y[-|\varphi|+t,t+|\varphi|]=y'[-|\varphi|+u,u+|\varphi|]$ then $\varphi(y)[t]=\varphi(y')[u]$.

/b/ Let $S\in C(x)$ and $\delta>0$. There is a finite code φ such that

/8/ $\quad d(Sy,\varphi y)<\delta \quad / d(z,z')=\lim_{m\to\infty} d(z[-m,m],z'[-m,m])/$

/9/ $\quad d(\varphi y,\varphi\tilde{y})>1-2\delta \quad$ for a.e. $y\in\mathcal{O}_x$

Proof The proof is straightforward and we use only ergodic theorem.

Following [13] we say x is a regular Morse sequence if there is $\varrho > 0$ such that

/10/ $\quad \varrho < p_t < 1-\varrho$ and $\quad \varrho < q_t < 1-\varrho \quad , \ t \geq 0.$

In addition we assume

/11/ $\quad \sup_{t\in N} \lambda_t = \lambda < \infty$

The following characterization of regular Morse sequences satisfying /11/ can be found in [14] .

<u>Proposition 5</u> Let $x=b^0 \times b^1 \times \dots$ be a regular Morse sequence and let /11/ holds. Then

$(\exists \delta>0)(\exists L>0)(\forall \eta\text{-block})(\forall t\in N)$ [if $\eta = c_t \times B$, $|B|=L$ appears in x at i within δ then $n_t | i$ and η appears in x at i]

In the sequel we will need some facts of combinatorial nature.

Let $x=b^0 \times b^1 \times \dots$ be a regular Morse sequence satisfying /11/ and let $S \in C(x)$. Let $\delta>0$, $L>0$ be determined by Proposition 5.

Let us take $\varepsilon > 0$ and assume $\varphi : X^{\wp}$ is a code of lenght k so that

/12/ $\quad d(\varphi y, Sy) < \varepsilon$ for a.e. $y \in \mathcal{O}_x$

Fix $y \in \mathcal{O}_x$ for which /12/ holds.

Next, we find $t \in N$ so large that

/13/ $\quad k/n_t < \varepsilon/2$

/14/ $\quad d(e_t, \hat{e}_t) > 1-3\varepsilon$ where e_t / $\hat{e}_t$ / is the code of c_t / $\tilde{c}_t$ / via φ i.e. $|e_t| = n_t - 2k$, $e_t[j] = \varphi(c_t[k+j, 2k+j-1])$, $j=0,\dots,n_t-2k-1$

/15/ $\quad (\forall m \geq n_t) \quad d(\varphi y[-m,m], Sy[-m,m] < \varepsilon.$

Assume in addition

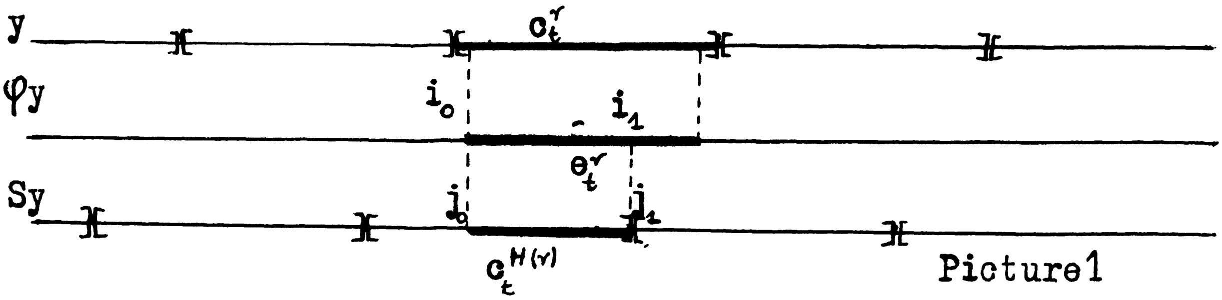
/16/ $\quad y \in D_u^{n_t}$

Now we shall define some map $H:\{0,1\}^{\wp}$ in the following way

$$d(e_t^r[i_0,i_1], c_t^{H(r)}[j_0,j_1]) = \min\{d(e_t^r[i_0,i_1], c_t[j_0,j_1]), \\ d(e_t^r[i_0,i_1], \tilde{c}_t[j_0,j_1])\}$$

where $e_t^r = e_t$ if $r=0$ or $\hat{e}_t$ otherwise and $|i_0-i_1| = |j_0-j_1| \geq \frac{1}{2}|e_t|$

/ see Picture 1/ .



Picture1

Let us observe that

/17/ $\quad d(e_t^r[i_o,i_1], c_t^{H(r)}[j_o,j_1]) < 20\varepsilon$, $r=0,1$

Indeed, otherwise we would have $d(e_t^r[i_o,i_1], c_t^s[j_o,j_1]) \geq 20\varepsilon$; $s=0,1$

Choose a sector of y, say $y[-m,m]$, $m > n_t$ such that

$y[-m,m]$ consists of p t-symbols and this sector contains

/18/ at least $(\frac{1}{2}-\varepsilon)p$ of c_t^r , $r=0,1$ calculated only in the

places of the form $-u+vn_t$, $v=0,\pm 1,\pm 2,\dots$

To see /18/ it is sufficient to use ergodic theorem and the fact that $\mu_t(r)=\frac{1}{2}$ for every $r=0,1$, $t=0,1,\dots$. Hence

$\varepsilon > d(\varphi y[-m,m], Sy[-m,m]) \geq (\frac{1}{2}-\varepsilon)p \;\; 20\varepsilon\, \frac{1}{2}|e_t^r|/(2m+1) \geq$

$\geq (\frac{1}{2}-\varepsilon)p\, 10\,\varepsilon\, |e_t^r|/pn_t = 5\varepsilon(1-2\varepsilon)(1-2k/n_t) \geq 5\varepsilon(1-2\varepsilon)(1-\varepsilon) \geq \varepsilon$

a contradiction.

Now, we show $H:\{0,1\}^{\circlearrowleft}$ is one-to-one. Indeed let us suppose $H(0)=H(1)$. Then

$d(e_t[i_o,i_1], \hat{e}_t[i_o,i_1]) \leq d(e_t[i_o,i_1], c_t^{H(0)}[j_o,j_1]) + d(\hat{e}_t[i_o,i_1], c_t^{H(1)}[j_o,j_1]) < 40\varepsilon$

But from /14/

$d(e_t[i_o,i_1], \hat{e}_t[i_o,i_1]) \geq (1-3\varepsilon)\frac{1}{2}|e_t^r|/|e_t^r| \geq \frac{1}{2}-2\varepsilon$, a contradiction.

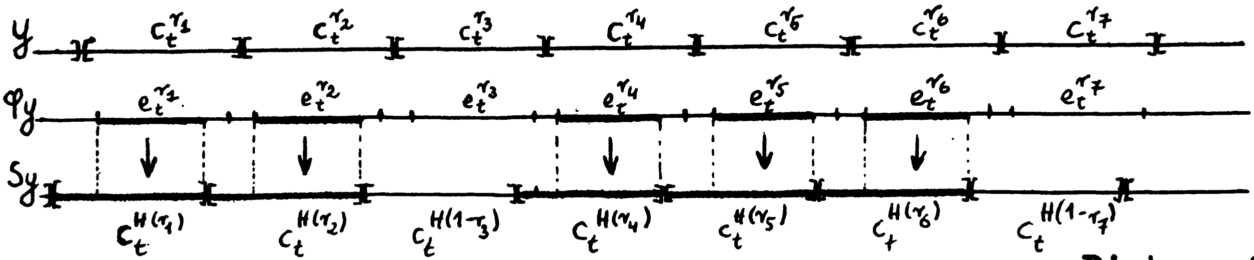
At present, we estimate $d(e_t^r[i_o,i_1], c_t^{H(1-r)}[j_o,j_1])$. We have

$d(e_t^r[i_o,i_1], e_t^{1-r}[i_o,i_1]) \leq d(e_t^r[i_o,i_1], c_t^{H(1-r)}[j_o,j_1]) + d(c_t^{H(1-r)}[j_o,j_1], e_t^{1-r}[i_o,i_1])$

Hence

/19/ $\quad d(e_t^r[i_o,i_1], c_t^{H(1-r)}[j_o,j_1] \geq \frac{1}{2}-22\varepsilon$

Let us consider again the sector $y[-m,m]$ satisfying /18/ and we match by arrow e_t^r with $c_t^{H(r)}$ / Picture 2/.



Picture 2

We wish to estimate the number R of e_t^r , r=0,1 without arrow. We have $d(\varphi y[-m,m], Sy[-m,m]) \geq R(\frac{1}{2}-22\varepsilon)\frac{1}{2}|e_t^r|/pn_t$, therefore

/20/ $R < 8\varepsilon p$

<u>Proof of Theorem1</u> From the invertibility of H we have

/21/ $c_t^{H(r)} = c_t^{H(0)+r}$ for r=0,1

Take now $T^s y^{H(0)}$ where $T^s y[-u+in_t, -u+(i+1)n_t-1]$ is always t-symbol. Then $d(T^s y^{H(0)}[-m+s, m-s], Sy[-m+s, m-s] < \frac{R}{p} < 8\varepsilon$

Find the greatest t_0 such that $y[-m,m]$ contains L t_0-symbols. Using the condition of boundness of $\{\lambda_t\}$ we get $t_0 \to \infty$ whenever $p \to \infty$. So choosing ε as small as we need and applying Proposition 5 we obtain $T^s y^{H(0)}[v, v+Ln_{t_0}-1] = Sy[v, v+Ln_{t_0}-1]$ for some $v \in Z$. Letting $p \to \infty$ we get at once $T^S \sigma^{H(0)} y = Sy$.

Let us set $A_h = \{y \in \mathcal{O}_x : Sy = T^S \sigma^h y\}$, h=0,1 . So either $\mu_x(A_0) > 0$ or $\mu_x(A_1) > 0$. But A_h is T-invariant and ergodicity of T forces A_h / with positive measure / to have full measure. Finally $S = T^s \sigma^h$.

<u>Corollary 2</u> For every regular Morse sequence with /11/ there are no roots of the shift induced by x.

<u>Corollary 3</u> For every regular Morse sequence with /11/ $C(T) \neq \overline{\{T^i : i \in Z\}}^w$

<u>Proof</u> In the case of the equality C(T) is uncountble.

<u>Final remarks</u> Let us now consider the class of all nonperiodic substitutions on two symbols of constant lenght / for definition and properties see [4] /

/22/ Θ: $$\begin{aligned} 0 &\longmapsto B=(b_0,\dots,b_{\lambda-1}) \\ 1 &\longmapsto C=(c_0,\dots,c_{\lambda-1}) \end{aligned}$$

There are two kinds of them:

/i/ discrete substitutions: if Θ defined in /22/ has the property $b_i=c_i$ for some i, $0\leq i\leq\lambda-1$

/ii/continuous substitutions: otherwise.

Their topological centraliter was calculated in [3]. It is equal to $\{T^k: k\in Z\}$ for /i/ and $\{T^k\sigma^j: k\in Z,\ j=0,1\}$ for /ii/ /here σ is again the mirror map/.

Now, we are able to show measure-theoretic centralizer for such a Θ. Let Θ be discrete substitution. Then Θ may be considered from the measure-theoretic point of view as a discrete, ergodic dynamical system with $Sp(\Theta)=G\{\lambda^t: t\geq 0\}$. From [19] it follows that $C(\theta)=End(G\{\lambda^t: t\geq 0\})$. It is easy to see that the last group is equal to the λ-adic integers.

Let Θ be a continuous substitution. Then the dynamical system arising from Θ is equal to (Θ_x, T, μ_x) where $x=B\times B\times\dots$ is a Morse sequence / if B does not start with zero we replace B by $B\times B$/, /[4]/. So from Theorem 1 $C^{top}(\theta)=C(\theta)=\{T^i\sigma^j, i\in Z,\ j=0,1$

Consider the class of Morse sequences over a fixed finite Abelian group G / see [16],[17] /. Let $x=b^0\times b^1\times\dots$ be such a one. Let us call it regular if $\sup_{t\in N} s_t=s<\infty$ where $s_t=\sup_{g\in G}\{|B|: B=0\sigma_g(0)\dots\sigma_{(|g|-1)}(0)\ 0\sigma_g(0)\dots 0\sigma_g(0)\dots\sigma_{mg}(0)$ and B appears in $x_t\}$ $t\geq 0$, $|g|$ denotes the order of g and $\sigma_g(i)=i+g$, $i,g\in G$. If $\{\lambda_t\}$ is bounded then Proposition 5 holds for these regular Morse sequences over G. The concept of finite code $\varphi: G^l$ and Proposition 4 go as in Section 6. Let us assume $S\in C(x)$ and in addition $S\sigma_g=\sigma_g S$ for every $g\in G$. Repeating considerations of Section 6 we see that the only formula which is not quite clear is the following $H(g)=H(0)+g$, $g\in G$. To prove it we take p as in /18/. There must exist an $i_0 \in Z$ such that $\sigma_g(y)[-u+i_0n_t, -u+(i_0+1)n_t-1] = \sigma_g(c_t)$ with an arrow for every $g \in G$ / it is a simple consequence of /18/,/20/ and $S\sigma_g = \sigma_g S$ /. This proves that if $S \in C(x)$, $S\sigma_g = \sigma_g S$, $g \in G$ then $S = T^i\sigma_g$ for some $i \in Z$, $g \in G$. To get $S\sigma_g = \sigma_g S$ it is sufficient to know that $(\mathcal{O}_x, T, \mu_x)$ has a simple spectrum. In general it is still unknown whether they have simple spectra or not. Recently Kwiatkowski have communicated me that he knows examples of Morse sequences /over any cyclic group/ of the form $x = B \times B \times \ldots$ having simple spectrum

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M. LEMANCZYK
Nicholas Copernicus University
Institute of Mathematics
TORUN
Poland

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