

ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Probabilités et applications

D. NUALART

M. SANZ

Malliavin calculus for two-parameter processes

Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 85, série *Probabilités et applications*, n° 3 (1985), p. 73-86

http://www.numdam.org/item?id=ASCFPA_1985__85_3_73_0

© Université de Clermont-Ferrand 2, 1985, tous droits réservés.

L'accès aux archives de la revue « Annales scientifiques de l'Université de Clermont-Ferrand 2 » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

MALLIAVIN CALCULUS FOR TWO-PARAMETER PROCESSES

D. NUALART and M. SANZ

Abstract. In this paper we apply the Malliavin Calculus to derive the existence of a density for the law of the solution of a stochastic differential equation with respect to a multidimensional two-parameter Wiener process.

AMS 1980 Subject Classification: 60H20, 60G30.

0. Introduction. In this paper we prove the existence of a density for the probability law on $\mathbb{R}^m$ induced by the solution of the stochastic integral equations.

$$X_z^i = x^i + \int_{[0,z]} [A_j^i(X_r) dW_r^j + B^i(X_r) dr] , \quad i = 1, \dots, m, \quad (0.1)$$

$z \in \mathbb{R}_+^2$, where $W_z = (W_z^1, \dots, W_z^d)$ is a d -dimensional two-parameter Wiener process, $x = (x^1, \dots, x^m) \in \mathbb{R}^m$, and assuming some conditions on the coefficients A_j^i and B^i . If these coefficients are smooth, it is known (cf. Cairoli [2], Hajek [4]) that (0.1) has a unique continuous solution, which has a particular Markov property. There exists a transition semigroup corresponding to these Markov processes, but it acts on continuous functions over sets of the form $\{(x,t), x \geq s\} \cup \{(s,y), y \geq t\}$ and we cannot expect that the probability law of X_z^i satisfy a second order partial differential equation.

In the case of an ordinary stochastic differential equation with respect to the Brownian motion, Malliavin has developed in [6] probabilistic techniques to show the existence and smoothness of density for the solution of these equations under Hörmander's conditions. Alternative approaches to Malliavin's theory were given by Shigekawa [8], Bismut [1] and Stroock [9]. It is not difficult to extend the Malliavin calculus to two-parameter Wiener functionals. However when we apply this calculus to the solutions of (0.1) some technical difficulties appear, in relation with the following facts:

(a) The inner products $\langle X_z^i, X_z^k \rangle$ (in the notation of Stroock) are not solutions of a similar system of equations, because the stochastic differentiation rules with respect to the two-parameter Wiener process involve the presence of double integrals over the set $\{(z, z') \in \mathbb{R}_+^2 \times \mathbb{R}_+^2, z = (x, y), z' = (x', y'), x \leq x', y \geq y'\}$ (cf. [10])

(b) The system (0.1) do not provide a flow of transformations of $\mathbb{R}^m$. Moreover if we consider a linear system of equations, the solution is not invertible, in general.

Here we have followed Shigekawa's presentation of Malliavin Calculus, and using this theory we have proved the existence of a density for the two-parameter Wiener functional X_z assuming that the vector space spanned by the vector fields $A_1, \dots, A_d, A_i^\nabla A_j, 1 \leq i, j \leq d, A_i^\nabla (A_j^\nabla A_k), 1 \leq i, j, k \leq d, \dots$, at the point x (where $A_i^\nabla A_j$ denotes the covariant derivative of A_j in the direction of A_i), is $\mathbb{R}^m$. This property is strictly weaker than the restricted Hörmander's conditions, as we show in an example. Actually we have proved (see [7]) that it also implies the smoothness of the density.

1. Some results on Malliavin Calculus. The set of parameters will be $T = [0, 1]^2$, with the partial ordering $(s_1, t_1) \leq (s_2, t_2)$ if and only if $s_1 \leq s_2$ and $t_1 \leq t_2$; $(s_1, t_1) < (s_2, t_2)$ means that $s_1 < s_2$ and $t_1 < t_2$. If $z_1 < z_2$, $(z_1, z_2]$ will denote the rectangle $\{z \in T, z_1 < z \leq z_2\}$. We put $R_z = [0, z]$, and $z_1 \boxtimes z_2 = (s_1, t_2)$ if $z_1 = (s_1, t_1)$ and $z_2 = (s_2, t_2)$. If $f: R_+^2 \rightarrow R$, $f((z_1, z_2])$ means $f(z_1) - f(z_1 \boxtimes z_2) - f(z_2 \boxtimes z_1) + f(z_2)$. The Lebesgue measure of a Borel set $B \subset R_+^2$ is denoted by $|B|$.

Our probability space (Ω, F, P) is the canonical space associated to the d -dimensional two-parameter Wiener process. We also consider the filtration $\{F_z, z \in T\}$, where F_z is generated by the functions $\{\omega(s), \omega \in \Omega, s \leq z\}$ and the null sets of F . The family $\{F_z, z \in T\}$ satisfies the usual conditions of [3].

The following subset of Ω plays an important role:

$$H = \{ \omega \in \Omega, \text{ there exists } \dot{\omega}^i \in L^2(T), i = 1, \dots, d, \text{ such that} \\ \omega^i(z) = \int_{R_z} \dot{\omega}^i(r) dr, \text{ for any } z \in T \text{ and for any } i \}.$$

H is a Hilbert space with the inner product

$$\langle \omega_1, \omega_2 \rangle_H = \int_T \sum_{i=1}^d \dot{\omega}_1^i(r) \dot{\omega}_2^i(r) dr.$$

A measurable function defined on (Ω, F, P) is called a Wiener functional. A Wiener functional $F: \Omega \rightarrow R$ is smooth if there exists some $n \geq 1$ and a C^2 -function f on R^n such that

(i) f and its derivatives up to the second order have at most polynomial growth order,

(ii) $F(\omega) = f(\omega(z_1), \dots, \omega(z_n))$ for some $z_1, \dots, z_n \in T$.

Every smooth functional is Fréchet-differentiable, and we have

$$DF(\omega_0)(\omega) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\omega_0(z_1), \dots, \omega_0(z_n)) \omega^i(z_i) .$$

We also need the operator L defined on smooth functionals as follows:

$$LF(\omega) = \sum_{j=1}^d \sum_{i,k=1}^n \frac{\partial^2 f}{\partial x_j^i \partial x_j^k}(\omega(z_1), \dots, \omega(z_n)) z_i \wedge z_k - DF(\omega)(\omega) ,$$

where $z_i \wedge z_k = (x_i \wedge x_k)(y_i \wedge y_k)$, if $z_i = (x_i, y_i)$, $i = 1, \dots, n$.

For any $p \geq 1$, L_H^p will denote the space of Wiener functionals $F: \Omega \longrightarrow H$ such that $E(\|F\|_H^p) < \infty$. If we fix $\omega \in \Omega$ and a smooth functional F , $DF(\omega): H \longrightarrow R$ is a continuous linear map, and, so, it may be considered as an element of H . In this sense we have $DF \in L_H^p$, for any $p \geq 1$.

Let $H(p_1, p_2; p_3)$, $p_1, p_2, p_3 \geq 1$, be the space of real valued Wiener functionals F such that there exists a sequence of smooth functionals $\{F_k, k \geq 1\}$ satisfying:

- (a) $F_k \longrightarrow F$ in L^{p_1} ,
- (b) $\{DF_k, k \geq 1\}$ is a Cauchy sequence in $L_H^{p_2}$, and
- (c) $\{LF_k, k \geq 1\}$ is a Cauchy sequence in L^{p_3} .

For a Wiener functional $F \in H(p_1, p_2; p_3)$ we define $DF = \lim_k DF_k$ and $LF = \lim_k LF_k$. $H(p_1, p_2; p_3)$ is a Banach space with the norm $\|F\|_{p_1} + \|DF\|_{p_2} + \|LF\|_{p_3}$. We set $H_\infty = \bigcap_{p \geq 2} H(p, p; p)$. If $F \in H_\infty$, we will say that a sequence of smooth functionals $\{F_k, k \geq 1\}$ is an approximating sequence for F if $\lim_k (\|F - F_k\|_p + \|DF - DF_k\|_p + \|LF - LF_k\|_p) = 0$, for any p .

Let $F^i \in H_\infty$ for $i = 1, \dots, n$, and let $u: R^n \longrightarrow R$ be twice

continuously differentiable function such that u and its first and second derivatives have at most polynomial growth order. If we set $F = (F^1, \dots, F^n)$, then $u \circ F \in H_\infty$, and the following differentiation rules hold:

$$D(u \circ F) = \left(\frac{\partial u}{\partial x_i} \circ F \right) DF^i,$$

$$L(u \circ F) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \circ F \right) \langle DF^i, DF^j \rangle_H + \left(\frac{\partial u}{\partial x_i} \circ F \right) LF^i.$$

The next result is the main theorem in Shigekawa's paper [8] adapted to the Wiener process with two parameters, and with a slight difference consisting in the use of the space $H(1,2;1)$. The proof given by Shigekawa can be extended to these new conditions without any change.

Theorem 1.1. Let $F = (F^1, \dots, F^m)$ be an R^m -valued two-parameter Wiener functional. We assume that F satisfies the following conditions:

- (i) $F^i \in H(1,2;1)$, for $i = 1, \dots, m$.
- (ii) $\langle DF^i, DF^j \rangle_H \in H(1,2;1)$, for $i, j = 1, \dots, m$.
- (iii) $\det(\langle DF^i, DF^j \rangle_H) \neq 0$ a.s.

Then, the probability law of F is absolutely continuous with respect to the Lebesgue measure.

In the next section we will employ this result to show the existence of a density for the law induced by the solution of a stochastic differential system.

2. Application to stochastic differential equations. Consider mappings

$$A: R^m \times R^n \longrightarrow R^n \otimes R^d \quad \text{and} \quad B: R^m \times R^n \longrightarrow R^n \quad \text{such that}$$

(i) All their components have bounded and continuous first order derivatives.

(ii) $x \longrightarrow A(x,0)$ and $x \longrightarrow B(x,0)$ are slowly increasing functions, that is $|A(x,0)| \leq k(1 + |x|^\alpha)$ and $|B(x,0)| \leq k(1 + |x|^\beta)$ for some positive integers α and β and positive constant k .

Lemma 2.1. Let $\alpha = \{\alpha_z, z \in T\}$ be a continuous and adapted n -dimensional stochastic process such that for each $p \geq 2$, $E \left\{ \sup_{z \in T} |\alpha_z|^p \right\} < \infty$. Consider a continuous and adapted m -dimensional process $X = \{X_z, z \in T\}$ such that $E \left\{ \sup_{z \in T} |X_z|^p \right\} < \infty$, for every $p \geq 2$, and fix $v \in T$. Then, there is a unique continuous and adapted n -dimensional process $Y = \{Y_z, z \geq v\}$ satisfying the stochastic differential system

$$Y_z^i = \alpha_z^i + \int_{[v, z]} [A_j^i(X_r, Y_r) dW_r^j + B^i(X_r, Y_r) dr], \quad i=1, \dots, n, \quad (2.1)$$

and the following property holds:

$$E \left\{ \sup_{z \geq v} |Y_z|^p \right\} < \infty, \text{ for any } p \geq 2.$$

The solution of (2.1) (with an arbitrary initial point v) can be approximated by polygonal paths, as it is stated in the next lemma.

For any $k \geq 1$ we consider the set S^k of points $(i2^{-k}, j2^{-k})$, $i, j = 0, \dots, 2^k$. If $z \in T$ put $S_z^k = \{u \leq z: u \in S^k, \text{ or } u = v \wedge z \text{ with } v \in S^k, \text{ or } u = z \wedge v \text{ with } v \in S^k \text{ or } u = z\}$. Define $\phi_k(z) = \sup \{u \in S^k, u \leq z\}$ and $\psi_k(z) = \inf \{u \in S^k, u \geq z\}$.

Lemma 2.2. Let $\alpha = \{\alpha(z), z \in T\}$ and $\alpha_k = \{\alpha_k(z), z \in T\}$, $k \geq 1$ be continuous and adapted n -dimensional processes satisfying

$$\lim_{k \rightarrow \infty} E \left\{ \sup_{z \in T} |\alpha_k(z) - \alpha(z)|^p \right\} = 0$$

for each $p \geq 2$, and let X be as in the preceding lemma. Consider the

process $Y_z = (Y_z^1, \dots, Y_z^n)$, $z \geq v$ defined by (2.1). Consider also the process $Y_z^{(k)} = (Y_z^{k,1}, \dots, Y_z^{k,n})$, $z \geq v$ given by

$$Y_z^{k,i} = \alpha_k^i(z) + \int_{[\psi_k(v) \wedge z, z]} [A_j^i(X_{\phi_k(r)}^{(k)}, Y_{\phi_k(r)}^{(k)}) dW_r^j + B^i(X_{\phi_k(r)}^{(k)}, Y_{\phi_k(r)}^{(k)}) dr],$$

where $X^{(k)} = \{X_z^{(k)}, z \in T\}$, $k \geq 1$ is a sequence of continuous and adapted m -dimensional stochastic processes such that for each $p \geq 2$, $E \{ \sup_{z \in T} |X_z^{(k)} - X_z|^p \} \xrightarrow{k \rightarrow \infty} 0$. Then

$$\lim_{k \rightarrow \infty} \sup_{v \in T} \{ E [\sup_{z \geq v} |Y_z - Y_z^{(k)}|^p] \} = 0, \forall p \geq 2.$$

This result is a generalization to the two-parameter case of lemma V.2.1 from [5].

Lemma 2.3. Let $X_z = (X_z^1, \dots, X_z^m)$, $z \in T$ be the process which satisfies

$$X_z^i = x^i + \int_{R_z} [A_j^i(X_r) dW_r^j + B^i(X_r) dr], \quad i = 1, \dots, m.$$

where $A: R^m \rightarrow R^{m \times d}$ and $B: R^m \rightarrow R^m$ have bounded and continuous derivatives up to the second order. Then $X_z^i \in H_\infty$, for $i=1, \dots, m$, $z \in T$.

Proof. For any $n \geq 1$ consider the process $X_z^{(n)} = (X_z^{n,1}, \dots, X_z^{n,m})$ defined by the recursive system

$$X_z^{n,i} = x^i + \int_{R_z} [A_j^i(X_{\phi_n(r)}^{(n)}) dW_r^j + B^i(X_{\phi_n(r)}^{(n)}) dr], \quad i = 1, \dots, m.$$

The random variables $X_z^{n,i}$, $z \in T$ are smooth functionals. We are going to prove that $\{X_z^{n,i}, n \geq 1\}$ is an approximating sequence for X_z^i .

First, by lemma 2.2 we have $\lim_n E \left\{ \sup_{z \in T} |X_z^{(n)} - X_z|^p \right\} = 0$,

for all $p \geq 1$. If $z \in (0, 1]^2$ and $u = \sup \{v \in S^n, v < z\}$,

$X_z^{n,i}$ is given by

$$X_z^{n,i} = X_{z \otimes u}^{n,i} + X_{u \otimes z}^{n,i} - X_u^{n,i} + A_j^i(X_u^{(n)}) W^j((u, z]) + B^i(X_u^{(n)}) |(u, z]|.$$

Fix an element $\omega \in \Omega$. Differentiating term by term we obtain

$$\begin{aligned} DX_z^{n,i}(\omega) &= DX_{z \otimes u}^{n,i}(\omega) + DX_{u \otimes z}^{n,i}(\omega) - DX_u^{n,i}(\omega) + \frac{\partial A_j^i}{\partial x_k}(X_u^{(n)}(\omega)) \\ &\cdot DX_u^{n,k}(\omega) \omega^j((u, z]) + A_j^i(X_u^{(n)}(\omega)) e^j \varepsilon_{(u, z]} + \frac{\partial B^i}{\partial x_k}(X_u^{(n)}(\omega)) \\ &\cdot DX_u^{n,k}(\omega) |(u, z]|, \end{aligned}$$

where $\varepsilon_{(u, z]} = \varepsilon_z - \varepsilon_{z \otimes u} - \varepsilon_{u \otimes z} + \varepsilon_u$, $\varepsilon_z = \int_T 1_{R_z}(u) du$, and e^j is the vector of R^d given by $(e^j)^i = \delta_i^j$. From now on we will omit the dependence on ω .

Denote by $U_z^{n,i}$ the derivative of $DX_z^{n,i}$ in the sense

$$\langle DX_z^{n,i}, h \rangle_H = \int_{R_z} \sum_{j=1}^d U_z^{n,i,j}(r) h^j(r) dr.$$

Then, $U_z^{n,i,j}(r) = 0$ if $r \notin R_z$, and for $r \in R_z$ we have

$$\begin{aligned} U_z^{n,i,j}(r) &= A_j^i(X_{\phi_n}(r)) + \int_{[\psi_n(r) \wedge z, z]} \left[\frac{\partial A_h^i}{\partial x_k}(X_{\phi_n}(s)) U_{\phi_n(s)}^{n,k,j}(r) \right. \\ &\quad \left. \cdot dW_s^h + \frac{\partial B^i}{\partial x_k}(X_{\phi_n}(s)) U_{\phi_n(s)}^{n,k,j}(r) ds \right]. \end{aligned}$$

For a fixed r let us consider the processes $\{U_z^{i,j}(r), z \geq r\}$ solution of

$$\begin{aligned} U_z^{i,j}(r) &= A_j^i(X_r) + \int_{[r, z]} \left[\frac{\partial A_h^i}{\partial x_k}(X_s) U_s^{k,j}(r) dW_s^h + \frac{\partial B^i}{\partial x_k}(X_s) \right. \\ &\quad \left. \cdot U_s^{k,j}(r) ds \right] \end{aligned} \quad (2.2)$$

$i = 1, \dots, m; j = 1, \dots, d$.

Applying lemma 2.2 to the processes $\{X_z^{n,i}, U_z^{n,i,j}(r); i=1, \dots, m; j=1, \dots, d; z \geq r\}$ and $\{X_z^i, U_z^{i,j}(r); i=1, \dots, m; j=1, \dots, d; z \geq r\}$ we obtain

$$\sup_{r \in T} E \left\{ \sup_{z \geq r} |U_z(r) - U_z^{(n)}(r)|^p \right\} \xrightarrow{n \rightarrow \infty} 0.$$

In consequence, $\{DX_z^{n,i}, n \geq 1\}$ is a Cauchy sequence in L_H^p for any $p \geq 1$, and the derivative $DX_z = (DX_z^1, \dots, DX_z^m)$ satisfies

$$\langle DX_z^i, h \rangle_H = \int_{R_z} \sum_{j=1}^d U_z^{i,j}(r) \dot{h}^j(r) dr, \quad \text{a.s.}, \quad (2.3)$$

for any $h \in H$.

It remains to prove that $\{LX_z^{n,i}, n \geq 1\}$ is a Cauchy sequence in L^p for all $p \geq 1, i = 1, \dots, m$. Applying the differentiation rules we have

$$LX_z^{n,i} = \beta_n^i(z) + \int_{R_z} \left[\frac{\partial A_j^i}{\partial x_k} (X_{\phi_n(s)}^{(n)}) LX_{\phi_n(s)}^{n,k} dW_s^j + \frac{\partial B^i}{\partial x_k} (X_{\phi_n(s)}^{(n)}) \cdot LX_{\phi_n(s)}^{n,k} ds \right],$$

where,

$$\beta_n^i(z) = \int_{R_z} \left\{ \left[\frac{\partial^2 A_j^i}{\partial x_k \partial x_h} (X_{\phi_n(s)}^{(n)}) \langle DX_{\phi_n(s)}^{n,k}, DX_{\phi_n(s)}^{n,h} \rangle_H - A_j^i(X_{\phi_n(s)}^{(n)}) \right] dW_s^j + \frac{\partial^2 B^i}{\partial x_k \partial x_h} (X_{\phi_n(s)}^{(n)}) \langle DX_{\phi_n(s)}^{n,k}, DX_{\phi_n(s)}^{n,h} \rangle_H ds \right\}.$$

Consider now the stochastic differential system

$$LX_z^i = \beta^i(z) + \int_{R_z} \left[\frac{\partial A_j^i}{\partial x_k} (X_r) LX_r^k dW_r^j + \frac{\partial B^i}{\partial x_k} (X_r) LX_r^k dr \right],$$

with

$$\begin{aligned} \beta^i(z) = & \int_{R_z} \left\{ \left[\frac{\partial^2 A_j^i}{\partial x_k \partial x_h} (X_r) \langle DX_r^k, DX_r^h \rangle_H - A_j^i(X_r) \right] dW_r^j \right. \\ & \left. + \frac{\partial^2 B^i}{\partial x_k \partial x_h} (X_r) \langle DX_r^k, DX_r^h \rangle_H dr \right\}. \end{aligned}$$

It can be checked that $E \left\{ \sup_{z \in T} |\beta_n^i(z) - \beta^i(z)|^p \right\} \xrightarrow{n \rightarrow \infty} 0$ for

any $p \geq 1$, and therefore, that $\{LX_z^{n,i}, n \geq 1\}$ is a Cauchy sequence in L^p , by lemma 2.2 applied to the processes $\{X_z^{n,i}, LX_z^{n,i}, i=1, \dots, m; z \in T\}$ and $\{X_z^i, LX_z^i, i = 1, \dots, m; z \in T\}$. $\square$

Theorem 2.4. Let $X_z = (X_z^1, \dots, X_z^m)$ be the solution of the stochastic differential system (0.1), where A_j^i and B^i have bounded and continuous derivatives of any order. Assume further that the following property holds:

(P) The vector space spanned by the vector fields $A_1, \dots, A_d, A_i^\nabla A_j, 1 \leq i, j \leq d, A_i^\nabla (A_j^\nabla A_k), 1 \leq i, j, k \leq d, \dots$, at the point x has full rank.

Then for any point (s, t) with $st \neq 0$, the law of the random vector X_{st} admits a density function.

Proof. We have to check conditions (i), (ii) and (iii) of theorem 1.1. The first condition follows from lemma 2.3. Let $U_z(r)$ be the process introduced in the proof of lemma 2.3, which satisfies (2.2), and call $S_{ij} = \langle DX_z^i, DX_z^j \rangle_H$. Then, from (2.3) we have

$$S_{ij} = \int_{R_z} \sum_{k=1}^d (\xi_1^i(r, z) A_k^1(X_r) \xi_1^j(r, z) A_k^{1'}(X_r)) dr,$$

where, for any r , the process $\{ \xi_j^i(r, z), z \geq r \}$ is defined as the solution of the stochastic differential system

$$\xi_j^i(r, z) = \delta_j^i + \int_{[r, z]} \left[\frac{\partial A_h^i}{\partial x_k}(X_u) \xi_j^k(r, u) dW_u^h + \frac{\partial B^i}{\partial x_k}(X_u) \xi_j^k(r, u) du \right].$$

By Burkholder and Hölder inequalities and Gronwall's lemma it is easy to obtain the following estimate:

$$\text{For all } r, r' \leq z \text{ and } p > 2, \quad E \{ |\xi(r, z) - \xi(r', z)|^p \} \leq C |r - r'|^{p/2} \quad (2.4)$$

Therefore, by means of Kolmogorov's continuity criterium, a version of $\{ \xi(r, z), r \leq z \}$ can be chosen with almost surely continuous paths.

For each $n \geq 1$ let $\xi_j^{n,i}(r, z)$ be the process defined recursively

by the equation

$$\begin{aligned} \xi_j^{n,i}(r,z) = & \delta_j^i + \int [\psi_n(r) \wedge z, z] \left[\frac{\partial A_n^i}{\partial x_k} (X_{\phi_n(s)}^{(n)}) \xi_j^{n,k}(r, \phi_n(s)) dW_s^h \right. \\ & \left. + \frac{\partial B^i}{\partial x_k} (X_{\phi_n(s)}^{(n)}) \xi_j^{n,k}(r, \phi_n(s)) ds \right]. \end{aligned}$$

The random variables $\{\xi_j^{n,i}(r,z), n \geq 1\}$ are smooth functionals and by a slight modification of lemma 2.3 one can see that they form an approximating sequence for $\xi_j^i(r,z)$, and that this approximation is uniform in r . The same conclusion is true for the random variables $\{A_j^i(X_{\phi_n(r)}^{(n)}), n \geq 1\}$ and $A_j^i(X_r)$.

Therefore, after having noticed that $\{S_{ij}^n = \int_{R_z} \sum_{k=1}^d (\xi_1^{n,i}(r,z) A_k^1(X_{\phi_n(r)}^{(n)}) \xi_1^{n,j}(r,z) A_k^{1'}(X_{\phi_n(r)}^{(n)})) dr, n \geq 1\}$ is a sequence of smooth functionals we deduce that $\{S_{ij}^n, n \geq 1\}$ is an approximating sequence for S_{ij} and so, condition (ii) of theorem 1.1 holds.

Set $C_0 = \{A_k, 1 \leq k \leq d\}$ and for $j \geq 1$, $C_j = \{A_k^\nabla, 1 \leq k \leq d, A \in C_{j-1}\}$. By property (P) there exists a positive integer j_0 such that the linear span of $\bigcup_{j=0}^{j_0} C_j$ at the point x has dimension m .

For each $\sigma \leq s, \omega \in \Omega$, denote by $K_\sigma(\omega)$ the linear span of $\{A_k(X_{x',t}(\omega)), x' \leq \sigma, k=1, \dots, d\}$, and $K_{0+}(\omega) = \bigcap_{\sigma \leq s} K_\sigma(\omega)$. We point out the following facts: (a) $K_\sigma(\omega)$ increases with σ . (b) By the Blumenthal zero-one law, $K_{0+}(\omega)$ doesn't depend on ω a.s. (c) Let $\rho = \inf \{\sigma, \dim K_\sigma > \dim K_{0+}\}$. ρ is a strictly positive stopping time with respect to the filtration $\{F_{\sigma t}, 0 \leq \sigma \leq s\}$, and $\forall \sigma < \rho(\omega)$, $K_\sigma(\omega) = K_{0+}(\omega)$.

Assume that condition (iii) of theorem 1.1 is not satisfied, that means $P\{\omega, \inf_{|\lambda|=1} \lambda^t S(\omega) \lambda = 0\} > 0$, where $S = (S_{ij})_{i,j=1, \dots, m}$, and

consequently $P \{ \exists \lambda, |\lambda| = 1, \int_{R_z} \sum_{k=1}^d (\lambda_i \xi_{11}^i(r, z) A_k^1(X_r))^2 dr = 0 \} > 0$.

Then, using (2.4) we have

$$P \{ \exists \lambda, |\lambda| = 1, \sum_{k=1}^d (\lambda_i A_k^i(X_{\sigma_t}))^2 = 0, \forall \sigma \leq s \} > 0.$$

By (b) and (c) this implies that there exists $\lambda, |\lambda| = 1$ such that

$$P \{ \omega, \lambda \text{ orthogonal to } K_O(\omega), \forall \sigma < \rho(\omega) \} > 0,$$

in consequence $\lambda_i A_k^i(x) = 0$ for all $k=1, \dots, d$.

Applying Itô's formula in the first coordinate (see [10]), we

have

$$\begin{aligned} A_k^i(X_{\sigma_t}) &= A_k^i(x) + \int_{R_{\sigma_t}} \left\{ \frac{\partial A_k^i}{\partial x_1}(X_{\cup t}) A_h^1(X_{\cup \tau}) dW_{\cup \tau}^h + \left[\frac{\partial A_k^i}{\partial x_1}(X_{\cup t}) B^1(X_{\cup \tau}) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \frac{\partial^2 A_k^i}{\partial x_1 \partial x_r}(X_{\cup t}) \sum_{j=1}^d A_j^1(X_{\cup \tau}) A_j^r(X_{\cup \tau}) \right] d\cup d\tau \right\}. \end{aligned}$$

For any $k=1, \dots, d$, $\{\lambda_i A_k^i(X_{\sigma \wedge \rho, t}), \sigma \leq s\}$ is a continuous semimartingale, which is equal to zero on a set of positive probability, in consequence on this set

$$\int_{R_{\sigma_t}} \left(\lambda_i \frac{\partial A_k^i}{\partial x_1}(X_{\cup t}) A_h^1(X_{\cup \tau}) \right)^2 d\cup d\tau = 0,$$

$\forall h, k = 1, \dots, d, \forall \sigma \leq \rho$.

$$\text{In particular, } \lambda_i \frac{\partial A_k^i}{\partial x_1}(X_{0t}) A_h^1(X_{0\tau}) = \lambda_i (A_h^\nabla A_k)^i(x) = 0, \forall h, k =$$

$1, \dots, d$.

Repeating the same argument as before to the continuous semimartingale $\{\lambda_i \frac{\partial A_k^i}{\partial x_1}(X_{\cup \wedge \rho, t}) A_h^1(X_{\cup \wedge \rho, t}), \cup \leq s\}$, we show that $\lambda_i A^i(x) = 0$ for all $A \in C_2$, and recursively, $\lambda_i A^i(x) = 0$, for all $A \in C_j$, $j \geq 2$, which is contradictory with the hypothesis (P). $\square$

In the one parameter case, the existence of a density for the solution of a stochastic differential equation can be proved under Hörman-

der's conditions. Suppose that the vector fields $B, A_1, \dots, A_d$ have bounded derivatives of any order greater than or equal to one. Then the following assumption suffices for the existence of a density:

(H) The vector space spanned by $A_1, \dots, A_d, [A_i, A_j], 1 \leq i, j \leq d, [A_i, [A_j, A_k]], 1 \leq i, j, k \leq d, \dots$, at the point x is $\mathbb{R}^m$.

Actually a more general condition, using the Lie brackets formed with the vector field B as generators would be sufficient.

Condition (P) is weaker than (H) and, in fact, theorem 2.4 can be applied to a family of situations that did not appear in the one parameter case. Consider, for instance, the following example. Assume that $m=2, d=1, A_1^1 = 1, A_1^2 = x^1$, and $x = (0,0)$. Then condition (H) does not hold, and the one parameter solution to $X_t^1 = W_t^1, X_t^2 = \int_0^t W_s^1 dW_s^1 = \frac{1}{2} [(W_t^1)^2 - t]$ satisfies $2X_t^2 = (X_t^1)^2 - t$. However, in the two-parameter case, theorem 2.4 can be used, and, for $st \neq 0$, the joint distribution of the random variables $X_{st}^1 = W_{st}^1, X_{st}^2 = \int_{R_{st}} W_z^1 dW_z^1$ has a density on $\mathbb{R}^2$. Remark that here the stochastic differentiation rules (cf. [10]) claim that $X_{st}^2 = - \int_{R_{st} \times R_{st}} l_D(z, z') dW_z dW_{z'} + \frac{1}{2} [(W_{st}^1)^2 - st]$, and X_{st}^2 is not a function of X_{st}^1 .

References.

- [1]. Bismut, J.M. (1981). Martingales, the Malliavin Calculus and hypoellipticity under general Hörmander's conditions. Z. Wahrsch. verw. Gebiete, pp 469-505.
- [2]. Cairoli, R. (1972). Sur une équation différentielle stochastique. CRAS 274, pp 1739-1742.
- [3]. Cairoli, R. and Walsh, J.B. (1975). Stochastic integrals in the plane. Acta Math. 134, pp 111-183.
- [4]. Hajek, B. (1982). Stochastic equations of hyperbolic type and a

- two-parameter Stratonovich calculus. Ann. Probability 10, pp. 451-463.
- [5] . Ikeda, N. and Watanabe, S. (1981). Stochastic differential equations and diffusion processes. Amsterdam-Oxford-New York: North-Holland and Tokyo: Kodansha.
- [6] . Malliavin, P. (1978). Stochastic Calculus of variations and hypoelliptic operators. Proceedings of the International Conference on Stoch. differential equations of Kyoto 1976, pp. 195-263 Tokyo: Kimokuniya and New York: Wiley.
- [7] . Nualart, D. and Sanz, M. (1984). Malliavin Calculus for two-parameter Wiener functionals. Preprint.
- [8] . Shigekawa, I. (1980). Derivatives of Wiener functionals and absolute continuity of induced measures. J. Math. Kyoto Univ. 20-2 pp. 263-289.
- [9] . Stroock, D. (1981). The Malliavin Calculus, a functional analytic approach. Journal of Functional Analysis 44, pp.212-257.
- [10] . Wong, E. and Zakai, M. (1978). Differentiation formulas for stochastic integrals in the plane. Stochastic Processes and their Applications 6, pp. 339-349.

D. Nualart and M. Sanz
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via, 585. 08007 Barcelona.
Spain.

Reçu en Septembre 1984