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On existence and behaviour of the solution in quasistatic processes for rate-type viscoplastic models

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1. Introduction

Everywhere in this paper we consider the case of small deformations , we denote by ε the small strain tensor and by σ the stress tensor; the dot above a quantity will represent the derivative with respect to the time variable of that quantity.

In order to describe the behaviour of real materials like rubbers, metals,pastes,rocks and so on, many authors proposed rate-type models i.e. models involving a relation between the stress rate $\dot{\sigma}$ and the strain rate $\dot{\varepsilon}$ .For example, a semilinear rate-type constitutive equation is an equation of the form

$$\dot{\sigma} = \mathcal{E}\dot{\varepsilon} + F(\sigma,\varepsilon) \tag{1.1}$$

in which $\mathcal{E}$ is a fourth order tensor and F is a given function. Various results and mechanical interpretation concerning models of the form (1.1) may be found for instance in the book of Cristescu and Suliciu [1] (see also the references quotated there).Existence and uniqueness results for initial and boundary value problems involving (1.1) for different forms of F were obtained for instance by Duvaut and Lions [2] ch.5, Suquet [3] , [4] , [5] ,Djaoua and Suquet [6] (the case when F depends only on σ),Ionescu and Sofonea [7], Ionescu [8] (the case when a full coupling in stress and strain is involved in F).

In the case when the plastic rate of deformation depends also on a parameter χ , (1.1) may be replaced by

$$\dot{\sigma} = E\dot{\varepsilon} + F(\sigma, \varepsilon, \chi) \tag{1.2}$$

Concrete examples for constitutive equations of the form (1.2) where proposed for instance by Cristescu [9] ,[10] for rock-like materials (in these papers χ is the work / or strain / hardening parameter). Existence and uniqueness results for problems involving (1.2) with different meaning of χ were given by Nečas and Kratochvil [11], John[12], Laborde[13] (the case when F does not depend on ε) and by Ionescu [8], [14] ,Sofonea [15], [16] ,Ionescu and Sofonea [17] (the case when F depends also on ε). Energy estimates for one-dimensional problems in the study of models (1.2) in which χ is the work hardening parameter were obtained by Suliciu and Sabac [18] .

The aim of this paper is to study a quasistatic problem for materials with a rate-type constitutive equation of the form

$$\dot{\sigma} = E(\chi)\dot{\varepsilon} + F(\sigma, \varepsilon, \chi) \tag{1.3}$$

and to deduce some suplementary results in the particular case of (1.2). Constitutive equations (1.3) may be used in order to model the behaviour of real materials for which both the elastic and the plastic rate of deformation depend on a parameter χ which may be interpreted for instance as the absolute temperature or an internal state variable.The paper is organized as follows : in section 2 the mechanical problem involving (1.3) is stated and some notations and preliminaries are given; in section 3 the existence and the uniqueness of the solution is proved reducing the studied problem to an ordinary differential equation in a Hilbert space (theorem 3.1) ; in section 4 the continuous dependence of the solution with respect the data is given (theorem 4.1) and a finite-time stability result is obtained (corollary 4.1); further one we consider the case of models (1.2) (i.e. $\mathcal{E}(\chi) = \mathcal{E}$) and we deduce the continuous dependence of the solution with respect the parameter χ (theorem 4.2); this last result is used in section 5 in order to study the connection between two uncoupled thermo-viscoplastic problems using the Cattaneo heat conduction law and the well known Fourier law.

2. Problem statement and preliminaries

Let Ω be a domain in $\mathbb{R}^N$ (N = 1,2,3) with a smooth boundary $\partial\Omega = \Gamma$ and let Γ_1 be an open subset of Γ such that meas $\Gamma_1 > 0$. We denote by $\Gamma_2 = \Gamma \setminus \overline{\Gamma}_1$, ν the outward unit normal vector on Γ and by $\mathcal{S}_N$ the set of second order symmetric tensors on $\mathbb{R}^N$. Let T be a real positive constant. We consider the following mixt problem :

$$\left.\begin{aligned} &\mathrm{Div}\,\sigma + b = 0 && (2.1)\\ &\dot{\sigma} = \mathcal{E}(\chi)\dot{\varepsilon}(u) + F(\sigma,\varepsilon(u),\chi) && (2.2)\end{aligned}\right\}\quad \text{in } \Omega \times (0,T)$$

$$u = f \qquad \text{on} \quad \Gamma_1 \times (0,T) \tag{2.3}$$

$$\sigma\nu = g \qquad \text{on} \quad \Gamma_2 \times (0,T) \tag{2.4}$$

$$u(0) = u_0 \;, \qquad \sigma(0) = \sigma_0 \qquad \text{in } \Omega \tag{2.5}$$

in which the unknows are the displacement function $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^N$ and the stress function $\sigma : \Omega \times [0,T] \longrightarrow \mathcal{S}_N$. This problem represents a quasistatic problem for rate-type models of the form (2.2) in which $\mathcal{E}$ is a fourth order tensor, F is a given function, $\varepsilon(u)$ defines the linearized strain tensor (i.e. $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u))$ and χ is a parameter; equation (2.1) is the equilibrium equation in which b represents the given body force and Div σ represents the divergence of the tensor-valued function σ ; the functions f and g in (2.3),(2.4) are the given boundary data and finally the functions u_0 and σ_0 in (2.5) are the initial data.

In the sequel we denote by " . " the inner product on the spaces $\mathbb{R}^N$, $\mathbb{R}^M$ $(M\in\mathbb{N})$, S_N and by $|\cdot|$ the Euclidean norm on these spaces. The following notations are also used : $H = [L^2(\Omega)]^N$, $H_1 = [H^1(\Omega)]^N$, $\mathcal{H} = [L^2(\Omega)]^{N\times N}_s$, $\mathcal{H}_1 = \{\sigma\in\mathcal{H} \mid \text{Div}\,\sigma\in H\}$, $Y = [L^2(\Omega)]^M$. The spaces H, H_1, $\mathcal{H}$, $\mathcal{H}_1$ and Y are real Hilbert spaces endowed with the cannonical inner products denoted by $<,>_H$, $<,>_{H_1}$, $<,>_{\mathcal{H}}$, $<,>_{\mathcal{H}_1}$ and $<,>_Y$ respectively.

Let $H_\Gamma = [H^{1/2}(\Gamma)]^N$ and $\gamma_1 : H_1 \longrightarrow H_\Gamma$ be the trace map.We denote by V the closed subspace of H_1 defined by $V = \{u\in H_1 \mid \gamma_1 u=0 \text{ on } \Gamma_1\}$ and let $V_\Gamma = \gamma_1(V)$. We also denote by H'_Γ and V'_Γ the duals of H_Γ and V_Γ. The operator $\varepsilon : H_1 \longrightarrow \mathcal{H}$ given by $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u)$ is linear and continuous and moreover, since meas $\Gamma_1 > 0$, Korn's inequality holds :

$$\|\varepsilon(u)\|_{\mathcal{H}} \geq C\|u\|_{H_1} \quad \text{for all } u\in V \qquad (2.6)$$

where C is a strictly positive constant (everywhere in this paper C will represent strictly positive generic constants that may depend on Ω, Γ_1, $\mathcal{E}$, F and T and do not depend on time or on input data.

If $\sigma\in\mathcal{H}_1$ there exists $\gamma_2\sigma\in H'_\Gamma$ such that

$$<\gamma_2\sigma,\gamma_1 u>_{H'_\Gamma,H_\Gamma} = <\sigma,\varepsilon(u)>_{\mathcal{H}} + <\text{Div}\,\sigma,u>_H$$

for all $u\in H_1$. By $\sigma\nu|_{\Gamma_2}$ we shall understand the restriction of $\gamma_2\sigma$ on V_Γ and we denote by $\mathcal{V}$ the closed subspace of $\mathcal{H}_1$ defined by $\mathcal{V} = \{\sigma\in\mathcal{H}_1 \mid \text{Div}\,\sigma=0,\ \sigma\nu|_{\Gamma_2}=0\}$. Here we consider V and $\mathcal{V}$ as real Hilbert spaces endowed with the inner products of H_1 and $\mathcal{H}_1$ respectively. It is well known that $\varepsilon(V)$ is the orthogonal complement of $\mathcal{V}$ in $\mathcal{H}$ hence

$$<\sigma,\varepsilon(u)>_{\mathcal{H}} = 0 \quad \text{for all } \sigma\in\mathcal{V} \text{ and } u\in V \qquad (2.7)$$

Finally, for every real Hilbert space X we denote by $\|\cdot\|_X$ the norm on X and by $C^j(0,T,X)$ $(j=0,1)$ the spaces defined as follows :

$$C^0(0,T,X) = \{ z : [0,T] \longrightarrow X \mid z \text{ is continuous} \}$$
$$C^1(0,T,X) = \{ z : [0,T] \longrightarrow X \mid \text{there exists } \dot{z} \text{ the derivative of } z \text{ and } \dot{z} \in C^0(0,T,X) \} .$$

$C^j(0,T,X)$ are real Banach spaces, endowed with the norms

$$\| z \|_{0,T,X} = \max_{t \in [0,T]} \| z(t) \|_X \quad \text{and} \quad \| z \|_{1,T,X} = \| z \|_{0,T,X} + \| \dot{z} \|_{0,T,X} .$$

3. An existence and uniqueness result

Let us consider the following assumptions :

$$\left.\begin{array}{l}
\mathcal{E} : \Omega \times \mathbb{R}^M \times \mathcal{S}_N \longrightarrow \mathcal{S}_N \text{ and :} \\
\text{(a)} \quad \mathcal{E}(x,\chi)\sigma.\tau = \sigma.\ \mathcal{E}(x,\chi)\tau \text{ for all } \chi \in \mathbb{R}^M,\ \sigma,\tau \in \mathcal{S}_N \text{ a.e. in } \Omega \\
\text{(b) there exists } \alpha > 0 \text{ such that } \mathcal{E}(x,\chi)\sigma.\sigma \geq \alpha|\sigma|^2 \text{ for all} \\
\quad \chi \in \mathbb{R}^M,\ \sigma \in \mathcal{S}_N \text{, a.e. in } \Omega \\
\text{(c) there exists } L>0 \text{ such that } |\mathcal{E}_{ijkh}(x,\chi_1)-\mathcal{E}_{ijkh}(x,\chi_2)| \leq \\
\quad \leq L\,|\chi_1-\chi_2| \text{ for all } \chi_1,\ \chi_2 \in \mathbb{R}^M, i,j,k,h=\overline{1,N}, \text{a.e. in } \Omega \\
\text{(d) } x \longrightarrow \mathcal{E}_{ijkh}(x,\chi) \text{ is a measurable function with respect} \\
\quad \text{to the Lebesgue measure of } \Omega \text{, for all } \chi \in \mathbb{R}^M, i,j,k,h=\overline{1,N} \\
\text{(e) there exists } \beta > 0 \text{ such that } |\mathcal{E}_{ijkh}(x,\chi)| \leq \beta \text{ for all} \\
\quad \chi \in \mathbb{R}^M, i,j,k,h=\overline{1,N} \text{, a.e. in } \Omega .
\end{array}\right\} \quad (3.1)$$

$$\left.\begin{array}{l}
F : \Omega \times \mathcal{S}_N \times \mathcal{S}_N \times \mathbb{R}^M \longrightarrow \mathcal{S}_N \text{ and :} \\
\text{(a) there exists } \tilde{L} > 0 \text{ such that } |F(x,\sigma_1,\varepsilon_1,\chi_1)-F(x,\sigma_2,\varepsilon_2,\chi_2)| \leq \\
\quad \leq \tilde{L}(\,|\sigma_1-\sigma_2| + |\varepsilon_1-\varepsilon_2| + |\chi_1-\chi_2|\,) \text{ for all } \sigma_1,\ \sigma_2,\ \varepsilon_1,\ \varepsilon_2 \in \\
\quad \in \mathcal{S}_N,\ \chi_1,\ \chi_2 \in \mathbb{R}^M \text{, a.e. in } \Omega . \\
\text{(b) } x \longrightarrow F(x,\sigma,\varepsilon,\chi) \text{ is a measurable function with respect to} \\
\quad \text{the Lebesgue measure on } \Omega \text{, for all } \sigma,\ \varepsilon \in \mathcal{S}_N \text{ and } \chi \in \mathbb{R}^M \\
\text{(c) } x \longrightarrow F(x,0,0,0) \in \mathcal{H} .
\end{array}\right\} \quad (3.2)$$

$$b \in C^1(0,T,H) \ , \ f \in C^1(0,T,H_\Gamma) \ , \ g \in C^1(0,T,V'_\Gamma) \tag{3.3}$$

$$u_0 \in H_1 \quad , \quad \sigma_0 \in \mathcal{H}_1 \tag{3.4}$$

$$\mathrm{Div}\, \sigma_0 + b(0) = 0 \text{ in } \Omega \ , \ \ u_0 = \gamma_1 f(0) \text{ on } \Gamma_1, \ \ \sigma_0 \nu = g(0) \text{ on } \Gamma_2 \tag{3.5}$$

$$\chi \in C^0(0,T,Y) \tag{3.6}$$

The main result of this section is given by :

Theorem 3.1. Suppose that the hypotheses (3.1)-(3.6) are fulfilled. Then there exists a unique solution $u \in C^1(0,T,H_1)$, $\sigma \in C^1(0,T,\mathcal{H}_1)$ of the problem (2.1)-(2.5).

Remark 3.1. Let us observe that if the problem (2.1)-(2.5) has a solution (u,σ) such that $u \in C^1(0,T,H_1)$, $\sigma \in C^1(0,T,\mathcal{H}_1)$ then the hypotheses (3.3)-(3.5) are fulfilled.

Remark 3.2. In the case when $\mathcal{E}$ does not depend on χ theorem 3.1. as well as theorem 4.1. of section 4 was proved by Ionescu and Sofonea [7]. Here we extend the technique presented in [7] in the case when $\mathcal{E} = \mathcal{E}(\chi)$.

In order to prove theorem 3.1 we need some preliminary results; we start with the following lemma whose proof can be easily obtained:

Lemma 3.1. Let (3.1) and (3.6) hold. Then for all $t \in [0,T]$ $\sigma \longrightarrow \mathcal{E}(\chi(t))\sigma$ defines a linear continuous invertible operator on $\mathcal{H}$ and if we denote by $\mathcal{E}^{-1}(\chi(t))$ his inverse , we have :

$$\| \mathcal{E}(\chi(t))\sigma\|_{\mathcal{H}} \leq \beta \|\sigma\|_{\mathcal{H}} \tag{3.7}$$

$$\|\mathcal{E}^{-1}(\chi(t))\sigma\|_{\mathcal{H}} \leq \frac{1}{\alpha} \|\sigma\|_{\mathcal{H}} \tag{3.8}$$

$$<\mathcal{E}(\chi(t))\sigma, \ \sigma>_{\mathcal{H}} \geq \alpha \|\sigma\|^2_{\mathcal{H}} \tag{3.9}$$

$$<\mathcal{E}^{-1}(\chi(t))\sigma, \ \sigma>_{\mathcal{H}} \geq \frac{\alpha}{\beta^2} \|\sigma\|^2_{\mathcal{H}} \tag{3.10}$$

for all $\sigma \in \mathcal{H}$. Moreover, for every $\sigma \in \mathcal{H}$ the maps $t \longrightarrow \mathcal{E}(\chi(t))\sigma$

and $t \longrightarrow \mathcal{E}^{-1}(\chi(t))\sigma$ are continuous from $[0,T]$ to H .

Let now $X = V \times V$ be the product space endowed with the cannonical inner product $<\ ,\ >_X$ and let $\tilde{u} \in C^1(0,T,H_1)$, $\tilde{\sigma} \in C^1(0,T,\mathcal{H}_1)$ be two functions such that

$$\tilde{u} = f \quad \text{on } \Gamma_1 \times (0,T) \tag{3.11}$$

$$\text{Div}\, \tilde{\sigma} + b = 0 \quad \text{in } \Omega \times (0,T) \tag{3.12}$$

$$\tilde{\sigma}\nu = g \quad \text{on } \Gamma_2 \times (0,T) \tag{3.13}$$

(the existence and regularity of $\tilde{u}$, $\tilde{\sigma}$ can be easily proved using (3.3) and the proprieties of the trace maps γ_1 and γ_2). Let $a : [0,T] \times X \times X \longrightarrow \mathbb{R}$ and $G : [0,T] \times X \longrightarrow X$ be given by

$$a(t,x,y) = <\mathcal{E}(\chi(t))\varepsilon(u),\varepsilon(v)>_{\mathcal{H}} + <\mathcal{E}^{-1}(\chi(t))\sigma,\tau>_{\mathcal{H}} \tag{3.14}$$

$$\left.\begin{aligned} <G(t,x),y>_X = {} & <\mathcal{E}^{-1}(\chi(t))F(\sigma+\tilde{\sigma}(t),\varepsilon(u)+\varepsilon(\tilde{u}(t)),\chi(t)),\tau>_{\mathcal{H}} - \\ & - <F(\sigma+\tilde{\sigma}(t),\varepsilon(u)+\varepsilon(\tilde{u}(t)),\chi(t)),\varepsilon(v)>_{\mathcal{H}} + \\ & + <\dot{\tilde{\sigma}}(t)-\mathcal{E}(\chi(t))\varepsilon(\dot{\tilde{u}}(t)),\varepsilon(v)>_{\mathcal{H}} + \\ & + <\varepsilon(\dot{\tilde{u}}(t))-\mathcal{E}^{-1}(\chi(t))\dot{\tilde{\sigma}}(t),\ \tau>_{\mathcal{H}} \end{aligned}\right] \tag{3.15}$$

for all $x=(u,\sigma)$, $y=(v,\tau) \in X$ and $t \in [0,T]$.

Let now denote by

$$\bar{u} = u - \tilde{u} \ , \quad \bar{\sigma} = \sigma - \tilde{\sigma} \quad , \ x = (\bar{u},\bar{\sigma}) \tag{3.16}$$

$$\bar{u}_0 = u_0 - \tilde{u}(0) \ , \ \bar{\sigma}_0 = \sigma_0 - \tilde{\sigma}(0) \quad , \ x_0 = (\bar{u}_0,\bar{\sigma}_0) \tag{3.17}$$

<u>Lemma 3.2.</u> The pair $(u,\sigma) \in C^1(0,T,H_1 \times \mathcal{H}_1)$ is a solution of (2.1)-(2.5) iff $x \in C^1(0,T,X)$ is a solution of the problem

$$a(t,\dot{x}(t),y) = <G(t,x(t)),y>_X \quad \text{for all } t \in [0,T] \tag{3.18}$$

$$x(0) = x_0 \tag{3.19}$$

Proof. Using (3.11)-(3.13), (3.16), (3.17) it is easy to see that $(u,\sigma) \in C^1(0,T,H_1 \times \mathcal{H}_1)$ is a solution of (2.1)-(2.5) iff $x \in C^1(0,T,X)$ and

$$\dot{\bar{\sigma}} = \mathcal{E}(\chi)\varepsilon(\dot{\bar{u}}) + F(\bar{\sigma}+\tilde{\sigma},\varepsilon(\bar{u})+\varepsilon(\tilde{u}),\chi)+\mathcal{E}(\chi)\varepsilon(\dot{\tilde{u}})-\dot{\tilde{\sigma}} \quad \text{in} \quad \Omega\times(0,T) \tag{3.20}$$

$$\bar{u}(0) = \bar{u}_0 \quad , \quad \bar{\sigma}(0) = \bar{\sigma}_0 \quad \text{in} \quad \Omega \quad . \tag{3.21}$$

Let us suppose that (3.20) and (3.21) are fulfilled. Using (2.7) we have

$$\langle \mathcal{E}(\chi)\varepsilon(\dot{\bar{u}}),\varepsilon(v)\rangle_{\mathcal{H}} = -\langle F(\bar{\sigma}+\tilde{\sigma},\varepsilon(\bar{u})+\varepsilon(\tilde{u}),\chi),\varepsilon(v)\rangle_{\mathcal{H}} - \langle \mathcal{E}(\chi)\varepsilon(\dot{\tilde{u}}),\varepsilon(v)\rangle_{\mathcal{H}} + $$
$$+ \langle \dot{\tilde{\sigma}},\varepsilon(v)\rangle_{\mathcal{H}} \quad ,$$

$$\langle \mathcal{E}^{-1}(\chi)\dot{\bar{\sigma}},\tau\rangle_{\mathcal{H}} = \langle \mathcal{E}^{-1}(\chi)F(\bar{\sigma}+\tilde{\sigma},\varepsilon(\bar{u})+\varepsilon(\tilde{u}),\chi),\tau\rangle_{\mathcal{H}} + \langle \varepsilon(\dot{\bar{u}}),\tau\rangle_{\mathcal{H}} - \langle \mathcal{E}^{-1}(\chi)\dot{\tilde{\sigma}},\tau\rangle_{\mathcal{H}}$$

for all $y=(v,\tau) \in X$ and $t \in [0,T]$. Using now (3.14) and (3.15) we get (3.18).

Conversely, let (3.18) hold and let

$$z(t) = \dot{\bar{\sigma}}(t)-\mathcal{E}(\chi(t))\varepsilon(\dot{\bar{u}}(t))-F(\bar{\sigma}(t)+\tilde{\sigma}(t),\varepsilon(\bar{u}(t))+\varepsilon(\tilde{u}(t)),\chi(t))-$$
$$- \mathcal{E}(\chi(t))\varepsilon(\dot{\tilde{u}}(t))+\dot{\tilde{\sigma}}(t) \tag{3.22}$$

for all $t \in [0,T]$. Taking $y=(v,0) \in X$ in (3.18) and using (2.7) we get

$$\langle z(t),\varepsilon(v)\rangle_{\mathcal{H}} = 0 \quad \text{for all} \quad v \in V \text{ and } t \in [0,T]. \tag{3.23}$$

Taking $y=(0,\tau) \in X$ in (3.18) and using again (2.7) we get

$$\langle \mathcal{E}^{-1}(\chi(t))z(t),\tau\rangle_{\mathcal{H}} = 0 \quad \text{for all} \quad \tau \in \mathcal{V} \quad \text{and} \quad t \in [0,T] \tag{3.24}$$

Since the orthogonal complement of $\varepsilon(V)$ in $\mathcal{H}$ is $\mathcal{V}$, from (3.23) we get $z(t) \in \mathcal{V}$ for all $t \in [0,T]$. Thus we may put $\tau = z(t)$ in (3.24) and from (3.10) we deduce $z(t)=0$ for all $t \in [0,T]$. Using now (3.22) we get (3.20).

Hence, we proved that (3.20) is equivalent to (3.18) and we finish the proof with the remark that (3.21) is equivalent to (3.19).

Lemma 3.3. For every $t \in [0,T]$ and $x \in X$ there exists a unique element $z \in X$ such that

$$a(t,z,y) = \langle G(t,x),y\rangle_X \quad \text{for all } y\in X \tag{3.25}$$

Moreover the operator $A : [0,T]\times X \longrightarrow X$ defined by $A(t,x) = z$ is continuous and there exists $C > 0$ such that

$$\|A(t,x_1)-A(t,x_2)\|_X \leqslant C\,\|x_1-x_2\|_X \quad \text{for all } x_1,x_2\in X \text{ and } t\in[0,T] \tag{3.26}$$

<u>Proof.</u> Let $t\in[0,T]$ and $x\in X$. Using lemma 1.1 and (2.6) we get that $a(t,.,.)$ is a bilinear continuous and coercive form on X hence the existence and uniqueness of z which satisfies (3.25) follows from Lax-Milgram 's lemma.

Let now consider $t_1,t_2\in[0,T]$, $x_i=(u_i,\ \sigma_i)\in X$ and let be $z_i=(w_i,\tau_i)\in X$ defined by $z_i = A(t_i,x_i)$, $i = 1,2$. Using (3.25) we have

$$a(t_1,z_1,z_1-z_2)-a(t_2,z_2,z_1-z_2)=\langle G(t_1,x_1)-G(t_2,x_2),z_1-z_2\rangle_X \tag{3.27}$$

and from (3.9),(3.10) and (2.6) we get

$$\left.\begin{aligned}
&a(t_1,\overset{z_1}{\smile},z_1-z_2)-a(t_2,z_2,z_1-z_2) \geqslant C\,\|z_1-z_2\|_X^2 \;-\\
&-\|E(\chi(t_1))-E(\chi(t_2))\varepsilon(w_2)\|_{H}\|z_1-z_2\|_X -\\
&-\|(E^{-1}(\chi(t_1))-E^{-1}(\chi(t_2)))\tau_2\|_{H}\|z_1-z_2\|_X
\end{aligned}\right\} \tag{3.28}$$

In a similar way, from (3.7),(3.8) and (3.2) after some manipulations we get

$$\left.\begin{aligned}
&\langle G(t_1,x_1)-G(t_2,x_2),z_1-z_2\rangle_X \leqslant C\Big[\|x_1-x_2\|_X+\|\tilde\sigma(t_1)-\tilde\sigma(t_2)\|_{H}+\\
&\|\dot{\tilde\sigma}(t_1)-\dot{\tilde\sigma}(t_2)\|_{H}+\|\tilde u(t_1)-\tilde u(t_2)\|_{H_1}+\|\dot{\tilde u}(t_1)-\dot{\tilde u}(t_2)\|_{H_1}+\|\chi(t_1)-\chi(t_2)\|_Y+\\
&+\|(E^{-1}(\chi(t_1))-E^{-1}(\chi(t_2)))F(\sigma_2+\tilde\sigma(t_2),\varepsilon(u_2)+\varepsilon(\tilde u(t_2)),\chi(t_2))\|_{H}+\\
&+\|(E(\chi(t_1))-E(\chi(t_2)))\varepsilon(\dot{\tilde u}(t_2))\|_{H}+\\
&+\|(E^{-1}(\chi(t_1))-E^{-1}(\chi(t_2)))\sigma(t_2)\|_{H}\Big]\|z_1-z_2\|_X
\end{aligned}\right\} \tag{3.29}$$

So, from (3.27)-(3.29) it results

$$\left.\begin{aligned}\|z_1-z_2\|_X \leqslant C\Big[&\|(E(\chi(t_1))-E(\chi(t_2)))\varepsilon(w_2)\|_{H}+\|E^{-1}(\chi(t_1))-E^{-1}(\chi(t_2)))\tau_2\|_{H}+\\ &+\|x_1-x_2\|_X+\|\tilde{\sigma}(t_1)-\tilde{\sigma}(t_2)\|_{H}+\|\dot{\tilde{\sigma}}(t_1)-\dot{\tilde{\sigma}}(t_2)\|_{H}+\|\tilde{u}(t_1)-\tilde{u}(t_2)\|_{H_1}+\\ &+\|\dot{\tilde{u}}(t_1)-\dot{\tilde{u}}(t_2)\|_{H_1}+\|\chi(t_1)-\chi(t_2)\|_Y+\\ &+\|(E^{-1}(\chi(t_1))-E^{-1}(\chi(t_2)))F(\sigma_2+\tilde{\sigma}(t_2),\varepsilon(u_2)+\varepsilon(\tilde{u}(t_2)),\chi(t_2))\|_{H}+\\ &+\|(E(\chi(t_1))-E(\chi(t_2)))\varepsilon(\dot{\tilde{u}}(t_2))\|_{H}+\|(E^{-1}(\chi(t_1))-E^{-1}(\chi(t_2)))\dot{\tilde{\sigma}}(t_2)\|_{H}\Big]\end{aligned}\right\}\quad(3.30)$$

Using now lemma 3.1, (3.6) and the regularity in time of the functions $\tilde{u}$ and $\tilde{\sigma}$,from (3.30) we get $z_1 \longrightarrow z_2$ in X when $t_1 \longrightarrow t_2$ in $[0,T]$ and $x_1 \longrightarrow x_2$ in X . Hence A is a continuous operator.Moreover, taking $t_1=t_2$ in (3.30) we get (3.26).

Proof of theorem 3.1.

Using (3.4) and (3.5) we get that x_o defined by (3.17) belongs to X and by lemma 3.3 and the classical Cauchy-Lipschitz theorem we get that there exists a unique solution $x \in C^1(0,T,X)$ of the following Cauchy problem :

$$\dot{x}(t) = A(t,x(t)) \qquad \text{for all} \quad t\in[0,T] \tag{3.31}$$

$$x(0) = x_o \tag{3.32}$$

Theorem 3.1 follows now from the definition of the operator A and lemma 3.2.

4. The continuous dependence of the solution upon the input data

The continuous dependence of the solution of (2.1)-(2.5) with respect to the data is given by :

Theorem 4.1. Let (3.1)-(3.2),(3.6) hold and let (u_i,σ_i) be the solutions of (2.1)-(2.5) for the data $b_i,f_i,g_i,u_{oi},\sigma_{oi}$, $i=1,2$ such that (3.3)-(3.5) hold. Then there exists $C > 0$ such that

$$\left.\begin{aligned}\|u_1-u_2\|_{1,T,H_1}+\|\sigma_1-\sigma_2\|_{1,T,H_1} \leqslant C\,(&\|u_{o1}-u_{o2}\|_{H_1}+\|\sigma_{o1}-\sigma_{o2}\|_{H_1}+\\ &+\|b_1-b_2\|_{1,T,H}+\|f_1-f_2\|_{1,T,H_\Gamma}+\|g_1-g_2\|_{1,T,V'_\Gamma})\end{aligned}\right\}\quad(4.1)$$

Proof. Let $(\tilde{u}_i,\tilde{\sigma}_i)$ be two functions which satisfy (3.11)-(3.13) for the data b_i,f_i,g_i , i=1,2 and

$$\bar{u}_i=u_i-\tilde{u}_i \quad , \quad \bar{\sigma}_i=\sigma_i-\tilde{\sigma}_i \quad , \quad x_i=(\bar{u}_i,\bar{\sigma}_i) \tag{4.2}$$

$$\bar{u}_{oi}=u_{oi}-\tilde{u}_i(0), \quad \bar{\sigma}_{oi}=\sigma_{oi}-\tilde{\sigma}_i(0) \quad , \quad x_{oi}=(\bar{u}_{oi},\bar{\sigma}_{oi}) \quad ,i=1,2 \tag{4.3}$$

As it results from the proof of theorem 2.1 we have

$$\dot{x}_i(t) = A_i(t,x_i(t)) \qquad \text{for all} \quad t\in[0,T] \tag{4.4}$$

$$x_i(0) = x_{oi} \tag{4.5}$$

where the operators A_i are defined by lemma 3.3 replacing $(\tilde{u},\tilde{\sigma})$ by $(\tilde{u}_i,\tilde{\sigma}_i)$ in (3.14),(3.15),i=1,2. In a similar way in which (3.30) was proved we obtain

$$\|A_1(t,y_1)-A_2(t,y_2)\|_X \leqslant C(\|y_1-y_2\|_X+\|\tilde{u}_1(t)-\tilde{u}_2(t)\|_{H_1}+\|\dot{\tilde{u}}_1(t)-\dot{\tilde{u}}_2(t)\|_{H_1}+$$
$$+\|\tilde{\sigma}_1(t)-\tilde{\sigma}_2(t)\|_{\mathcal{H}}+\|\dot{\tilde{\sigma}}_1(t)-\dot{\tilde{\sigma}}_2(t)\|_{\mathcal{H}}) \quad \text{for all} \quad t\in[0,T] \text{and} \quad y_1,y_2\in X$$

which implies

$$\|A_1(t,x_1(t))-A_2(t,x_2(t))\|_X \leqslant C(\|x_1(t)-x_2(t)\|_X+\|\tilde{u}_1-\tilde{u}_2\|_{1,T,H_1}+$$
$$+\|\tilde{\sigma}_1-\tilde{\sigma}_2\|_{1,T,\mathcal{H}_1}) \quad \text{for all} \quad t\in[0,T] \tag{4.6}$$

Using (4.4) and (4.6) we get

$$<\dot{x}_1(t)-\dot{x}_2(t),x_1(t)-x_2(t)>_X \leqslant C(\|x_1(t)-x_2(t)\|_X+\|\tilde{u}_1-\tilde{u}_2\|_{1,T,H_1}+$$
$$+\|\tilde{\sigma}_1-\tilde{\sigma}_2\|_{1,T,\mathcal{H}_1})\ \|x_1(t)-x_2(t)\|_X \qquad \text{for all} \quad t\in[0,T] \ ;$$

hence by (4.5) and a Gronwall-type lemma it follows

$$\|x_1(t)-x_2(t)\|_X \leqslant C(\|x_{01}-x_{02}\|_X+\|\tilde{u}_1-\tilde{u}_2\|_{1,T,H_1}+\|\tilde{\sigma}_1-\tilde{\sigma}_2\|_{1,T,\mathcal{H}_1}) \tag{4.7}$$

for all $t\in[0,T]$.

Using again (4.4) and (4.6) , from (4.7) we deduce

$$\|\dot{x}_1(t)-\dot{x}_2(t)\|_X \leqslant C(\|x_{01}-x_{02}\|_X+\|\tilde{u}_1-\tilde{u}_2\|_{1,T,H_1}+\|\tilde{\sigma}_1-\tilde{\sigma}_2\|_{1,T,\mathcal{H}_1}) \tag{4.8}$$

for all $t\in[0,T]$

hence from (4.2),(4.3),(4.7) and (4.8) we get

$$\|u_1-u_2\|_{1,T,H_1}+\|\sigma_1-\sigma_2\|_{1,T,\mathcal{H}_1} \leq C(\|u_{o1}-u_{o2}\|_{H_1}+\|\sigma_{o1}-\sigma_{o2}\|_{\mathcal{H}_1}+ \tag{4.9}$$
$$+\|\tilde{u}_1-\tilde{u}_2\|_{1,T,H_1}+\|\tilde{\sigma}_1-\tilde{\sigma}_2\|_{1,T,\mathcal{H}_1}).$$

Using standard arguments from (3.11)-(3.13) we get

$$\|\tilde{u}_1-\tilde{u}_2\|_{1,T,H_1}+\|\tilde{\sigma}_1-\tilde{\sigma}_2\|_{1,T,\mathcal{H}_1}\leq C(\|b_1-b_2\|_{1,T,H}+ \|f_1-f_2\|_{1,T,H_\Gamma}+$$
$$+\|g_1-g_2\|_{1,T,V'_\Gamma})$$

hence (4.9) implies (4.1).

In particular, from theorem 4.1 we deduce

Corollary 4.1. Let the hypotheses of theorem 4.1 hold. If $b_1=b_2$, $f_1=f_2$, $g_1=g_2$ then

$$\|u_1-u_2\|_{1,T,H_1}+\|\sigma_1-\sigma_2\|_{1,T,\mathcal{H}_1}\leq C(\|u_{o1}-u_{o2}\|_{H_1}+\|\sigma_{o1}-\sigma_{o2}\|_{\mathcal{H}_1}) \tag{4.10}$$

Remark 4.1. From (4.10) we deduce the finite-time stability of every solution on (2.1)-(2.5) (for definitions in the field see for instance Hahn [19] ch.5). Some unidimensional examples can be considered in order to prove that generally stability does not hold (see Ionescu and Sofonea [17]).

Further one we consider the case when $\mathcal{E}$ in (2.1) does not depend on χ and we replace (3.1) by the following assumption :

$$\left.\begin{array}{ll} \text{(a)} & \mathcal{E}(x)\sigma.\tau=\sigma.\mathcal{E}(x)\tau \quad \text{for all } \sigma,\tau\in\mathcal{S}_N \text{, a.e. in } \Omega \\ \text{(b)} & \text{there exists } \alpha>0 \text{ such that } \mathcal{E}(x)\sigma.\sigma\geq\alpha|\sigma|^2 \text{ for all } \sigma\in\mathcal{S}_N, \\ & \text{a.e. in } \Omega \;, \\ \text{(c)} & \mathcal{E}_{ijkh}\in L^\infty(\Omega) \quad \text{for all } i,j,k,h=\overline{1,N} \end{array}\right\} \tag{4.11}$$

We have the following result :

Theorem 4.2. Let (4.11),(3.2)-(3.5) hold and let (u_i,σ_i) be the solutions of (2.1)-(2.5) for $\chi=\chi_i$, i=1,2 such that (3.6) hold. Then there exists $C > 0$ such that

$$\|u_1-u_2\|_{1,T,H_1} + \|\sigma_1-\sigma_2\|_{1,T,\mathcal{H}_1} \leq C\,\|\chi_1-\chi_2\|_{0,T,Y} \tag{4.12}$$

Proof. Let $(\tilde{u},\tilde{\sigma})$ be two functions which satisfy (3.11)-(3.13) and

$$\bar{u}_i=u_i-\tilde{u} \quad , \quad \bar{\sigma}_i=\sigma_i-\tilde{\sigma} \quad , \quad x_i=(\bar{u}_i,\bar{\sigma}_i) \quad , \quad i=1,2 \tag{4.13}$$

As it results from the proof of theorem 3.1 we have

$$\dot{x}_i(t) = A_i(t,x_i(t)) \qquad \text{for all} \quad t\in[0,T] \tag{4.14}$$

$$x_i(0) = x_0 \tag{4.15}$$

where x_0 is given by (3.13) and the operators A_i are defined by lemma 3.3 replacing χ by χ_i in (3.14),(3.15), $i=1,2$. In a similar way in which (3.30) and (4.6) were proved we obtain

$$\|A_1(t,x_1(t))-A_2(t,x_2(t))\|_X < C(\|x_1(t)-x_2(t)\|_X+\|\chi_1(t)-\chi_2(t)\|_Y)$$

for all $t\in[0,T]$ hence from (4.14),(4.15) using a standard technique we get

$$\|x_1(t)-x_2(t)\|_X\leq C\int_0^t \|\chi_1(s)-\chi_2(s)\|_Y ds \tag{4.16}$$

$$\|\dot{x}_1(t)-\dot{x}_2(t)\|_X\leq C(\|x_1(t)-x_2(t)\|_X+\|\chi_1(t)-\chi_2(t)\|_Y) \tag{4.17}$$

for all $t\in[0,T]$. Theorem 4.2 follows now from (4.13),(4.16) and (4.17).

5. A convergence result in the study of uncoupled thermo - viscoplastic processes

In this section we study uncoupled thermo-viscoplastic processes i.e. problems of the form (2.1)-(2.5) in which the parameter χ is denoted by θ and it is interpreted as the absolute temperature. In order to model heat-propagation processes, differents laws relating the temperature field θ and the heat flux q can be considered. One of them is the well known Fourier law

$$q = k\nabla\theta \tag{5.1}$$

in which $k > 0$ is the heat conduction coefficient and $\nabla\theta$ is the temperature gradient.As well known,this law implies a very unpleasant feature: a thermal disturbance at any point in the body is felt instantaneously at ever other point; or in therms less precise then evocative, the speed of the propagation of thermal signals is infinite.To remove the aforementioned paradox Cattaneo [20] by means of statistical consideration,proposed a generalization of the Fourier law in the homogenuous and isotropic case which is

$$\xi\dot{q} + q = k\nabla\theta \tag{5.2}$$

where $\xi > 0$ is the thermal relaxation time and it represents the time-lag required to create steady-state heat conduction in an element of volume when a temperature gradient is suddenly imposed on that element (for detailed references on this subject see for instance Cristescu and Suliciu [1] p.190).

Let us now consider a heat propagation processes in $\Omega\times(0,T)$ and let us denote by θ the temperature field in the context of Fourier law (5.1) and by θ_ξ the temperature field in the context of Cattaneo law (5.2) . Under appropriate hypothesses on the date we may assume that $\theta,\theta_\xi\in C^0(0,T,L^2(\Omega))$. Suppose now that (4.9),(3.2)-(3.5) hold and let (u,σ) be the solution of (2.1)-(2.5) for $\chi=\theta$ and (u_ξ,σ_ξ) be the solution of (2.1)-(2.5) for $\chi=\theta_\xi$ (we take $\mathcal{E}(\chi)=\mathcal{E}$ in (2.2) and M=1 in the definition of the space Y). Using theorem 4.2 we get

$$\|u_\xi-u\|_{1,T,H_1} + \|\sigma_\xi-\sigma\|_{1,T,\mathcal{H}_1} \leqslant C(\|\theta_\xi-\theta\|_{0,T,L^2(\Omega)})$$

and since $\theta_\xi \longrightarrow \theta$ in $C^0(0,T,L^2(\Omega))$ when $\xi\longrightarrow 0$ (see for example [15], [21]) we get

$$u_\xi \longrightarrow u \quad \text{in } C^1(0,T,H_1) \quad , \quad \sigma_\xi \longrightarrow \sigma \quad \text{in } C^1(0,T,\mathcal{H}_1) \quad \text{when } \xi\longrightarrow 0 \tag{5.3}$$

The physical signifiance of the previous convergence result is the following : at high temperatures (room temperatures for the materials considered) when the relaxation time ξ becomes very short (i.e. when we are dealing with "higly conductive heat materials") the uncoupled thermo-viscoplastic problem (2.1)-(2.5) can be considered in the context of Fourier's theory. This means that the relaxation time ξ does not influence the quasistatic processes in "higly conductive heat" materials.

Remark 5.1. A relatively simple example of model of the form (1.2) satisfying (3.2) may be found for instance in [15], [21]. Moreover, the convergence result (5.3) improves a result obtained in [21] .

References

[1] Cristescu N and Suliciu I 1982 *Viscoplasticity* (Bucharest: Ed.Tehnica, Martinus Nijhoff,The Netherlands)

[2] Duvaut G and Lions L 1972 *Les inéquations en mécanique et en physique* (Paris : Dunod)

[3] Suquet P 1981 Sur les équations de la plasticité; existence et régularité des solutions.*J.Mec.Téor.Appl.* 20 3-39

[4] Suquet P 1981 Evolution problems for a class of dissipative materials *Quart.Appl.Math.* 38 391-414

[5] Suquet P 1982 Plasticité et homogenisation, *Ph D Thesis* University of Paris VI

[6] Djaoua M and Suquet P 1984 Evolution quasistatique des milieux visco-plastiques de Maxwell-Norton. *Math.Methodes Appl.Sci.* 6 192-205

[7] Ionescu I R and Sofonea M 1988 Quasistatic processes for elastic-viscoplastic materials *Quart. Appl.Math.* 2 229-43.

[8] Ionescu I R 1988 Dynamic processes for a class of elastic-viscoplastic materials.*Preprint series in Math.* INCREST , Bucharest ,64

[9] Cristescu N 1987 Elastic/ Viscoplastic constitutive equations for ROCK. *Int.J.Rock.Mech.Min.Sci.Geomech.* 24 5 272-82

[10] Cristescu N 1989 *Rock Rheology* (Boston:Kluwer Academic Publ.)

[11] Nečas J and Kratochvil J 1973 On existence of the solution of boundary value problems for elastic-inelastic solids.*Comment.Math.Univ.Carolinae* 14 755-60

[12] John O 1974 On the solution of displacement boundary value problem for elastic-inelastic materials.*Appl.Math.* 19 65-71

[13] Laborde P 1979 On viscoplasticity with hardening.*Numer.Funct.Anal.Optim.* 1 315-39

[14] Ionescu I R 1990 Functional and numerical methods in viscoplasticity. *Ph D Thesis* Univ. of Bucharest (in Romanian)

[15] Sofonea M 1988 Functional methods in thermo-elasto-visco-plasticity. *Ph D Thesis* Univ. of Bucharest (in Romanian).

[16] Sofonea M 1990 Some remarks on the behaviour of the solution in dynamic processes for rate-type models (to appear in *ZAMP*)

[17] Ionescu I R and Sofonea M *Functional and numerical methods in viscoplasticity* (to appear in Oxford University Press 1992)

[18] Suliciu I and Sabac M 1988 Energy estimates in onedimensional rate-type viscoplasticity. *J.Math.Anal.Appl.* 131 2 354-72

[19] Hahn W 1967 *Stability of motion* (Berlin: Springer-Verlag)

[20] Cattaneo C 1948 Sulla conduzione del calore. *Atti Sem.Mat.Fis.Univ. Modena* 3 83- 101

[21] Sofonea M 1989 On existence and behaviour of the solution of two uncoupled thermo-elastic-visco-plastic problems.*Ann.Univ.Bucharest* 38 1 56-65

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