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Consuelo MARTINEZ LOPEZ

INTRODUCTION

The usual order relation in Boolean rings is extended to commutative semiprime rings, [1], when it is expressed as $a \leqslant b$ if and only if $ab = a^2$. In this case $\leqslant$ makes A an ordered semigroup and the ring is isomorphic to a direct product of division rings if and only if $\leqslant$ is an order relation such that the ring is hyperatomic and orthogonally complete.

Chacron [4] extended the above result to associative non-commutative rings, using that a reduced associative ring R can be embedded into a direct product of skewdomains. Abian, in [2], obtained the same results for not necessarily associative or commutative rings satisfying the property (a) given by :

(a) A has no nilpotent element of index 2, and a product of elements of A which is equal to zero remains equal to zero no matter how its factors are associated.

Finally, Myung and Jimenez, in [6], extended the same results to any alternative ring without nonzero nilpotent elements and they showed that the same results do not hold for Jordan rings, because the ring Q of real quaternions under the product $a.b = \frac{1}{2}(ab + ba)$ becomes a Jordan ring Q^+ without nonzero nilpotent elements, but the relation $\leqslant$ is not a partial order on Q^+ . Also Q^+ is a Jordan division ring. In [5], we define a new relation in Jordan rings by :

$$a \leqslant b \text{ if and only if } ab = a^2 \text{ and } a^2b = ab^2 = a^3$$

(if A is associative, this relation coincides with Abian's relation) and we prove that $\leqslant$ is a partial order in a Jordan ring R without nonzero nilpotent elements and satisfying the property (P) given by :

(P) If $(x,x,y) = 0$ then $(xy,x,y) = 0$ for any $x,y \in R$

A structure theorem similar to the above mentioned ones for the associative and alternative cases, is then obtained.

Also, a result of Bunce assures that every JB-algebra satisfies the property (P). So in every JB-algebra there are two order relations : the usual order relation defined by the positive cone, $A_+ = A^2$ and the new relation which we have defined.

1. PRELIMINARIES

If R is a Jordan ring in which $2x = 0$ implies $x = 0$ for all $x \in R$, we define the following relation :

$$x \leqslant y \text{ if and only if } xy = x^2,\ x^2y = xy^2 = x^3$$

This is equivalent to : $x \leqslant y$ if and only if $xy = x^2$ and x and y generate an associative subalgebra.

It is clear that if $\leqslant$ is a partial order in R, then there are no nilpotent elements ($\neq 0$) in R. Also $\leqslant$ is always a reflexive relation and is antisymmetric when R has no nonzero nilpotent elements.

<u>Theorem</u> 1. Let R be a Jordan ring without nonzero nilpotent elements and satisfying property (P) given by :

(P) If $(x,x,y) = 0$ then $(xy,x,y) = 0$ for any $x,y \in R$

Then $\leqslant$ is a partial order in R.

<u>Theorem</u> 2. Let R be a special Jordan ring whose special universal envelope is an associative algebra without nilpotent elements. Then $\leqslant$ is a partial order on R.

<u>Observation</u>. The above result cannot be modified in the sense that there is

a special Jordan algebra without nonzero nilpotent elements with a special universal envelope having nonzero nilpotent elements.

Consider the JB-algebra R of symmetric real matrix with the Jordan product $M.N = \frac{1}{2}(MN + NM)$. Evidently R has no nonzero nilpotent elements. If the special universal envelope A was a reduced associative algebra, then for an idempotent E of R, E would also be an idempotent of A. But in a reduced associative algebra the idempotents commute with every element. That is not the case with

$$E = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \text{ because if } M = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \quad EM = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \neq \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} = ME.$$

So R Jordan algebra without nilpotent elements does not imply that the special universal envelope is a reduced associative algebra.

2. NEW ORDER IN JB-ALGEBRAS

After theorem 1 of [5], in order to see that the relation $\leqslant$ above defined is a partial order in any JB-algebra, it is sufficient to prove that any JB-algebra satisfies the condition (P).

This is a consequence of the following result of Bunce (cf. [3]).

<u>Lemma</u> 3. ([3]) Let A be a JB-algebra and a,b elements of A. Then the following conditions are equivalent :

i) $U_a(b) = a^2.b$;
ii) a and b operator commute in A, that is, $L_aL_b = L_bL_a$ on A ;
iii) The JB-subalgebra C(a,b) of A generated by a and b is associative ;

So, we have :

<u>Theorem</u>. Every JB-algebra A satisfies the condition (P). Therefore the relation $\leqslant$ defined above is a partial order on A.

<u>Proof</u>. If $(x,x,y)=0$, then $x.(x.y)=x^2.y$ and so $U_x(y) = 2x.(y.x)-x^2.y = x^2.y$. By Bunce's result C(x,y) is associative. In particular $(x.y,x,y) = 0$. Since the condition $\|x^2\| = \|x\|^2$ assures that every JB-algebra has no nonzero

nilpotent elements, it is clear, by theorem 1, that the relation $\leqslant$ is a partial order.

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