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## ON SOME REPRESENTATIONS OF HYPERGROUPS

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### Abstract

A class of hypermatrices to represent hypergroups is introduced.  
Application on class of P-hypergroups is given.

### 1. INTRODUCTION

The problem of representations of hypergroups by hypermatrices can be tackled in two ways : First, by using hypermatrices and with the usual multiplication. Second, by using usual matrices with a hypermultiplication of matrices. The former case requires a special hyperring and the latter a permanent hyperoperation. Both above problems are almost open. In this paper we mainly deal with the first problem.

The *hypergroup* in the sense of Marty is the largest class of multivalued systems that satisfies the group-like axioms:  $\langle H, \cdot \rangle$  is a hypergroup if  $\cdot : H \times H \longrightarrow \mathfrak{p}(H)$  is an associative hyperoperation which satisfies the reproduction axiom  $hH = Hh = H$ , for every  $h$  of  $H$ .

D-hypergroups, cogroups, polygroups, canonical hypergroups, complete hypergroups, join spaces, etc. are special classes of the hypergroup of Marty [1],[2],[4],[6], but also there are some related hypergroups introduced and studied as in [12].

The *hyperring in the general sense* [10] is the largest class of multivalued systems that satisfies the ring-like axioms :  $\langle R, +, \cdot \rangle$  is a hyperring in the general sense if  $\langle R, + \rangle$  is a hypergroup of Marty,  $(\cdot)$  is associative hyperoperation and the distributive law  $x(y+z) \subset xy+xz$ ,  $(x+y)z \subset xz+yz$  is satisfied for every  $x,y,z$  of  $R$ . *Additive hyperring*

is the one of which only  $(+)$  is a hyperoperation, *multiplicative hyperring* is the one which only  $(\cdot)$  is a hyperoperation. If the equality in the distributive law is valid then the hyperring is called *strong* or *good*. The most known class of hyperrings is the additive hyperring in the sense of Krasner.

In the following we shall use the generalization of the hyperring by dropping the reproduction axiom:  $\langle R, +, \cdot \rangle$  will be called *semihyperring* if  $(+), (\cdot)$  are associative hyperoperations where  $(\cdot)$  is distributive with respect to  $(+)$ . The rest definitions are analogous. We remark that the definitions presending in the following as well the results on hyperrings are also true for semihyperrings. Note that in the definition of semihyperring we do not require the commutativity even for the hyperoperation  $(+)$ .

Remark: It is not known yet a general definition of a hyperfield that contains all the known classes of hyperfields.

*Hypermatrices* are called the matrices with entries, elements of a semihyperring. The product of two hypermatrices  $(a_{ij}), (b_{ij})$  is the hyperoperation given in the usual manner

$$(a_{ij})(b_{ij}) = \{ (c_{ij}) : c_{ij} \in \sum_{k=1}^n a_{ik} b_{kj} \}$$

Our problem is the following one: For given hypergroup  $H$ , find a semihyperring  $R$  such that to have a representation of  $H$  by hypermatrices with entries from  $R$ . Recall that if  $M_R = \{(a_{ij}) : a_{ij} \in R\}$ , then a map  $T: H \rightarrow M_R : h \rightarrow T(h)$  is called a *representation* if

$$T(h_1 h_2) \subset T(h_1) T(h_2) \text{ for every } h_1, h_2 \text{ of } H.$$

We are more interested in the case when the *good condition*

$$T(h_1) \cdot T(h_2) = \{ T(h) : h \in h_1 h_2 \} = T(h_1 h_2)$$

is valid for every  $h_1, h_2$  of  $H$ , in which case we can obtain an induced representation  $T^*$  for the hypergroup algebra of  $H$ , see [11].

## 2. THE FUNDAMENTAL EQUIVALENCE RELATIONS

Let  $\langle H, \cdot \rangle$  be a hypergroup. The *fundamental equivalence relation*

$\beta_H^*$  is the transitive closure of the relation  $\beta_H$  defined by setting

$$a \beta_H b \text{ iff } \exists x_1, \dots, x_n \in H : \{a, b\} \subset \prod_{i=1}^n x_i$$

An equivalent definition is the following:

$a \beta_H^* b$  iff  $\exists x_0, x_1, \dots, x_k; h_{i_1}, \dots, h_{i_k} \in H$  with  $x_0 = a, x_k = b$ ,  
and  $I_1, \dots, I_k$  finite sets of indices such that

$$\{x_{j-1}, x_j\} \subset \prod_{i_j \in I_j} h_{i_j}, \quad j=1, \dots, k.$$

The basic property of the fundamental relation is that it is the minimum equivalence relation such that  $H/\beta_H^*$  is a group, the *fundamental group* (see [2],[3],[9]). So, denoting by  $F_H(x)$  the *fundamental class of the element x*, we have the *fundamental property*:

$F_H(a).F_H(b) = F_H(ab) = F_H(x)$  for every  $x \in ab$ . The kernel of the canonical map  $\phi: H \rightarrow H/\beta_H^*$  is called *core* and it is denoted by  $\omega_H$ , if  $|\omega_H|=n$  then  $H$  is called *n-hypergroup*.

Remark The fundamental equivalence relations on hypergroupoids, semihypergroups, hypergroups etc. can lead us to stricter algebraic domains from given ones, see [3].

Now let  $\langle R, +, \cdot \rangle$  be a hyperring in the general sense and  $n \neq 0$  a natural number. The *n-fundamental relation*, denoted by  $\underline{n}$ , is defined as follows [10]:

- (i)  $a \underline{n} a$  for every  $a$  of  $R$
- (ii)  $a \underline{n} b, a \neq b$  iff  $\exists x_0 = a, x_1, \dots, x_k = b \in R$ , and  $i_1, \dots, i_k \in \mathbb{N} \setminus \{0, 1\}$ :

$$\{x_{s-1}, x_s\} \subset \sum_{\nu=1}^{i_s-1} \left( \prod_{\mu=1}^{i_s} x_{s\mu\nu} \right), \quad s=1, \dots, k \quad \text{and} \quad x_{s\mu\nu} \in R.$$

The *n-fundamental relation* is an equivalence relation and let us denote by  $F_n$  the *n-quotient set* and by  $F_n(x)$  the *n-fundamental class of x*. The 1-fundamental relation coincides with the fundamental relation  $\beta^*$  defined only on the multiplication.

### THEOREM 1

The  $\langle F_n, \oplus, \circ \rangle$  is an additive hyperring where

$$F_n(x) \oplus F_n(y) = \{F_n(z) : z \in F_n(x) + F_n(y)\},$$

$$F_n(x) \circ F_n(y) = \{F_n(z) : z \in F_n(x) \cdot F_n(y)\}.$$

### Proof

The above hyperoperations are associative, the fundamental property for the multiplication is satisfied i.e.  $F_n(x) \circ F_n(y) = \{F_n(z)\}, \forall z \in xy$ . The distributivity is not strong. For a complete proof see [10].

THEOREM 2

A necessary condition in order to have a representation  $T$  of the hypergroup  $H$  by  $n \times n$  hypermatrices over the hyperring  $R$  is the following:

For any fundamental class  $F_n(x)$ ,  $x \in H$ , we must have elements  $a_{ij}$  of  $R$ ,  $i, j=1, \dots, n$  such that

$$T(F_H(x)) \subset \{ (a'_{ij}) : a'_{ij} \in F_n(a_{ij}), i, j=1, \dots, n \}.$$

Proof See [10].

3. A CLASS OF HYPERMATRICES

Let  $(G, \cdot)$  be a group with  $|G|=n$ , and  $\{ X_g \}_{g \in G}$  be a disjoint family of sets indexed of the set  $G$ . We set  $X = \bigcup_{g \in G} X_g$  and we consider one more element, the zero element  $0$ , and set  $X^0 = X \cup \{0\}$ . On  $X^0$  we define the (hyper)multiplication as follows

$$0 \cdot x = x \cdot 0 = 0 \text{ for all } x \in X^0$$

$$x_r \cdot x_s = X_{rs} \text{ for all } x_r \in X_r \text{ and } x_s \in X_s .$$

This hyperoperation is associative. The  $X^0$  becomes a semihyperring, strong distributive, with  $0$  the additively absolute scalar identity and where the (hyper)addition on the rest elements can be defined in some ways as the following ones (c.f. [10])

- (a)  $x_r + x_s = \{ x_r, x_s \}$
- (b)  $x_r + x_s = X_r \cup X_s$  for all  $x_r \in X_r$  and  $x_s \in X_s$
- (c)  $x_r + x_s = X$

THEOREM 3

All the above semihyperrings have

$$F_1 = \{ \{0\} \} \cup \{ X_g : g \in G \} \text{ and } F_n = \{ \{0\}, X \} , n > 1 .$$

Proof

The  $0$  is not contained in any sum of products with any other element so  $F_n(0) = \{0\}$  for every  $n$ .

One observes that the 1-fundamental relation coincides with the  $\beta^*$  relation on the multiplication  $(\cdot)$ . Therefore the 1-fundamental classes are  $\{0\}$  and the family  $\{ X_g \}_{g \in G}$ .

Now let  $n > 1$ . We observe that the smallest hyperoperation is the case (a) so it has the smallest equivalence classes. Therefore it is enough to prove the theorem only in this case.

If  $e$  be the unit element of  $G$  we have for all  $r, s \in G$  and  $x_r \in X_r, x_s \in X_s$

$$\{x_r, x_s\} \subset X_r \cup X_s = X_r + X_s = \underbrace{X_{re} + \dots + X_{re}}_{n-1 \text{ times}} + X_{se} = \underbrace{x_r x_e + \dots + x_r x_e}_{n-1 \text{ times}} + x_s x_e$$

That means that  $x_r \cdot x_s = x_r x_s$ . So  $F_n(x_r) = F_n(x_s) = X$ . Q.E.D.

**Remark** If  $G$  is imbedable in the multiplicative group of a field then also the addition can be defined by setting  $x_r + x_s = X_{r+s}$  then  $X$  becomes a hyperring. In this case  $F_n(x_r) = X_r, \forall x_r \in X_r$ , and  $F_n(0) = \{0\}$ .

Let  $f: g \rightarrow M_g$  be the regular representation of the group  $G$  by  $n \times n$  permutation matrices. For every permutation matrix  $M_g = (g_{ij})$  we consider the set of *associated* hypermatrices

$$\underline{M}_g = \{ A = (a_{ij}) : a_{ij} = 0 \text{ if } g_{ij} = 0 \text{ and } a_{ij} \in X_g \text{ if } g_{ij} = 1 \}$$

We observe that  $|\underline{M}_g| = n^{|X_g|}$ .

**THEOREM 4**

The set  $\underline{M} = \bigcup_{g \in G} \underline{M}_g$  is a hypergroup with respect to the usual multiplication of hypermatrices.

**Proof**

Let  $r, s \in G$  and  $A = (a_{ij}) \in \underline{M}_r, B = (b_{ij}) \in \underline{M}_s$  we have

$$A \cdot B = \{ C = (c_{ij}) : c_{ij} \in \sum_{k=1}^n a_{ik} b_{kj} \} = \underline{M}_{rs}$$

because every non zero product of the form  $a_{ik} b_{kj}$  is equal to  $X_{rs}$ , and the permutation matrix corresponding to the product  $A \cdot B$  is  $M_{rs}$ .

Therefore if  $A \in \underline{M}_r, B \in \underline{M}_s, C \in \underline{M}_t$  we have

$$A(BC) = A(\underline{M}_{st}) = \bigcup_{D \in \underline{M}_{st}} AD = \underline{M}_{rst}$$

Similarly  $(AB)C = \underline{M}_{rst}$ , so  $(.)$  is associative.

Moreover for all  $A \in \underline{M}_g$  we have

$$\underline{A} \cdot \underline{M} = \bigcup_{g \in G} \underline{A} \cdot \underline{M}_g = \bigcup_{g \in G} \underline{M}_{rg} = \bigcup_{g \in G} \underline{M}_g = \underline{M}$$

and  $\underline{M} \cdot \underline{A} = \underline{M}$ . Therefore  $\langle \underline{M}, . \rangle$  is a hypergroup.

**Remark**

It is obvious that  $|\underline{M}/\beta_{\underline{M}}^*| = n$ . More precisely for all  $A$  of  $\underline{M}_r$  we have  $F_{\underline{M}}(A) = \underline{M}_r$ . Moreover since  $AB = \underline{M}_{rs}$  for every  $A$  of  $\underline{M}_r$  and  $B$  of  $\underline{M}_s$  we obtain that  $\langle \underline{M}, . \rangle$  is a complete hypergroup [2].

We also notice that if  $|X_g| = v = \text{constant}$  then we have  $|\underline{M}| = n^{v+1}$ .

4. A REPRESENTATION

Let  $\langle H, \cdot \rangle$  be a hypergroup with  $H/\beta_H^*$  be the fundamental group which we suppose that is of finite order  $n$ . Let  $F_H(x)$  be the fundamental class of the element  $x$  and let  $h_1, \dots, h_n$  be selected elements of each class. We consider the semihyperring  $\langle X^0, \oplus, \circ \rangle$  introduced above where  $G = H/\beta_H^*$ ,  $X_{h_i} = F_H(h_i)$ ,  $i=1, \dots, n$ , so  $X^0 = H^0 = H \cup \{0\}$ . Therefore we actually have  $G \cong \{h_1, \dots, h_n\}$ .

The hyperoperation  $\circ$  is given by the relations  $0 \circ x = x \circ 0 = 0$ , for  $x \in H^0$  and  $x \circ y = F_H(x) \cdot F_H(y) = F_H(xy) = F_H(z)$ , where  $z \in xy$  and  $x, y \in H$ .

The hyperoperation  $\oplus$  is one of the above (a), (b) or (c).

For the group  $G$  and the above semihyperring we consider the hypergroup  $\underline{M}$  of the associated hypermatrices.

We have, if  $H$  be finite,  $|\underline{M}_{h_i}| = n^{|F_H(h_i)|}$  and  $|\underline{M}| = \sum_{i=1}^n n^{|F_H(h_i)|}$ .

We consider any map  $T: H \rightarrow \underline{M}$  such that if  $F_H(x) = F_H(h_i)$  we have  $T(F_H(x)) \subset \underline{M}_{h_i}$ . Then  $T$  is a representation of  $H$  by hypermatrices.

Since  $\underline{M}$  is complete it is possible to represent hypergroups with order greater than  $|H|$ . We give such a construction in the following:

Let us consider the maximal  $k_i$ 's such that

$$|F_H(h_i)|^{k_i} \leq n^{|F_H(h_i)|} \text{ and let } k = \min \{k_1, \dots, k_n\}.$$

We consider the union  $K$  of the cartesian products  $(F_H(h_i))^k$ , i.e.

$$K = \{ (x_1, \dots, x_k) \in (F_H(h_i))^k : i=1, \dots, n \}.$$

In  $K$  we define a hyperoperation  $*$  as follows :

$$\text{if } (x_1, \dots, x_k) \in (F_H(h_i))^k \text{ and } (y_1, \dots, y_k) \in (F_H(h_j))^k \text{ then}$$

$$(x_1, \dots, x_k) * (y_1, \dots, y_k) = (F_H(h_i h_j))^k.$$

It is immediate that any map  $T: K \rightarrow \underline{M}$  such that

$$T((F_H(h_i))^k) \subset \underline{M}_{h_i}$$

is a representation of  $K$  by hypermatrices.

In the special case for which  $|F_H(h_i)| = \nu$  for every  $i$  then the above set  $K$  is isomorphic to the set

$$K' = \{ (h_i, x_1, \dots, x_\nu) : i=1, \dots, n \text{ with } x_1, \dots, x_\nu \in F_H(h_i) \}$$

with hyperproduct

$$(h_i, x_1, \dots, x_\nu) * (h_j, y_1, \dots, y_\nu) = \{ (h_i h_j, z_1, \dots, z_\nu) : z_1, \dots, z_\nu \in F_H(h_i h_j) \} =$$

$$= (h_i h_j, F_H(h_i h_j), \dots, F_H(h_i h_j)).$$

### 5. APPLICATION ON P-HYPERGROUPS

A large class of hypergroups is the following one [7],[8]:  
 Let  $(S,.)$  be a semigroup and  $P \subset S, P \neq \emptyset$ . The  $\langle S, P^* \rangle$  is called *P-semihypergroup* if the associative *P-hyperoperation*  $P^*$  is defined by setting  $xP^*y = xPy$  for all  $x,y$  of  $S$ . If  $\langle S, P^* \rangle$  is a hypergroup then it is called *P-hypergroup*. One defines *P-hyperoperations* on semihypergroups instead of semigroups as well. So the following theorem is also true for this case.

#### THEOREM 5

Let  $(S,.)$  be a semigroup, then the *P-semihypergroups*  $\langle S, P^* \rangle$  are hypergroups iff  $(S,.)$  is a group.

#### Proof

If  $(S,.)$  is a group then obviously  $\langle S, P^* \rangle$  is a hypergroup.

Now let us suppose that  $(S,.)$  is not a group, then we have an element  $x$  of  $S$  such that  $xS \subsetneq S$ .

We remark that the greater *P-hyperoperation* is for  $P=S$ , i.e. for all  $u,v \in S$  and  $P \subset S$  we have  $uP^*v \subset uSv$ . Therefore it is enough to prove that  $\langle S, S^* \rangle$  is not a hypergroup. Let us take an element  $y \in xS$  and suppose that the reproduction axiom is valid for  $\langle S, S^* \rangle$  then  $y \in xS^*S = xSS \subset xS$  which is a contradiction. Q.E.D.

We can obtain a representation (isomorphic) on a class of *P-hypergroups* using the construction given in the above sections 3,4 as follows:

Let  $X_g = G$  for all  $g \in G$  and  $(G,.)$  be a group, then in the above construction  $|\underline{M}| = n^{n+1}$ . We consider the cartesian product  $G^{n+1}$  and we take the set

$P = \{(e, g_1, \dots, g_n) : g_1, \dots, g_n \in G\} = \{e\} \times G^n$   
 where  $e$  be the identity of  $G$ . Then in  $G^{n+1}$  the *P-hypergroup* is defined where the *P-hyperoperation*  $P^*$  is given by

$$(g, g_1, \dots, g_n)P^*(g', g'_1, \dots, g'_n) = \{gg'\} \times G^n,$$

We consider the map

$$T: G^{n+1} \longrightarrow \underline{M} : (g, g_1, \dots, g_n) \longrightarrow A = (a_{ij})$$

where  $A \in \underline{M}_g$  is such that in the permutation matrix  $M_g$  we set  $a_{ij} = g_i$  for all  $a_{ij} \neq 0$ . This map (which is not unique) is obviously an isomorphic representation of the *P-hypergroup*  $\langle G^{n+1}, P^* \rangle$  by hypermatrices. It is clear that the good condition is valid.



## 6. P-HYPEROPERATION ON MATRICES

In this section we use ordinary matrices but we introduce a new hyperoperation on them.

### DEFINITION

Let  $M_{m \times n}$  be the set of  $m \times n$  matrices with entries from a given ring.

Let also  $P = \{ P_i : i \in I \} \subset M_{m \times n}$  we can define a P-hyperoperation  $P^{t*}$  on  $M_{m \times n}$  extending the Ree's multiplication, see [5], as follows

$$AP^{t*}B = AP^tB, \forall A, B \in M_{m \times n}$$

where  $P^t = \{ P_i^t : i \in I \}$  is the set of transpose matrices of the set P. This hyperoperation is defined always and is obviously associative. Therefore  $\langle M_{m \times n}, P^{t*} \rangle$  is a semihypergroup, which we shall call also *P-semihypergroup*.

### Remark

The set  $M_{m \times n}$  is not a semigroup for  $m \neq n$ , but this P-hyperoperation is actually defined on the set of all matrices (for every m and n) where an associative partial operation (the multiplication of matrices) is defined. Therefore P-hyperoperations can be defined on subsets of sets equipped with partial associative operations.

Using the P-hyperoperation on the set  $M_n$  of square matrices we can represent all P-hypergroups as follows:

Let  $(G, \cdot)$  be a group with  $|G|=n$  and let

$$T: G \longrightarrow M_n : g \longrightarrow T(g) = M_g$$

be the ordinary representation of  $G$  by permutation matrices and we set  $M = \{ M_g : g \in G \}$ . Let  $P \subset G$  and we consider the P-hypergroup  $\langle G, P^* \rangle$ . In  $M_n$  we take the set  $\{ M_p^t : p \in P \} = \underline{P}$  then the map T is a representation of the P-hypergroup  $\langle G, P^* \rangle$  on the set  $M_n$  using the P-hyperoperation  $\underline{P}^*$ .

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