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MAXIMUM LIKELIHOOD ESTIMATION FOR DISCRETE-TIME PROCESSES WITH FINITE STATE SPACE

A LINEAR CASE

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PRELIMINARY REMARK : This report, presented at the Ecole d'Ete de Calcul des Probabilités, Saint-Flour, July, 1980, is abstracted from reference [1]. The reader should go to this reference for formal proofs of the results presented and for some additional results.

THE PROBLEM :

Consider a process X_o, X_1, X_2, ..., which takes values on a finite state space $S = \{1, 2, \ldots, s\}$. Denote by $\mathcal{F}_n$ the past of the process up to and including the n-th transition, i.e., $\mathcal{F}_n = \sigma\{X_o, X_1, \ldots, X_n\}$. Given $\mathcal{F}_n$, we can consider the probability of the process going to state j in time (n + 1), which we can write as

$$p_n^j = \text{Prob}\ \{X_{n+1} = j \ / \ \mathcal{F}_n\}.$$

In what follows we assume that p_n^j is known as a function of an unknown parameter α, write it as $p_n^j(\alpha)$, and look at the behavior of the MLE (maximum likelihood estimate) of α.

We make the following restrictive assumptions :

A1 : α is real, and the true value $\alpha°$ is known to belong to a bounded interval $I = [\ \underline{\alpha}\ ,\ \bar{\alpha}\ ]$

A2 : $p_n^j(\alpha)$ is linear in α, that is

$$p_n^j(\alpha) = a_n^j\alpha + b_n^j$$

A3 : The effect of α is restricted by the following condition : $\exists K > 0$ such that for each n, for each possible evolution of the process up to time n, we have that for every $j \in S$, either $p_n^j(\alpha) = 0 \quad \forall\alpha \in I$, or $p_n^j(\alpha) \geqslant K \quad \forall\alpha \in I$.

We can interpret A3 as stating that, at time n, we know which states might be occupied at time (n+1), and we know that the probability of occupancy of such states is at least K. Even though current work seems to indicate that assumptions A1 and A2 can be relaxed, A3 seems to be essential in the developments that follow.

AN EXAMPLE :

The example that motivated this study has already been analyzed in references [2] and [3] by different methods, and was suggested to this author by one of the authors in [3], Professor P. Varaiya.

Consider a Markov Chain with state space $S = \{1, 2, \ldots, s\}$ whose transition probabilities $\{p_{ij}, i, j \in S\}$ depend both on an unknown parameter α and on a control action u that we can apply on each transition. That is,

$$p_{ij} = p_{ij}(\alpha, u) \ .$$

After each transition we can estimate α from our observations on the chain. Call α_n this estimate. We say that u is an adaptive control if $u_n = u(X_n, \alpha_n)$, that is, if the control we exert at time n depends both on the state we occupy at that time and on the estimate we have of what the true value of α may be.

As the control depends on the whole past of the chain (through α_n) the resulting process is no longer Markov. We can imagine more complex ways of selecting u_n based, for example, both on α_n and the estimated variance of α_n, etc.

Assumptions A1 and A2 keep almost the same form in this special case, and A3 takes the simpler form :

$$\exists K > 0 \ni \forall i, j, u, \quad \text{either } p_{ij}(\alpha, u) = 0 \quad \forall\alpha \in I$$
$$\text{or} \quad p_{ij}(\alpha, u) \geq K \quad \forall\alpha \in I.$$

NOTATION AND MAIN RESULTS :

We define the likelihood of a given $\alpha \in I$ at time n as

$$\text{Prob}\{X_o, X_1, \ldots, X_n/X_o\}(\alpha) = \prod_{m=o}^{n-1} p_m(\alpha) ,$$

and the log-likelihood

$$L_n(\alpha) = \sum_{m=o}^{n-1} \log[p_m(\alpha)] = \sum_{m=o}^{n-1} \log[a_m\alpha + b_m],$$

where we have denoted $p_m = p_m^{Xm+1}$, $a_m = a_m^{Xm+1}$, $b_m = b_m^{Xm+1}$.

Observe that $p_m, a_m, b_m \in \mathfrak{F}_{m+1}$, while p_m^j , a_m^j , $b_m^j \in \mathfrak{F}_m$ $\forall j \in S$.

Assumptions A2, A3 allow us to consider the derivatives of $Ln(\alpha)$, which take a simple form :

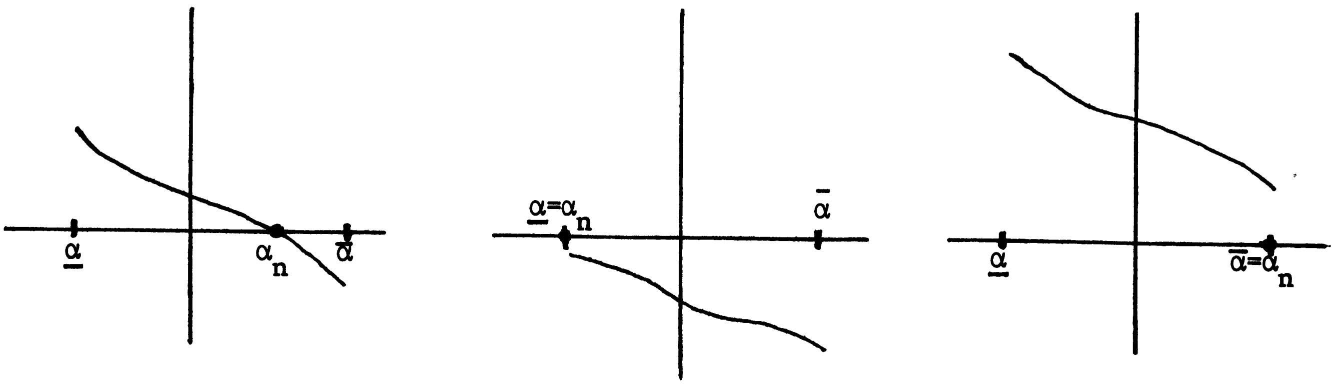
$$L_n'(\alpha) = \frac{dLn}{d\alpha}(\alpha) = \sum_{m=o}^{n-1} \frac{a_m}{a_m\alpha + b_m} ;$$

$$L_n''(\alpha) = \frac{d^2Ln}{d\alpha^2}(\alpha) = \sum_{m=o}^{n-1} \left[\frac{a_m}{a_m\alpha + b_m}\right]^2 \leq 0$$

We can now define the MLE of $\alpha°$ at time n, α_n, to be the smallest element of $I = [\underline{\alpha}, \overline{\alpha}]$ such that

$$L_n(\alpha_n) \geq L_n(\alpha) \quad \forall\alpha \in I.$$

To study the asymptotic behavior of α_n it now suffices to look at the asymptotic behavior of L'_n , as the following picture suggests :



L'_n has a zero in I. α_n is at the zero

$L'_n < 0$ on I $\alpha_n = \underline{\alpha}$

$L'_n > 0$ on I $\alpha_n = \bar{\alpha}$

Define

$$D_m(\alpha) = \frac{a_m}{a_m\alpha + b_m} \quad ; \; D_m(\alpha) \in \mathcal{F}_{m+j}. \quad \text{Observe } L'_n(\alpha) = \sum_{m=o}^{n-1} D_m(\alpha).$$

To simplify notation we assume, without loss of generality, that $\alpha^\circ = 0$, and also $-1 \in I$, $1 \in I$.

Now

$$E_m(\alpha) = E\left[D_m(\alpha) \;/\; \mathcal{F}_m\right] = \sum_{j=1}^{i} \frac{a_m^j}{a_m^j\alpha + b_m^j} \; b_m^j \; .$$

Using the fact that, since $\sum_{j=1}^{i} P_m^j = 1$, $\sum_{j=1}^{i} a_m^j = 0$ and $\sum_{j=1}^{i} b_m^j = 1$,

it can be easily shown that $E_m(0) = 0 \;\; \forall m$ and, furthermore, for all m;

$$\alpha < 0 \Longrightarrow E_m(\alpha) \geqslant 0,$$

$$\alpha > 0 \Longrightarrow E_m(\alpha) \leqslant 0.$$

Take now $\alpha < 0$ fixed, the argument being symmetric for $\alpha > 0$. Then $E_m(\alpha) \geqslant 0$ and $L'_n(\alpha)$ turns out to be a submartingale. We can decompose this submartingale by considering, for

$$Y_m = D_m - E_m,$$

the martingale

$$M_n = \sum_{m=o}^{n-1} Y_m , \qquad (M_n \in \mathcal{F}_n, \forall n)$$

and the increasing process

$$A_n = \sum_{m=o}^{n-1} E_m , \qquad (A_n \in \mathcal{F}_{n-1}, \forall n)$$

(where we have dropped the α's in $Y_m(\alpha)$, etc.)

After some manipulations, it is easy to show that

$$\frac{1}{K} |\alpha| \sum_{m=o}^{n-1} \{ \sum_{j=1}^{i} (a_m^j)^2 \} \geqslant A_n \geqslant |\alpha| \sum_{m=o}^{n-1} \{ \sum_{j=1}^{i} (a_m^j)^2 \}$$

Therefore, the asymptotic behavior of A_n is easily determined by the asymptotic behavior of $\sum_{m=o}^{n-1} \{ \sum_{j=1}^{i} (a_m^j)^2 \}$ as n goes to ∞. Suppose

$L_n' = A_n + M_n$ had the same asymptotic behavior as A_n. Then we would have

$$\sum_{m=o}^{n-1} \{ \sum_{j=1}^{i} (a_m^j)^2 \} \xrightarrow[n \to \infty]{} \infty \overset{?}{\Longrightarrow} L_n'(\alpha) \longrightarrow \infty,$$

and our $\alpha < 0$ could not be the MLE for n large (see the first or third drawings in our previous picture).

To have L_n' of the same order of A_n we would need

$$\frac{L_n'}{A_n} \to 1 \text{ , i.e., } \frac{M_n}{A_n} \to 0, \text{ when } A_n \to \infty.$$

To see under what conditions this would happen, consider

$$V_m = \text{Var} \left[D_m / \mathcal{F}m \right] \leqslant E \left[D_m^2 / \mathcal{F}m \right].$$

After some algebra, one can show that

$$\frac{V_m}{E_m} \leqslant \frac{1}{K |\alpha|} , \text{ i.e.,} \qquad V_m \leqslant \frac{1}{K |\alpha|} E_m .$$

Let's call $B_n = \sum_{m=o}^{n-1} V_m$, the process of "variations" of M_n. This process plays an important role in defining the behavior of M_n, since it is, somehow, the natural time scale for M_n. In particular, we can find in Neveu [4] that, almost surely, for each realization,

$$\begin{cases} \lim_{n\to\infty} B_n < \infty \implies M_n \text{ converges to a finite limit } M \\ \lim_{n\to\infty} B_n = \infty \implies \dfrac{M_n}{B_n} \xrightarrow[n\to\infty]{} 0 . \end{cases}$$

By the bound we got on Vm, we have that $B_n \leqslant \frac{1}{K|\alpha|} A_n$, so that we conclude that, for almost all realizations,

- If $A_n \to \infty$ and B_n remains bounded, then $\frac{M_n}{A_n} \to \frac{M}{\infty} = 0$
- If $A_n \to \infty$ and $B_n \to \infty$, then $\frac{|M_n|}{A_n} \leqslant \frac{1}{K|\alpha|} \frac{|M_n|}{B_n} \to 0.$

In either case, $\frac{M_n}{A_n} \to 0$, whence $\frac{L'_n}{A_n} \xrightarrow{a.s.} 1$ and we can now write, with no question mark,

$$\sum_{m=o}^{n-1} \{ \sum_{j=1}^{i} (a_m^j)^2 \} \xrightarrow[n\to\infty]{} \infty \implies L'_n(\alpha) \xrightarrow{a.s.} \infty$$

(an analogous result states that $L'_n(\alpha) \to -\infty$ a.s. if $\alpha > 0$).

We already suggested an argument for showing that, if $L'_n(\alpha) \to \infty$, then α cannot be the MLE for $\alpha°$.

This suggests the following theorem, whose proof can be found in [1].

<u>THEOREM</u> : Except for a negligible set of realizations, if the sequence $\{\alpha_n\}$ of MLE's has an accumulation point $\alpha^* \neq \alpha° = 0$, then

$$\sum_{m=o}^{\infty} \{ \sum_{j=1}^{i} (a_m^j)^2 \} < \infty.$$

Corollary :

Under the conditions of the theorem,

$$a_m^j \xrightarrow[m \to \infty]{} 0 \ , \quad j = 1, 2, \ldots, s.$$

Some other results can be proven using our knowledge of the limit behaviour of L'_n. The most important seems to be the following.

Proposition :

$\alpha_n \xrightarrow[n]{a.s.} \alpha^*$, where α^* may depend on the realization.

APPLICATION :

We can now apply the results we obtained to the example that motivated this work, a Markov Chain with transition probabilities.

$$p_{ij} (\alpha, u) = a_{ij} (u) \alpha + b_{ij}(u).$$

Let's assume

A4 : u_n, the control in force after X_n has been observed, is of the adaptive form $u_n = \varphi(\alpha_n, X_n)$; and $a_{ij}(\varphi(\alpha, i))$ is continuous in α, for every $i, j \in S$.

Proposition :

Under A1 to A4, and except for a negligible set of realizations, if the sequence of MLE's has an accumulation point $\alpha^* \neq \alpha^\circ = 0$, then

$$a_{ij}(\varphi(\alpha^*, i)) = 0 \text{ for every state } i \text{ that is reached infinitely often, for every } j \in S.$$

A5 : The chain is irreducible for all pairs (α,u) in the sense that : $\forall i,j \in S \ \ \exists i_1,\ldots,i_k \in S$ such that $p_{i_{\ell-1}, i_\ell}(\alpha,u) > 0$, $\ell = 0,\ldots, k + 1$, where $i_o = i$, $i_{k+1} = j$.

For a chain satisfying A5 Feller shows that all states are reached infinitely often. This result carries over to our situation (using A3), and we obtain our final

Proposition :

Under A1 to A5, and except for a negligible set of realizations,

$\alpha_n \to \alpha^*$ where α^* satisfies

$$p_{ij}(\alpha^*,\varphi(\alpha^*,i)) = p_{ij}(\alpha^\circ,\varphi(\alpha^*,i)), \quad \forall i, j \in S.$$

This last result can be phrased as saying that α^* is such that, if we were to take the control $U(i) = \varphi(\alpha^*,i)$, depending only on the present state of the chain, then α° and α^* would be indistinguishable, since they would produce equal values of all the p_{ij} 's.

This characterizes the set of all possible limit points α^* of the sequence of maximum likelihood estimators.

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