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HAYRI KOREZLIOGLU

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TWO-PARAMETER GAUSSIAN MARKOV PROCESSES AND THEIR RECURSIVE LINEAR FILTERING

Hayri KOREZLIOGLU(*)

SUMMARY : A class of two-parameter Gaussian Markov processes is characterized and conditions are given for a process of this class to have a representation in terms of Wiener processes. When the signal is a Gaussian Markov process having a representation in terms of a two-parameter Wiener process and the noise is another Wiener process independent of the signal, and when the observation is defined as in the one-parameter Kalman filtering model, recursive linear filtering equations satisfied by the estimation of the signal are obtained by using the linear filtering method of Hilbert space-valued Gaussian processes.

INTRODUCTION

In this work, we develop and improve the short exposé [1] on the linear filtering of two-parameter Gaussian Markov processes.

In § 1, we give the definition of a Markov property for two parameter Gaussian processes and express conditions on the covariance functions of such processes having the defined Markov property in order that they may possess a representation in terms of Wiener processes.

Shortly after the Saint-Flour seminars in 1978, Dr. D. NUALART kindly communicated to us his work in collaboration with M. SANZ, on the characterization of Gaussian Markov fields [2]. We show, in § 1, that our characterization is equivalent to theirs.

In § 2, we consider the following "signal and observation" model. The signal X is a Gaussian Markov process, defined by

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(*) Ecole Nationale Supérieure des Télécommunications
46 rue Barrault, 75634 PARIS CEDEX 13.

$$X_{s,t} = \varphi_{s,t} \int_0^s \int_0^t \varphi_{u,v}^{-1} G_{u,v}\, dB_{u,v}$$

where B is a Wiener process and φ and G are non-random functions. The observation Y is given by

$$Y_{s,t} = \int_0^s \int_0^t H_{u,v} X_{u,v}\, dv\, du + W_{s,t}$$

where W is a Wiener process independent of B and H is a non-random function. We regard various processes appearing in this model as taking their values in an L^2-space when indexed by their first parameter. Then, by using a result due to OUVRARD [3] on linear filtering of Hilbert space-valued Gaussian processes, we obtain the horizontal filtering equation (in the first parameter) satisfied by the estimation $\hat{X}^h_{s,t}(t_o)$ of $X_{s,t}$ in terms of the observations $\{Y_{u,v}: u \leqslant s, v \leqslant t_o\}$, with $t \leqslant t_o$. By commuting the roles of the two parameters, we obtain the vertical filtering equation (in the second parameter) satisfied by the estimation $\hat{X}^v_{s,t}(s_o)$ of $X_{s,t}$ in terms of the observations $\{Y_{u,v}: u \leqslant s_o, v \leqslant t\}$, with $s \leqslant s_o$. Finally, for the case $s_o = s$ and $t_o = t$, we give the expression of the second differential, in s and t, of $\hat{X}_{s,t} = \hat{X}^h_{s,t}(t_o) = \hat{X}^v_{s,t}(s_o)$ in terms of innovation processes appearing in the horizontal and vertical filtering equations.

§ 0 NOTATIONS

The random processes considered here will be indexed by $[o,S] \times [o,T] \subset \mathbb{R}^2$, where S and T are finite positive numbers(*). For a second order centered process $X = \{X_{s,t}: (s,t) \in [o,S] \times [o,T]\}$, defined on a probability space $(\Omega, \underline{\underline{A}}, P)$, $\underline{\underline{H}}^X_{s,t}$ will denote the smallest Hilbert subspace of $L^2(\Omega, \underline{\underline{A}}, P)$ generated by $\{X_{u,v}: u \leqslant s, v \leqslant t\}$ and $\underline{\underline{F}}^X_{s,t}$, the smallest σ-subalgebra generated by $\{X_{u,v}: u \leqslant s, v \leqslant t\}$ and all P-negligible sets of $\underline{\underline{A}}$. If $Z \in L^2(\Omega, \underline{\underline{A}}, P)$ and if $\underline{\underline{H}}$ is a Hilbert subspace of

(*) Throughout this work, the index set $[o,S] \times [o,T]$ can be replaced by $\mathbb{R}_+ \times \mathbb{R}_+$, with little precaution. In particular, this substitution would not bring any modification to the text of § 1.

$L^2(\Omega,\underline{\underline{A}},P)$, we shall denote by $(Z/\underline{\underline{H}})$ the projection of Z onto $\underline{\underline{H}}$. If $Y,Z \in L^2(\Omega,\underline{\underline{A}},P)$, we shall denote by (Z/Y) the projection of Z onto the subspace (of dimension 1) generated by Y.

For a random process $X = \{X_{s,t}:(s,t)\in[o,S]\times[o,T]\}$, we extend the domain of the parameters to negative numbers by putting $X_{s,t} = o$, when at least one of the parameters is negative. We denote by $\underline{\underline{H}}^X_{S,t} \vee \underline{\underline{H}}^X_{s,T}$ the smallest Hilbert space generated by $\underline{\underline{H}}^X_{S,t}$ and $\underline{\underline{H}}^X_{s,T}$, Notice that $\underline{\underline{H}}^X_{S,t} \vee \underline{\underline{H}}^X_{s,T} = \underline{\underline{H}}^X_{s,T}$ if $t < 0$ and $\underline{\underline{H}}^X_{S,t} \vee \underline{\underline{H}}^X_{s,T} = \underline{\underline{H}}^X_{S,t}$ if $s < o$.

We shall write $(u,v) \leqslant (s,t)$ for $u \leqslant s$, $v \leqslant t$, and $(u,v) < (s,t)$ for $u \leqslant s$, $v \leqslant t$, $(u,v) \neq (s,t)$.

For two Hilbert spaces H_1 and H_2 such that $H_1 \subset H_2$, $H_2 \ominus H_1$ will denote the orthogonal complement of H_1 in H_2.

§ 1. GAUSSIAN SIMPLE MARKOV PROCESSES

In this paragraph, $X = \{X_{s,t}:(s,t)\in[o,S]\times[o,T]\}$ will be a centered Gaussian process, defined on a given probability space $(\Omega,\underline{\underline{A}},P)$.

DEFINITION 1 : X will be called a horizontal Markov process, if

(1) for all s,t and $u \leqslant s$, $(X_{s,t}/H^X_{u,T}) = (X_{s,t}/X_{u,t})$

and a vertical Markov process, if

(2) for all s,t and $v \leqslant t$, $(X_{s,t}/H^X_{S,v}) = (X_{s,t}/X_{s,v})$

X will be called a simple Markov process, if it is a horizontal and a vertical Markov process.

We identify, here, random variables with their P-equivalence classes.

Conditions (1) and (2) are respectively equivalent to the following ones.

$$(1)' \quad \begin{cases} \text{for all } (u,v) \text{ and } (s,t) \text{ such that } u \leqslant s,\ v \geqslant t \\ (X_{s,t}/\underline{\underline{H}}^X_{u,v}) = (X_{s,t}/X_{u,t}) \end{cases}$$

$$(2)' \quad \begin{cases} \text{for all } (u,v) \text{ and } (s,t) \text{ such that } u \geqslant s,\ v \leqslant t \\ (X_{s,t}/\underline{\underline{H}}^X_{u,v}) = (X_{s,t}/X_{s,v}) \end{cases}$$

<u>PROPOSITION 1</u> : If X is a simple Markov process, then

$$(3) \quad \begin{cases} \text{for all } (u,v) \text{ and } (s,t) \text{ such that } (u,v) \leqslant (s,t) \\ (X_{s,t}/\underline{\underline{H}}^X_{u,v}) = (X_{s,t}/X_{u,v})^{(*)}. \end{cases}$$

Proof : For $(u,v) \leqslant (s,t)$, we have, according to (1) and (2)',

$$\begin{aligned}(X_{s,t}/\underline{\underline{H}}^X_{u,v}) &= ((X_{s,t}/\underline{\underline{H}}^X_{u,T})/\underline{\underline{H}}^X_{u,v}) = (aX_{u,t}/\underline{\underline{H}}^X_{u,v}) \\ &= a(X_{u,t}/\underline{\underline{H}}^X_{u,v}) = abX_{u,v}\end{aligned}$$

where a and b are adequate constants. Therefore, condition (3) is satisfied. ∎

The following corollary is an obvious consequence of the above proposition.

<u>COROLLARY</u> : If X is a simple Markov process, then for any increasing path C in $[o,S] \times [o,T]$, $\{X_{s,t} : (s,t) \in C\}$ is a one-parameter Markov process with respect to the filtration $\{\underline{\underline{F}}^X_{s,t} : (s,t) \in C\}$, hence, with respect to its natural filtration.

Let K be the covariance function of X and let Φ be defined by

$$(4) \begin{cases} \Phi((s,t),(u,v)) = \begin{cases} K((s,t),(u,v))K^{-1}((u,v),(u,v)) & \text{if } K((u,v),(u,v)) \neq o \\ o \quad \text{if} \quad K((u,v),(u,v)) = o \end{cases} \\ \text{for} \quad (u,v) < (s,t) \text{ and} \\ \Phi((s,t),(s,t)) = 1. \end{cases}$$

(*) This implication is proved by Pascal LEFORT in his current research work on the Markov property of two-parameter processes. In [1], we had defined a simple Markov process as a process satisfying conditions (1), (2) and (3).

Notice that we have

(5) $(X_{s,t}/X_{u,v}) = \Phi((s,t),(u,v))X_{u,v}$ for $(u,v) \leqslant (s,t)$.

Using this equation, one can show that conditions (1), (2) and (3) are equivalent to following conditions (6), (7) and (8), respectively.

(6) $\begin{cases} \text{For all } (s,t),\ (u,v) \text{ and } p \text{ such that } u \leqslant p \leqslant s \\ K((s,t),(u,v)) = \Phi((s,t),(p,t))K((p,t),(u,v)) \end{cases}$

(7) $\begin{cases} \text{For all } (s,t),\ (u,v) \text{ and } q \text{ such that } v \leqslant q \leqslant t \\ K((s,t),(u,v)) = \Phi((s,t),(s,q))K((s,q),(u,v)) \end{cases}$

(8) $\begin{cases} \text{For all } (u,v),\ (p,q) \text{ and } (s,t) \text{ such that } (u,v) \leqslant (p,q) \leqslant (s,t) \\ K((s,t),(u,v)) = \Phi((s,t),(p,q))K((p,q),(u,v)) \end{cases}$

Condition (8) implies that

(9) $\Phi((s,t),(u,v))=\Phi((s,t),(p,q))\Phi((p,q),(u,v))$ for $(u,v) \leqslant (p,q) \leqslant (s,t)$

and, according to the above corollary, if X is a simple Markov process, then for any increasing path C, the restriction of Φ to C^2 is the Markov transition function of the process $\{X_{s,t} : (s,t) \in C\}$.

PROPOSITION 2 : For a simple Markov process X, the following orthogonality relation holds.

(10) $\underline{\underline{H}}^X_{S,t} \ominus \underline{\underline{H}}^X_{s,t} \perp \underline{\underline{H}}^X_{s,T} \ominus \underline{\underline{H}}^X_{s,t}$ for all (s,t).

This is equivalent to saying that the σ-algebras $\underline{\underline{F}}^X_{S,t}$ and $\underline{\underline{F}}^X_{s,T}$ are conditionally independent with respect to $\underline{\underline{F}}^X_{s,t}$.

Proof : $\underline{\underline{H}}^X_{S,t} \ominus \underline{\underline{H}}^X_{s,t}$ is generated by $\{X_{p,q} - (X_{p,q}/\underline{\underline{H}}^X_{s,t}) : p > s, q \leqslant t\}$ and $\underline{\underline{H}}^X_{s,T} \ominus \underline{\underline{H}}^X_{s,t}$ is generated by $\{X_{u,v} - (X_{u,v}/\underline{\underline{H}}^X_{s,t}) : u \leqslant s, v > t\}$. Since, $(X_{p,q}/\underline{\underline{H}}^X_{s,t}) = (X_{p,q}/\underline{\underline{H}}^X_{s,T})$ and $X_{u,v} - (X_{u,v}/\underline{\underline{H}}^X_{s,t}) \in \underline{\underline{H}}^X_{s,T}$ for

$p > s$, $q \leqslant t$, $u \leqslant s$, $v > t$, we have $X_{p,q} - (X_{p,q}/\underline{\underline{H}}^X_{s,t}) \perp X_{u,v} - (X_{u,v}/\underline{\underline{H}}^X_{s,t})$. From this, the relation (9) follows. The equivalence of this relation with the conditional independance of $\underline{\underline{F}}^X_{S,t}$ and $\underline{\underline{F}}^X_{s,T}$ with respect to $\underline{\underline{F}}^X_{s,t}$ is a known property of Gaussian spaces (cf. [4]). ∎

<u>PROPOSITION 3</u> : If X is a simple Markov process, then, for $(u,v) < (s,t)$, the random variable

(11) $\hat{X}_{s,t} = \Phi((s,t),(u,t))X_{u,t} + \Phi((s,t),(s,v))X_{s,v} - \Phi((s,t),(u,v))X_{u,v}$

is the projection of $X_{s,t}$ onto $\underline{\underline{H}}^X_{S,v} \vee \underline{\underline{H}}^X_{u,T}$.

Proof : Since $\hat{X}_{s,t} \in \underline{\underline{H}}^X_{S,v} \vee \underline{\underline{H}}^X_{u,T}$, we only have to prove the orthogonality of $\tilde{X}_{s,t} = X_{s,t} - \hat{X}_{s,t}$ to $\underline{\underline{H}}^X_{s,v} \vee \underline{\underline{H}}^X_{u,T}$. Notice that this last space has the following orthogonal decomposition.

$$\underline{\underline{H}}^X_{S,v} \vee \underline{\underline{H}}^X_{u,T} = (\underline{\underline{H}}^X_{S,v} \ominus \underline{\underline{H}}^X_{u,v}) \oplus (\underline{\underline{H}}^X_{u,T} \ominus \underline{\underline{H}}^X_{u,v}) \oplus \underline{\underline{H}}^X_{u,v} .$$

Then

$$(\tilde{X}_{s,t}/\underline{\underline{H}}^X_{S,v} \vee \underline{\underline{H}}^X_{u,T}) = (\tilde{X}_{s,t}/\underline{\underline{H}}^X_{S,v}) + (\tilde{X}_{s,t}/\underline{\underline{H}}^X_{u,T}) - (\tilde{X}_{s,t}/\underline{\underline{H}}^X_{u,v})$$

It can easily be verified that each of the projections of the right-hand side is zero. ∎

In [2], the Markov property of X was defined as follows :

(12) $\begin{cases} \text{For all } (u,v) \text{ and } (s,t) \text{ such that } (u,v) < (s,t), \\ (X_{s,t}/\underline{\underline{H}}^X_{S,v} \vee \underline{\underline{H}}^X_{u,T}) = aX_{u,t} + bX_{s,v} + cX_{u,v} \\ \text{where } a, b \text{ and } c \text{ are adequate constants.} \end{cases}$

With our convention of § 0, in extending the domain of the parameters to negative numbers, one can deduce that, if X satisfies condition (12), then it satisfies conditions (1) and (2) ; thus X is a simple Markov process. Conversely, if X is a simple Markov process, according to Proposition 3, it satisfies condition (12). Therefore, conditions (1) and (2) together are equivalent to condition (12).

<u>From now on, we suppose that X is a centered Gaussian simple Markov process</u>.

We make the following hypothesis on the covariance function K of X.

<u>HYPOTHESIS H_1</u> : K is continuous on $([o,S] \times [o,T])^2$, $K((s,t),(s,t))$ is strictly positive for $st > o$, $K((s,o),(s,o))$ is either identically null or strictly positive on $]o,S]$ and $K((o,t),(o,t))$ is either identically null or strictly positive on $]o,T]$.

Under this hypothesis $K((s,t),(u,v))$ is strictly positive for all s,t,u,v such that $K((s,t),(s,t))K((u,v),(u,v)) > o$. In fact, any pair of different points (s,t) and (u,v) can be joined by a path consisting of at most one horizontal and one vertical line segments on each of which K is a one-parameter covariance function. Therefore, the proof of the mentioned property reduces to that of the one-parameter case (cf. [4], p. 55).

Suppose now that the function Φ, defined by (4), verifies the following hypothesis.

<u>HYPOTHESIS H_2</u> : There exists a strictly positive continuous function $\varphi_{s,t}$, defined on $[o,S] \times [o,T]$, such that

(13) $$\Phi((s,t),(u,v)) = \varphi_{s,t}\, \varphi^{-1}_{u,v}$$

for all (u,v) for which $K((u,v)(u,v)) > o$.

(We can always suppose $\varphi_{o,o} = 1$ by dividing φ by a constant, without changing (13). That is what we shall do).

Notice that, if K is strictly positive on $([o,S] \times [o,T])^2$, the function φ defined by

$$\varphi_{s,t} = K((s,t),(o,o))K^{-1}((o,o),(o,o))$$

verifies H_2. More generally, if, for $(p,q) < (s,t)$ in D, $\Phi((s,t),(p,q))$ has a strictly positive continuous limit $\varphi_{s,t}$, as $(p,q) \to (o,o)$ and if φ has a continuous and strictly positive extension to the entire domain $[o,S] \times [o,T]$, then this extension, denoted again by φ, verifies the hypothesis H_2. In fact, for $(p,q) < (u,v) < (s,t)$ in D, we have

$$\begin{aligned}\Phi((s,t),(p,q))\Phi^{-1}((u,v),(p,q)) &= K((s,t),(p,q))K^{-1}((u,v),(p,q)) = \\ &= K((s,t),(u,v))K^{-1}((u,v),(u,v))K((u,v),(p,q))K^{-1}((u,v),(p,q)) \\ &= \Phi((s,t),(u,v)).\end{aligned}$$

Therefore, for $(u,v) < (s,t)$,

$$\begin{aligned}\varphi_{s,t}\varphi_{u,v}^{-1} &= \lim_{(p,q) \to (o,o)} [\Phi((s,t),(p,q))\Phi^{-1}((u,v),(p,q))] \\ &= \Phi((s,t),(u,v))\end{aligned}$$

where the limit is to be taken for $(p,q) < (u,v)$ in D.

PROPOSITION 4 : Let M be defined by

(14) $$M_{s,t} = K((s,t),(s,t))\varphi_{s,t}^{-2}, \quad (s,t) \in [o,S] \times [o,T].$$

Then

(15) $$K((s,t),(u,v)) = \varphi_{s,t} M_{s\wedge u,t\wedge v} \varphi_{u,v}$$

for all $((s,t),(u,v)) \in ([o,S] \times [o,T])^2$, where $\wedge$ stands for the infinum.

Proof : The proof can be obtained by a direct verification of the equality

$$K((s,t),(u,v)) = \varphi_{s,t} K((s\wedge u,t\wedge v),(s\wedge u,t\wedge v))\varphi_{s\wedge u,t\wedge v}^{-2}\varphi_{u,v}$$

for all possible configurations of the line segment joining (u,v) to (s,t). ∎

Now, we can characterize the process X in terms of a Gaussian strong martingale.

PROPOSITION 5 : The process defined by $Z_{s,t} = \varphi_{s,t}^{-1} X_{s,t}$ is a (centered) Gaussian strong martingale with respect to the filtration $\{\underline{\underline{F}}_{s,t}^{X} : (s,t) \in [o,S] \times [o,T]\}$. Hence, Z has independent increments and $E(Z_{s,t} Z_{u,v}) = M_{s\wedge u, t\wedge v}$, where M is defined by (14). In particular,

$U_{s,t} = Z_{s,t} - Z_{o,t} - Z_{s,o} + Z_{o,o}$ defines a Gaussian strong martingale U with respect to the same filtration. Similarly, $U'_{s,o} = Z_{s,o} - Z_{o,o}$ and $U''_{o,t} = Z_{o,t} - Z_{o,o}$ define two Gaussian martingales U' and U", respectively, with respect to the filtrations $\{\underline{\underline{F}}_{s,o}^{X} : s \in [o,S]\}$ and $\{\underline{\underline{F}}_{o,t}^{X} : t \in [o,T]\}$. The random variable $X_{o,o}$ and the processes U, U' and U" are mutually independent.

Proof : We refer to [5] for the definition of a strong martingale. Let $Z(](u,v),(s,t)])$ be the increment of Z between (u,v) and (s,t) with $(u,v) < (s,t)$, i.e.

$$Z(](u,v),(s,t)]) = \varphi_{s,t}^{-1} X_{s,t} - \varphi_{s,v}^{-1} X_{s,v} - \varphi_{u,t}^{-1} X_{u,t} + \varphi_{u,v}^{-1} X_{u,v} .$$

In order to prove that Z is a strong martingale we only have to prove that the projection of the above increment onto $\underline{\underline{H}}_{S,v}^{X} \vee \underline{\underline{H}}_{u,T}^{X}$ is null. But, $\varphi_{s,t} Z(](u,v),(s,t)]) = X_{s,t} - \hat{X}_{s,t}$, where $\hat{X}_{s,t}$ is given by (11), and Proposition 3 says that the projection of $X_{s,t} - \hat{X}_{s,t}$ and, hence, of $Z(](u,v),(s,t)])$ onto $\underline{\underline{H}}_{S,v}^{X} \vee \underline{\underline{H}}_{u,T}^{X}$ is zero.

The fact that U is a strong martingale is due to the equality of the increments of U to those of Z.

As a consequence of Proposition 3, $\{Z_{s,o} : s \in [o,S]\}$ and hence U' is a martingale with respect to $\{\underline{\underline{F}}_{s,o}^{X} : s \in [o,S]\}$. A similar argument holds for U". The independence of $X_{o,o}$, U', U" and U is due to the fact that these quantities are increments of Z and that Z has independent increments. ∎

We can say more about the representation of X in terms of martingales, if the function M, defined by (14), verifies the following hypothesis.

<u>HYPOTHESIS H_3</u> : There exist square integrable functions G, G' and G" such that

$$M_{s,t} - M_{o,t} - M_{s,o} + M_{o,o} = \int_o^s \int_o^t G^2_{u,v}\, dv\, du$$

$$M_{s,o} - M_{o,o} = \int_o^s (G'_u)^2\, du$$

$$M_{o,t} - M_{o,o} = \int_o^t (G''_v)^2\, dv.$$

<u>THEOREM 1</u> : Under hypotheses H_1, H_2 and H_3, there exist one-parameter Wiener processes $B' = \{B'_s : s \in [o,S]\}$ and $B'' = \{B''_t : t \in [o,T]\}$ and a two-parameter Wiener process $B = \{B_{s,t} : (s,t) \in [o,S] \times [o,T]\}$ such that $X_{o,o}$, B', B" and B are mutually independent and

$$X_{s,t} = \varphi_{s,t}[X_{o,o} + \int_o^s G'_u dB'_u + \int_o^t G''_v dB''_v + \int_o^s \int_o^t G_{u,v} dB_{u,v}]. \tag{16}$$

Conversely, if φ, G', G", G, $X_{o,o}$, B', B" and B are as above, then the process X defined by (16) is a centered Gaussian simple Markov process verifying hypotheses H_1, H_2 and H_3. In both cases, X has a modification with continuous trajectoires.

Proof : Notice first that the right hand side of (16) devided by $\varphi_{s,t}$ should represent $Z_{s,t}$ defined in Proposition 5. Consider the martingale U of the same proposition. U is null on the coordinate axes and has independent increments. If $|G_{s,t}| > o$ for almost all (s,t), then a Wiener measure B can be defined by $dB_{s,t} = G^{-1}_{s,t}\, dU_{s,t}$ and we have $U_{s,t} = \int_o^s \int_o^t G_{u,v}\, dB_{u,v}$. If not, let $D = \{(s,t) : |G_{s,t}| > o\}$ and define a Wiener measure $\bar{B}$ on D by $d\bar{B}_{s,t} = G^{-1}_{s,t}\, dU_{s,t}$ and take any other Wiener measure $\bar{\bar{B}}$ on $[o,S] \times [o,T] \setminus D$, independent of X ; then, put $B = \bar{B} + \bar{\bar{B}}$. U has again the same representation in terms of B. The Wiener process B considered in the theorem is generated by the Wiener measure B of the proof. The process B and the corresponding stochastic integral $\int_0^s \int_0^t G_{u,v}\, dB_{u,v}$ have continuous modifications (cf. [6]).

The construction of B' and B" can be made in the same way, by starting, respectively, from the martingales U' and U" of Proposition 5. The independence of $X_{o,o}$, B, B' and B" is a consequence of this proposition.

The proof of the converse part of the theorem is a matter of direct verification. ■

§ 2. LINEAR FILTERING

We consider the following "state and observation" model for the filtering problem.

The state process or the signal X is a continuous Gaussian simple Markov process defined by

$$(1) \qquad X_{s,t} = \varphi_{s,t} \int_0^s \int_0^t \varphi^{-1}_{u,v}\, G_{u,v}\, dB_{u,v} \;, \quad (s,t) \in [o,S] \times [o,T]$$

where φ is a continuous strictly positive random function having continuous first partial derivatives $\frac{\partial \varphi_{s,t}}{\partial s}$, $\frac{\partial \varphi_{s,t}}{\partial t}$; B is a continuous Wiener process and G is a square-integrable non-random function.

The observation process Y is defined by

$$(2) \qquad Y_{s,t} = \int_0^s \int_0^t H_{u,v}\, X_{u,v}\, dv\, du + W_{s,t} \;,$$

where H is a continuous non-random function and W is a continuous Wiener process, independent of B.

For some regularity properties of the probability space $(\Omega,\underline{\underline{A}},P)$ on which the processes considered in the above model are defined, we shall identify it with the canonical space of (B,W).

We shall put $\underline{\underline{G}}_s = \bigcap_{u>s} \underline{\underline{F}}^Y_{u,T}$ and denote by $\underline{\underline{G}}$ the filtration $\{\underline{\underline{G}}_s, s\in[o,S]\}$ and by L^2 the space $L^2(\Omega\times[o,S]\times[o,T], \underline{\underline{A}}\otimes\underline{\underline{B}}, dP\otimes ds\otimes dt)$, where $\underline{\underline{B}}$ is the Borel σ-algebra of $[o,S]\times[o,T]$. Let $\underline{\underline{B}}_{[o,S]}$ (resp. $\underline{\underline{B}}_{[o,T]}$) the Borel σ-algebra of $[o,S]$ (resp. $[o,T]$). If $\underline{\underline{P}}'$ denotes the σ-algebra of $\underline{\underline{G}}$-predictable sets of $\underline{\underline{A}}\otimes\underline{\underline{B}}_{[o,S]}$, then we shall denote by $\underline{\underline{P}}$ the product σ-algebra $\underline{\underline{P}}'\otimes\underline{\underline{B}}_{[o,T]}$ and by $L^2(\underline{\underline{P}})$ the Hilbert subspace of L^2 generated by all $\underline{\underline{P}}$-measurable elements of L^2. The space $L^2([o,T],\underline{\underline{B}}_{[o,T]}, dt)$ will be denoted by $L^2(dt)$.

<u>DEFINITION 2</u> : For a process Z in L^2, the $\underline{\underline{G}}$-<u>predictable projection</u> Z^p of Z will be defined by the conditional expectation of Z with respect to the σ-algebra $\underline{\underline{P}}$ and the measure $dP\otimes ds\otimes dt$.

The predictable projection in the senseof the above definition can be constructed as follows, by using the notion of predictable projection in the senseof [7]. Let Z be a process defined by

$$(3) \qquad Z_{s,t} = U\, 1_{]u,S]}(s) 1_{]v,T]}(t)$$

where U is a bounded random variable and let $\{U_s : s\in[o,S]\}$ be the right-continuous version of the martingale $\{E(U/\underline{\underline{G}}_s) : s\in[o,S]\}$. Let us put

$$(4) \qquad Z^p_{s,t} = U_{s-}\, 1_{]u,S]}(s)\, 1_{]v,T]}(t) \, ,$$

where $U_{s-}\, 1_{]u,S]}(s)$ coincides with the $\underline{\underline{G}}$-predictable projection of $U\, 1_{]u,S]}(s)$ in the senseof [7]. Notice that, for almost all s, $Z^p_{s,t}$ is a version of $E(Z_{s,t}/\underline{\underline{G}}_s)$. On the other hand, the space $L^2(\underline{\underline{P}})$ is generated by processes of type

$$L_{s,t} = V\, 1_{]u',S]}(s)\, 1_{]v',T]}(t) ,$$

where V is a bounded $\underline{\underline{G}}_u$-measurable random variable. (There is no loss of generality in choosing $V \geqslant o$). By using the fact that the measure $dP \otimes ds$ commutes with predictable projections (cf. [7], T 30, p. 107), we find

$$E \int_o^S \int_o^T (Z_{s,t} - Z^p_{s,t}) L_{s,t}\, dt\, ds = o.$$

Therefore, the process Z^p is the predictable projection of Z in the sense of the above definition. Now, let Z be an arbitrary element of L^2 and let $\{Z_n : n \in \mathbb{N}\}$ be a sequence consisting of processes that are linear combinations of processes of type (3) and converging to Z in L^2. (The space L^2 is generated by processes of type (3)). To each Z_n there corresponds a process Z^p_n defined as linear combinations of processes of type (4) and the sequence $\{Z^p_n, n \in \mathbb{N}\}$ converges to the predictable projection Z^p of Z as defined in Definition 2. Therefore, there exists a subsequence $\{Z^p_{k_n}, n \in \mathbb{N}\}$ such that, for almost all (s,t), $\{Z^p_{k_n,s,t}, n \in \mathbb{N}\}$ converges to $Z^p_{s,t}$ in $L^2(\Omega, \underline{\underline{A}}, P)$. From this, we deduce that, for each $Z \in L^2$ and for almost all (s,t), $Z^p_{s,t}$ is a version of the conditional expectation $E(Z_{s,t} / \underline{\underline{G}}_s)$.

Now, let us consider the following <u>horizontal evolution equation</u> for X.

$$(5) \qquad X_{s,t} = \int_o^s \frac{\partial \varphi_{u,t}}{\partial u} \varphi^{-1}_{u,t} X_{u,t}\, du + \int_o^s \int_o^t \varphi_{u,t} \varphi^{-1}_{u,v} G_{u,v}\, dB_{u,v} .$$

X, Y, W and the system noise, represented here by the last term in (5), have continuous trajectoires. Therefore, for fixed s, $X_{s,.}$, $W_{s,.}$ and $\int_o^s \int_o^{\cdot} \varphi_{u,.} \varphi^{-1}_{u,v} G_{u,v}\, dB_{u,v}$ can be considered as taking their values in $L^2(dt) = L^2([o,T], \underline{\underline{B}}_{[o,T]}, dt)$.

If $U = \{U_{s,t} : (s,t) \in [o,S] \times [o,T]\}$ is a process such that, for all s and almost all $\omega \in \Omega$, $U_{s,.}(\omega) \in L^2(dt)$, we shall denote by $U = \{U_s, s \in [o,S]\}$ the corresponding $L^2(dt)$-valued process. We shall denote by $\| . \|$ and $<.,.>$, respectively, the norm and the scalar product on $L^2(dt)$.

The covariance operator of W_s is $s\mathfrak{W}$, where $\mathfrak{W}$ is the nuclear operator on $L^2(dt)$, the kernel of which is $t \wedge \tau$, i.e. $\forall f,g \in L^2(dt)$
$E[<W_s, f><W_s, g>] = s<\mathfrak{W}f,g> = s \int_0^T \int_0^T t \wedge \tau \, f(t) g(\tau) dt\, d\tau$.
The operator $\mathfrak{W}$ is called the covariance operator of W. We define the bounded linear operator H_s on $L^2(dt)$ by

$$(H_s f)(t) = \int_0^t H_{s,v} f_v \, dv \quad , \quad f \in L^2(dt)$$

Consequently, equation (2) can be written as

$$(6) \qquad Y_s = \int_0^s H_u X_u \, du + W_s \quad .$$

DEFINITION 3 : Let X^p be the $\underline{\underline{G}}$-predictable projection of the state process X. Then the process ν defined by

$$(7) \qquad \nu_{s,t} = Y_{s,t} - \int_0^s \int_0^t H_{u,v} X^p_{u,v} \, dv\, du$$

will be called the horizontal innovation process of Y

Notice that, as an $L^2(dt)$-valued process, ν can be defined by

$$(8) \qquad \nu_s = Y_s - \int_0^s H_u X^p_u \, du \; .$$

PROPOSITION 6 : The $L^2(dt)$-valued process $\nu = \{\nu_s : s \in [o,S]\}$ is a $\underline{\underline{G}}$-Brownian motion with covariance operator $\mathfrak{W}$. In particular, $\{\nu_{s,t} : (s,t) \in [o,S] \times [o,T]\}$ is a two-parameter Wiener process such that for all $s'<s$, $\{\nu_{s,t} - \nu_{s',t} : t \in [o,T]\}$ is independent of $\underline{\underline{G}}_{s'}$.

Proof : We refer to [8] for the definition and properties of a Hilbert space-valued Brownian motion. Notice first that ν can be written as

$$(9)\qquad \begin{cases} \nu_{s,t} = \int_0^s \int_0^t H_{u,v} \tilde{X}_{u,v} \, dv \, du + W_{s,t} \\ \nu_s = \int_0^s H_u \tilde{X}_u \, du + W_s \end{cases}$$

where $\tilde{X}_{u,v} = X_{u,v} - X^p_{u,v}$. It is a matter of easy verification that ν is square-integrable, i.e. $E\|\nu_s\|^2 < \infty$, and it is strongly continuous as an $L^2(dt)$-valued process. To prove that ν is a $\underline{\underline{G}}$-Brownian motion, according to [8], it is enough to show that, for all $f \in L^2(dt)$, $\{<\nu_s, f> : s \in [o,S]\}$ is a $\underline{\underline{G}}$-martingale, and for all $f,g \in L^2(dt)$ and all $s' < s$

$$E[<\nu_s - \nu_{s'}, f><\nu_s - \nu_{s'}, g>/\underline{\underline{G}}_{s'}] = (s-s')<\mathfrak{W} f, g>.$$

These two properties can be proved in exactly the same way as in the proof of the well known innovation theorem of the one-parameter filtering problem ([9], Lemma 2.2). The second part of the proposition is an immadiate consequence of the first. ∎

Let us consider the process $\{E(X_{s,t}/\underline{\underline{G}}_s) : t \in [o,T]\}$, for fixed s. As $\{X_{s,t} : t \in [o,T]\}$ is continuous in the quadratic mean, the same goes for this process. It has therefore, a measurable version (in t) which is what we shall consider below.

<u>PROPOSITION 7</u> : Let M be defined by

$$(10)\qquad M_{s,t} = E(X_{s,t}/\underline{\underline{G}}_s) - \int_0^s \frac{\partial \varphi_{u,t}}{\partial u} \varphi^{-1}_{u,t} X^p_{u,t} \, du.$$

Then $M = \{M_s : s \in [o,S]\}$ is a square-integrable $L^2(dt)$-valued Gaussian $\underline{\underline{G}}$-martingale. Moreover, for almost all t, $\{M_{s,t} : s \in [o,S]\}$ also is a Gaussian $\underline{\underline{G}}$-martingale.

Proof : The square-integrability of M_s is easy to show by direct computation. For $s'<s$ and $f\in L^2(dt)$, we have

$$E[<M_s-M_{s'},f>/\underline{\underline{G}}_s] = E[<\int_{s'}^{s}\int_0^{\cdot}\varphi_{u,.}\,\varphi_{u,v}^{-1}\,G_{u,v}\,dB_{u,v},f>/\underline{\underline{G}}_s] = o.$$

This shows that M is an $L^2(dt)$-valued $\underline{\underline{G}}$-martingale. The proof of the last part of the proposition is similar. ■

In the sequel, we shall only consider the right continuous modification for the process $\{E(X_{s,.}/\underline{\underline{G}}_s), s\in[o,S]\}$, taken as an $L^2(dt)$-valued process, that we shall denote by $X^o = \{X_s^o : s\in[o,S]\}$. We shall rather write equation (10) as follows :

$$(11)\qquad M_{s,t} = X_{s,t}^o - \int_0^s \frac{\partial\varphi_{u,t}}{\partial u}\,\varphi_{u,t}^{-1}\,X_{u,t}^p\,du,$$

keeping in mind that $X_{s,t}^o$ is a version of $E(X_{s,t}/\underline{\underline{G}}_s)$ and, as an $L^2(dt)$-valued process X^o is right-continuous. Notice that $X_{s,t}^o = X_{s,t}^p$ a.s. for almost all (s,t).

Let P_s be the covariance operator of $\widetilde{X}_s = X_s - X_s^o$, defined by

$$E[<\widetilde{X}_s,f><\widetilde{X}_s,g>] = <P_s f,g>,\qquad f,g\in L^2(dt).$$

Then P_s is a symmetric nuclear operator the kernel of which is defined by

$$P_s(t,\tau) = E[(X_{s,t}-X_{s,t}^o)\,(X_{s,\tau}-X_{s,\tau}^o)]$$
$$= E[(X_{s,t}-X_{s,t}^p)\,(X_{s,\tau}-X_{s,\tau}^p)]$$

for almost all s,t,τ.

PROPOSITION 8 : Let K be defined by

$$(12)\qquad K_s(t,\tau) = P_s(t,\tau)\,H_{s,\tau}$$

for almost all s,t,τ. Then the martingale M defined by (11) thas the following representation

(13) $$M_{s,t} = \int_0^s \int_0^T K_u(t,v)\, d\nu_{u,v}$$

for almost all t.

Proof : For the proof of the representation (13), we refer to the representation theorem 2.6 in [3]. Let M be an $L^2(dt)$-valued square-integrable $\underline{\underline{G}}$-martingale. Then there exists a $\underline{\underline{G}}$-predictable process Ψ with values in the Hilbert space of (not necessarily bounded) linear operators in $L^2(dt)$ such that $\int_0^S E[\mathrm{tr}(\Psi_u \mathfrak{W} \Psi_u^*)]\, du < \infty$, where Ψ^* is the adjoint of Ψ_u, and that

(14) $$M_s = \int_0^s \Psi_u d\nu_u .$$

(Cf. [10] for the definition of this kind of stochastic integrals).

Let the representation of the martingale M of Proposition 7 be given by (14). Then, for almost all t,

(15) $$M_{s,t}\ \nu_{s,\tau} - \int_0^s \int_0^\tau P_u(t,v) H_{u,v}\, dv\, du$$

is a $\underline{\underline{G}}$-martingale in terms of s. To see this, it is enough to apply the Ito differentiation rule to $M_{s,t}\ \nu_{s,\tau}$ and take the conditional expectation with respect to $\underline{\underline{G}}_{s'}$ for s'<s. This part of the proof is similar to that of Proposition 2.11 in [3]. Considered as an operator, the integral term of (15) can be written as $\int_0^s P_u\ H_u^*\, du$. This, with the above mentioned representation theorem, implies that M_s has the following representation

$$M_s = \int_0^s P_u H_u^* \mathfrak{W}^+ d\nu_u ,$$

as given in Theorem 2.12 in [3], where $\mathfrak{W}^+$ is the pseudo-inverse of $\mathfrak{W}$ defined by

$$\mathfrak{W}^+ f = \lim_{n\to\infty} \left[\left(\int_0^n \exp[(s-n)\mathfrak{W}^2]\, ds\right) \mathfrak{W} f\right] \quad , \qquad f \in \mathrm{rg}\, \mathfrak{W}$$

with the limit taken in $L^2(dt)$. Notice that $P_u H_u^{::} \mathfrak{W}^+$ is a non-random operator. Then, for almost all t, $M_{s,t}$ is an element of $\underline{\underline{H}}^{\nu}_{s,T}$. Therefore, by taking the mathematical expectation of (15), which equals o, we obtain

$$E(M_{s,t}\,\nu_{s,\tau}) = \int_0^s \int_0^\tau K_u(t,v)dv\,du = E[(\int_0^s \int_0^T K_u(t,v)d\nu_{u,v})\nu_{s,\tau}]$$

From this, follows representation (13). ■

Let us consider now the equation

$$(16) \qquad U_{s,t} = \int_0^s \frac{\partial\varphi_{u,t}}{\partial u} \varphi^{-1}_{u,t} U_{u,t}\,du + \int_0^s \int_0^T K_u(t,v)d\nu_{u,v}$$

and let Z be a measurable version of

$$(17) \qquad Z_{s,t} = \varphi_{s,t} \int_0^s \int_0^T \varphi^{-1}_{u,t} K_u(t,v)d\nu_{u,v}\ .$$

The existence of a measurable version of Z is guaranteed by the measurability of its covariance function. By substituting U by Z in (16), it can be verified that Z satisfies equation (16).

We want to show that if equation (16) is satisfied a.e. $dP \otimes ds \otimes dt$ by two different processes U and U' in L^2, then U = U' a.e. $dP \otimes ds \otimes dt$. In this case, we can also replace $K_u(t,v)$ in (16) and (17) by any other measurable function equal to K for almost all u, v, t.

Suppose that we have two processes U and U' in L^2 satisfying equation (16) a.e. $dP \otimes ds \otimes dt$. Then Y = U-U' satisfies the equation

$$Y_{s,t} = \int_0^s \frac{\partial\varphi_{u,t}}{\partial u} \varphi^{-1}_{u,t} Y_{u,t}\,du$$

a.e. $dP \otimes ds \otimes dt$. Then we have

$$E(Y^2_{s,t}) \leqslant \int_0^s F^2_{u,t}\,du \int_0^s E(Y^2_{u,t})\,du$$

$$\leqslant A \int_0^s E(Y^2_{u,t})\,du \qquad \text{a.e.} \qquad ds \otimes dt\ ,$$

where $F_{u,t} = \frac{\partial\varphi_{u,t}}{\partial u}\,\varphi^{-1}_{u,t}$ and A is an upper bound, independent of (s,t), of the integral in which F appears. Let us put

$$f_{s,t} = \int_0^s E(Y^2_{u,t})\,du.$$

Then the above inequality becomes

$$\frac{\partial f_{s,t}}{\partial s} - A\,f_{s,t} \leqslant o \qquad \text{a.e.} \qquad ds \otimes dt\ ,$$

which gives, after multiplication by e^{-As} ,

$$\frac{\partial}{\partial s}\,(e^{-As}\,f_{s,t}) \leqslant o \qquad \text{a.e.} \qquad ds \otimes dt.$$

Therefore,

$$\int_0^S \frac{\partial}{\partial s}\,(e^{-As}\,f_{s,t})\,ds = e^{-AS}\,f_{S,t} \leqslant o \qquad \text{a.e.} \qquad dt\ .$$

This is possible only if, for almost all t, $f_{S,t} = o$. Consequently we have $\int_0^S \int_0^T E(Y^2_{s,t})dt\,ds = o$.

Equation (11) and representation (13) show that X^p and X^o satisfy equation (16) a.e. $dP \otimes ds \otimes dt$. Therefore, $Z = X^p = X^o$ a.e. $dP \otimes ds \otimes dt$. Then, according to (7), we have

$$Y_{s,t} = \int_0^s \int_0^t H_{u,v}\,Z_{u,v}\,dv\,du + \nu_{s,t} \qquad \text{a.s.}$$

This implies that $Y_{s,t}$ is an element of $\underline{\underline{H}}^{\nu}_{s,T}$. Hence $\underline{\underline{H}}^{Y}_{s,T} \subseteq \underline{\underline{H}}^{\nu}_{s,T}$. Consequently, the filtration $\underline{\underline{G}}$ coïncides with $\{\underline{\underline{F}}^{\nu}_{s,T} : s\in[o,S]\}$ and is, therefore, continuous. All these imply that $\underline{\underline{G}}_s$ is generated by $\underline{\underline{H}}^{Y}_{s,T} = \underline{\underline{H}}^{\nu}_{s,T}$.

Moreover, by using the continuity of $\underline{\underline{G}}$, one can show that, as function of (s,t), $E(X_{s,t}/\underline{\underline{G}}_s)$ is continuous in the quadratic mean. Hence, $P_s(t,\tau)$ can be chosen as a continuous function of (s,t,τ). That is what we shall do in the sequel. In this case, the process Z defined by (17) is continuous in the quadratic mean. Therefore, $Z_{s,t}$ is a version of $E(X_{s,t}/\underline{\underline{G}}_s)$ for all (s,t).

Notice that, for any fixed t, $\{Z_{s,t} : s \in [o,S]\}$ has a continuous version satisfying (16) and, being continuous in the quadratic mean, for any fixed s, $\{Z_{s,t} : t \in [o,T]\}$ has a measurable version. Therefore, there exists a measurable process $\hat{X} = \{\hat{X}_{s,t} : (s,t) \in [o,S] \times [o,T]\}$ such that, for any fixed t, $\{\hat{X}_{s,t} : s \in [o,S]\}$ has continuous trajectoires and, for all (s,t), $\hat{X}_{s,t} = Z_{s,t}$ a.e. hence a version of $E(X_{s,t}/\underline{\underline{G}}_s)$, (cf. [11]).

Noting that nothing would change in the above conclusions if T were replaced by $t_o \in [o,T]$, we shall summarize the main results by the following theorem.

<u>THEOREM 2</u> : The process $\{E(X_{s,t}/F^Y_{s,t_0}) : (s,t) \in [o,S] \times [o,t_0]\}$ is continuous in the quadratic mean and has a measurable modification $\hat{X}^h(t_0) = \{\hat{X}^h_{s,t}(t_0) : (s,t) \in [o,S] \times [o,t_0]\}$ such that, for any fixed t, $\{\hat{X}^h_{s,t}(t_0) : s \in [o,S]\}$ is continuous. Let $P^h_s(t,\tau;t_0)$ and $K^h_s(t,\tau;t_0)$ be defined respectively by

$$P^h_s(t,\tau;t_0) = E[(X_{s,t} - \hat{X}^h_{s,t}(t_0))(X_{s,\tau} - \hat{X}^h_{s,\tau}(t_0))] \quad , \tag{18}$$

$$K^h_s(t,\tau;t_0) = P^h_s(t,\tau;t_0)\, H_{s,\tau} \quad , \tag{19}$$

and let the horizontal innovation process of height t_0 be defined by

$$\nu^h_{s,t}(t_0) = Y_{s,t} - \int_0^s \int_0^t H_{u,v}\, \hat{X}^h_{u,v}(t_0)\, dv\, du \quad . \tag{20}$$

Then $\hat{X}^h(t_0)$ satisfies the following horizontal filtering equation

$$\hat{X}^h_{s,t}(t_0) = \int_0^s \frac{\partial \varphi_{u,t}}{\partial u} \varphi^{-1}_{u,t}\, \hat{X}^h_{u,t}(t_0)\, du + \int_0^s \int_0^{t_0} K^h_u(t,v;t_0)\, d\nu^h_{u,v}(t_0 \tag{21}$$

and has the following explicit expression

(22) $$\hat{X}^h_{s,t}(t_0) = \varphi_{s,t} \int_0^s \int_0^{t_0} \varphi^{-1}_{u,t}\, K^h_u(t,v;t_0)\, d\nu^h_{u,v}(t_0).$$

Moreover, the kernel $P^h_s(t,\tau;t_0)$ satisfies the following Riccati equation.

(23) $$\frac{\partial P^h_s(t,\tau;t_0)}{\partial s} = \frac{\partial \varphi_{s,t}}{\partial s}\varphi^{-1}_{s,t}\, P^h_s(t,\tau;t_0) + P^h_s(t,\tau;t_0)\,\frac{\partial \varphi_{s,\tau}}{\partial s}\varphi^{-1}_{s,\tau}$$
$$-\int_0^{t_0} P^h_s(t,v;t_0)\, H^2_{s,v}\, P^h_s(v,\tau;t_0)\, dv + \varphi_{s,t}\left(\int_0^{t\wedge\tau} \varphi^{-2}_{s,v}\, G^2_{s,v}\, dv\right)\varphi_{s,\tau},$$

with $P^h_s(t,\tau;t_0) = o$ for $st\tau = o$.

Proof : We only have to establish the Riccati equation. By putting $\tilde{X} = \hat{X} - X^h(t_0)$ and considering equations (5), (21), (2) and (20), we get

$$\tilde{X}_{s,t} = \int_0^s \frac{\partial \varphi_{u,t}}{\partial u}\varphi^{-1}_{u,t}\, \tilde{X}_{u,t}\, du + \int_0^s \int_0^t \varphi_{u,t}\, \varphi^{-1}_{u,v}\, G_{u,v}\, dB_{u,v}$$
$$- \int_0^s \int_0^{t_0} K^h_u(t,v;t_0)[H_{u,v}\, \tilde{X}_{u,v}\, dv\, du + dW_{u,v}] ,$$

and we obtain equation (23) by applying the Ito differentiation formula to $\tilde{X}_{s,t}\, \tilde{X}_{s,\tau}$ as function of s and by taking the mathematical expectation. ■

By interchanging the roles of s and t,we obtain similar results for the vertical filtering problem.

<u>THEOREM 2'</u> : The process $\{E(X_{s,t}/F^Y_{s_0,t}) : (s,t)\in[o,s_0]\times[o,T]\}$, with $o < s_0 \leqslant S$, is continuous in the quadratic mean and has a measurable modification
$\hat{X}^v(s_0) = \{\hat{X}^v_{s,t}(s_0) : (s,t)\in[o,s_0]\times[o,T]\}$ such that, for any fixed s, $\{\hat{X}^v_{s,t}(s_0) : t\in[o,T]\}$ is continuous. Let $P^v_t(s,\sigma;s_0)$ and $K^v_t(s,\sigma;s_0)$ be defined respectively by

(18') $$P^v_t(s,\sigma;s_0) = E[X_{s,t} - \hat{X}^v_{s,t}(s_0))(X_{\sigma,t} - \hat{X}^v_{\sigma,t}(s_0))]$$

(19') $$K^{v}_{t}(s,\sigma;s_o) = P^{v}_{t}(s,\sigma;s_o)\, H_{\sigma,t}\,,$$

and let the vertical innovation process of width s_o be defined by

(20') $$\nu^{v}_{s,t}(s_o) = Y_{s,t} - \int_o^s \int_o^t H_{u,v}\, \hat{X}^{v}_{u,v}(s_o)\, dv\, du\ .$$

Then $\hat{X}^{v}(s_o)$ satisfies the following vertical filtering equation

(21') $$\hat{X}^{v}_{s,t}(s_o) = \int_o^t \frac{\partial \varphi_{s,v}}{\partial v} \varphi^{-1}_{s,v}\, \hat{X}^{v}_{s,v}(s_o)\, dv + \int_o^{s_o}\int_o^t K^{v}_{v}(s,u;s_o) d\nu^{v}_{u,v}(s_o)$$

and has the following explicit expression

(22') $$\hat{X}^{v}_{s,t}(s_o) = \varphi_{s,t} \int_o^{s_o}\int_o^t \varphi^{-1}_{s,v}\, K^{v}_{v}(s,u;s_o) d\nu^{v}_{u,v}(s_o)\ .$$

The kernel $P^{v}_{t}(s,\sigma;s_o)$ satisfies the Riccati equation :

(23') $$\frac{\partial P^{v}_{t}(s,\sigma;s_o)}{\partial t} = \frac{\partial \varphi_{s,t}}{\partial t} \varphi^{-1}_{s,t}\, P^{v}_{t}(s,\sigma;s_o) + P^{v}_{t}(s,\sigma;s_o)\, \frac{\partial \varphi_{\sigma,t}}{\partial t} \varphi^{-1}_{\sigma,t}$$
$$- \int_o^{s_o} P^{v}_{t}(s,u;s_o)\, H^2_{u,t}\, P^{v}_{t}(u,\sigma;s_o) du + \varphi_{s,t} \left(\int_o^{s\wedge\sigma} \varphi^{-2}_{u,t}\, G^2_{u,t}\, du\right)\varphi_{\sigma,t}\,,$$

with $P^{v}_{t}(s,u;s_o) = o$ for $tus = o$.

We may call any version of $E(X_{s,t}/F^{Y}_{s,t})$ a <u>causal estimation</u> of $X_{s,t}$ in terms of Y. It is clear that $\hat{X}^{h}_{s,t}(t)$ and $\hat{X}^{v}_{s,t}(s)$ are causal estimations of $X_{s,t}$.

Let us put $\hat{X}_{s,t} = \hat{X}^{h}_{s,t}(t) = \hat{X}^{v}_{s,t}(s)$ a.s.. For any fixed t, the process $\{\hat{X}_{s,t} : s \in [o,S]\}$ has a continuous version defined by

(24) $$\hat{X}_{s,t} = \varphi_{s,t} \int_o^s \int_o^t \varphi^{-1}_{u,t}\, K^{h}_{u}(t,v;t) d\nu^{h}_{u,v}(t)$$

and satisfying the equation

$$(25) \qquad \hat{X}_{s,t} = \int_0^s \frac{\partial \varphi_{u,t}}{\partial u} \varphi_{u,t}^{-1} \hat{X}_{u,t}\, du + \int_0^s \int_0^t K_u^h(t,v;t)\, d\nu_{u,v}^h(t) .$$

Similarly, for any fixed s, the process $\{\hat{X}_{s,t} : t \in [o,T]\}$ has a continuous version defined by

$$(24') \qquad \hat{X}_{s,t} = \varphi_{s,t} \int_0^s \int_0^t \varphi_{s,v}^{-1} K_v^v(s,u;s)\, d\nu_{u,v}^v(s)$$

and satisfying the equation

$$(25') \qquad \hat{X}_{s,t} = \int_0^t \frac{\partial \varphi_{s,v}}{\partial v} \varphi_{s,v}^{-1} \hat{X}_{s,v}\, dv + \int_0^s \int_0^t K_v^v(s,u;s)\, d\nu_{u,v}^v(s) .$$

We do not know yet whether or not $\hat{X}$ has a continuous version as a two-parameter process.

For numerical applications, it would be interesting to express $\hat{X}_{s+ds,t+dt}$, for $ds > o$ and $dt > o$, in terms of $\hat{X}_{s+ds,t}$, $\hat{X}_{s,t+dt}$ and $\hat{X}_{s,t}$, that is, to have a two-parameter recursive filtering equation. We could find this equation in [1] only by extending, in a rather formal way, the results obtained in [12] for the case of discrete parameters. This equation is the following

$$(26) \qquad d(\varphi_{s,t}^{-1} \hat{X}_{s,t}) = \varphi_{s,t+dt}^{-1} K_s^h(t+dt,t;t+dt)\, d\nu_{s,t}^h(t+dt)$$
$$+ \int_0^t d_t[\varphi_{s,t}^{-1} K_s^h(t,v;t)]\, d\nu_{s,v}^h(t) + \int_0^s d_s[\varphi_{s,t}^{-1} K_t^v(s,u;s)]\, d\nu_{u,t}^v(t)$$

where d denotes the second differential in s and t, d_s and d_t denote respectively the first differentials in s and in t. The first term of the right hand side can be replaced by $\varphi_{s+ds,t}^{-1} K_t^v(s+ds,s;s+ds)\, d\nu_{s,t}^v(s+ds)$.

At the view of equation (26), it seems difficult to represent $\hat{X}_{s,t}$ as the sum of stochastic integrals in terms of various innovations. For the non-linear filtering of two-parameter semi-martingales such a representation was obtained in [13]. We hope to be able to extend the method of [13] to the Gaussian case in a forthcoming publication.

We would like to mention that linear filtering equations of type (21), (21') and (26) were given by E. WONG in [14] where the martingale representation (13) was introduced without proof. Apart from providing all the tools for establishing the linear filtering equations, the extension of the method used in [13] to the Gaussian case will, we hope, also contain the proof of such martingale representation theorems.

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