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LINEAR OPERATORS AS MEASURE PRESERVING TRANSFORMATIONS

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Abstract. We examine conditions under which a bounded linear operator in a separable complex Banach space accepts an invariant probability Borel measure. Also we define a class of operators for which the measure preserving transformations they define have always complete point spectrum.

§1. We consider a complex separable Banach space B and we denote by T a continuous linear operator in B and by m a probability measure defined on the Borel σ -algebra of B . The support of m is the subset of B consisting of the elements whose every neighborhood has non-zero measure. We say that $T:B \rightarrow B$ accepts an invariant probability measure if there exists m as above whose support spans B and for which $m(T^{-1}(\cdot)) = m(\cdot)$. We are interested in characterizing the operators that have this property and also in determining the type of measure preserving transformation (m.p.t.) arising in each case. The case where B is finite dimensional can be solved completely and we consider it first. In §2 we solve the problem for a class of operators in infinite dimensions which includes the isometries and finally we apply the results to an example. The present work is a continuation of the results in [3].

We consider first some notions from the theory of m.p.t. If h is a m.p.t. in a probability space its eigenvalues are the complex numbers $\{c\}$ for which the equation $f(h(\cdot))=cf(\cdot)$ has non-trivial complex-valued measurable solutions $f(\cdot)$. We note that the eigenvalues of a m.p.t. h form always a subgroup of the unit circle group and they coincide with the eigenvalues of the isometry V induced by h in each of the (complex) L_p spaces, $1 < p < \infty$, over the probability space, defined by $Vf(\cdot)=f(h(\cdot))$. We say that h has complete point spectrum, if L_2 is spanned by the eigenvectors of $V:L_2 \rightarrow L_2$. These are in a sense the simplest m.p.t. being completely characterized by the spectrum of $V:L_2 \rightarrow L_2$. We will need the following result whose proof we omit as it is similar to a corresponding result in [4.p214].

Lemma 1. h is a m.p.t. in a probability space. If a collection of eigenfunctions of h generates the σ -algebra of the space then h has complete point spectrum. Also the eigenvalues of h are given by the subgroup of the unit circle group generated by the eigenvalues of the collection.

Considering now the case where B is finite dimensional we have:

Theorem 1. B is a finite dimensional complex Banach space and $T:B \rightarrow B$ a linear operator. Then

- (i) T accepts an invariant m as above iff B

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is spanned by eigenvectors of T having eigenvalues of norm 1.

(ii) If T accepts an invariant m then the m.p.t. defined by T has complete point spectrum given by the subgroup of the unit circle group generated by the eigenvalues of T .

Proof: (i+) T preserves the Haar measure on the torus defined by the cartesian product of the unit circle group in each one-dimensional eigenspace. (i+) Assuming the existence of an invariant m whose support spans B we consider also the duals B^*, T^* . Any eigenvalue of T^* must have norm 1 because it is also an eigenvalue of the m.p.t. defined by T . Assume now that an eigenvalue c of T^* does not have index 1. Then there exist x^*, y^* in B^* such that $T^*x^* = cx^*$ and $T^*y^* = cy^* + x^*$. It follows that $T^{*n}y^* = c^n y^* + nc^{n-1}x^*$, or considering the elements of B^* as functions on B that $y^*(T^n(\cdot)) = c^n y^*(\cdot) + nc^{n-1}x^*(\cdot)$ and in particular that

$$(*) \quad n|x^*(\cdot)| < |y^*(T^n(\cdot))| + |y^*(\cdot)|, \quad n=1, 2, \dots$$

We can find $\epsilon > 0$ so that for $A = \{z: |x^*(z)| \geq \epsilon\}$ we have $m(A) > 0$ because of the assumption on the support of m . We can also find $M > 0$ so that for $A' = \{z: |y^*(z)| \leq M\}$ we have $m(A' \cap A) > 0$. Setting $A = A' \cap A$ it follows from the measure preserving property of h that for some integer $n > 2M/\epsilon$ we have $m(T^{-n}A \cap A) > 0$ and then the inequality (*) above is contradicted for each $z \in T^{-n}A \cap A$. Indeed for the expression on the left side we have $z \in A \rightarrow z \in A' \rightarrow n|x^*(z)| \geq n\epsilon > 2M$ while for that on the right side we have $z \in A \rightarrow z \in A_2$ and $T^n z \in A_1 \rightarrow z \in A_2$ and

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$T^n z \in A_2 \Rightarrow |y^*(z)| \leq M$ and $|y^*(T^n z)| \leq M$. It follows that T^* has a spanning set of eigenvectors having eigenvalues of norm 1. Hence so does T , which proves (i). As for part (ii) we note that the functions $\{x^*(.): x^* \in B^*\}$ generate the Borel σ -algebra of B . By above so do the functions $\{x^*(.): x^* \text{ an eigenvector of } T^*\}$ which are also eigenfunctions of the m.p.t. T having the same eigenvalues. The result then follows from Lemma 1. Q.E.D.

§2. In infinite dimensions the problem considered above with T assumed to be a linear contraction appears in [1] in the following form: "Solve the equation $X(h(.)) = TX(.)$ where h is an ergodic m.p.t. in a probability space (S, Σ, μ) , $T: B \rightarrow B$ is a linear contraction and $X: S \rightarrow B$ is Borel measurable." Using this setting we can construct m.p.t. (B, T, m) of various types as follows:

Example. T is taken to have the property that there exists a sequence $\{x_i: i=0, 1, 2, \dots\}$ with $Tx_0 = 0$, $Tx_{i+1} = x_i$ and $\sum |x_i| < \infty$. Let also h be any m.p.t. on a probability space (S, Σ, μ) and $f \in L_1(S, \Sigma, \mu)$. We define

$$X(.) = \sum f(h^2(.)) x_i$$

Then $X(h(.)) = TX(.)$ and therefore $m(.) = \mu(X^{-1}(.))$ defines a Borel probability measure on B , invariant under T . The m.p.t. so defined inherits many of the properties of h . In particular it does not have any eigen-

values if h does not, so that Theorem 1 does not hold generally in infinite dimensions. We do note however that all of the unit circle group belongs to the point spectrum of the operator T .

Next we extend Theorem 1 to a class of operators in infinite dimensions, considering however only invariant probability measures for which the norm function on B is integrable, as is also the case in the example above. We mention that if B is a Banach space then a set of functionals $\{x^*\} \subset B^*$ is called total if $x^*(x) = 0$ for every $x^* \in \{x^*\}$ implies $x = 0$, or equivalently if $\{x^*\}$ spans B^* in its B -topology.

Theorem 2. B is a separable complex Banach space and $T: B \rightarrow B$ a continuous linear operator with the property that a total set of functionals has bounded orbits under T^* . Then:

(i) T accepts an invariant m of integrable norm iff B is spanned by eigenvectors of T having eigenvalues of norm 1.

(ii) If T leaves m of integrable norm invariant, then the m.p.t. defined by T has complete point spectrum given by the subgroup of the unit circle group generated by the eigenvalues of T . Concerning the structure of T we add:

(iii) If T leaves m of integrable norm invariant then the eigenvalues of the operators T, T^* are countable, they coincide and they all have norm 1. Also the eigenvectors of T^* span B^* in its B -topology.

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Proof: (i+) Let $\{x_i : i \in I\}$ be a countable collection of eigenvectors of T spanning B with eigenvalues $\{c_i : i \in I\}$ of norm 1. We consider the torus group $S = \prod C_i$, $i \in I$, where C_i is the unit circle group, equipped with the Haar measure, and the m.p.t. $h: S \rightarrow S$ given by multiplication by the element $(c_i : i \in I) \in S$. The projection functions $\varphi_i: S \rightarrow C_i$ are eigenfunctions of h with eigenvalues c_i correspondingly. The function $X: S \rightarrow B$ defined by $X(\cdot) = \sum \varphi_i(\cdot) x_i / 2^{|i|}$ is strongly integrable and satisfies $X(h(\cdot)) = TX(\cdot)$. Also the essential range of X spans B . Clearly the measure induced in B by X satisfies all the requirements.

For the rest of the proof we use the following construction. Assuming that T accepts an invariant m of integrable norm we consider the bounded linear operator $K: L_\infty(B, m) \rightarrow B$ defined by the strong integral $Kf = \int zf(z) dm$. The adjoint $K^*: B^* \rightarrow L_1 L_\infty^*$ is defined by $K^*x^* = x^*(\cdot) \in L_1$. We have

$$(*) \quad K^*T^* = VK^*$$

where $V: L_1 \rightarrow L_1$ is the isometry induced by the m.p.t. T . We take now $x^* \in B^*$ for which the orbit $\{x^*, T^*x^*, T^{*2}x^*, \dots\}$ under T^* is norm bounded. Considering B^* with its B -topology we have that the closure $C(x^*)$ of the orbit is compact and the restriction of K^* to $C(x^*)$ is injective and continuous. Indeed K^* is injective because the support of m spans B . Also from [5] we have that K is compact and therefore the restriction of K^* to the norm bounded subset $C(x^*)$ of B^* equipped with the B -topology is continuous [2, p.486]. Denoting by S the image of $C(x^*)$ we have : (a) S is compact in the norm

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topology of L_1 as the continuous image of a compact set.
 (b) S is invariant under the isometry V by (*) and the fact that $C(x^*)$ is invariant under T^* , (c) $X=K^{*-1}: S \rightarrow B^*$ is norm bounded and continuous in the B -topology because $K^*: C(x^*) \rightarrow S$ is a homeomorphism. Also we have

$$(**) \quad X(V(.))=T^*X(.)$$

Having a compact metric space S and an isometry $V: S \rightarrow S$ it is well known that there exists a Borel probability measure μ with support S which is invariant under V and such that the measure preserving transformation so defined has complete point spectrum. In fact it is also ergodic because $V: S \rightarrow S$ has a dense orbit by construction [1]. Let now $\{c_i\}$ be the collection of eigenvalues of the m.p.t. V and $\{f_i\}$ the corresponding collection of eigenfunctions with $|f_i|=1$ a.e. The weak integrals

$$x_i^* = \int \bar{f}_i X d\mu$$

are well defined as elements of B^* in the sense that $x_i^*(x) = \int \bar{f}_i X(x) d\mu$ for every $x \in B$. Also we have

$$T^*x_i^* = c_i x_i^* \quad \text{by } (**)$$

and finally we note that x^* lies in the subspace of B^* spanned by the collection $\{x_i^*\}$, in the B -topology. In particular we have, by the assumption on T^* , that B^* is spanned in the B -topology by the eigenvectors of T^* , which proves the last part of (iii). Let now $\{x_j^*\}$ be the collection of eigenvectors of T^* . It is a total set of functionals by above and therefore their images under K^* form a spanning set for $K^*B^* = \{x^*(.): x^* \in B^*\} \subset L_1$. From the separability of B it follows that the collection $K^*B^* \subset L_1$

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generates the Borel σ -algebra of B [6,p.74]. Therefore so does the collection $\{x_j(.) : x_j \text{ eigenvector of } T^*\}$ which consists of eigenfunctions of the m.p.t. T . It follows now from Lemma 1 that the m.p.t. T has complete point spectrum generated by the eigenvalues of T^* which in particular must all have norm 1. This proves the first part of (ii).

(i \rightarrow) Assuming that T accepts an invariant m of integrable norm we have shown above that the m.p.t. T has c.p.s. Let $\{\varphi_i(.)\}$ be the eigenfunctions of the m.p.t. T in $L_\infty(B,m)$. We define the strong integrals

$$x_i = \int \bar{\varphi}(z) z dm \quad (=K\bar{\varphi}_i)$$

We have that $\{x_i\}$ are eigenvectors of T having eigenvalues of norm 1. Also they span B because the functions $\{\varphi_i(.)\}$ L_∞ form a total set of functionals for L_1 and $x^*(x_i)=0$ for all x_i implies $x^*(.)=0$ μ -a.e. which implies $x^*=0$ by the injectivity of K^* .

(iii) Since the eigenvectors of T span B and those of T^* span B^* in the B -topology it follows easily that the eigenvalues of T, T^* must coincide, and also they are all of norm 1 because those of T^* are so. This also proves the last part of (ii). Q.E.D.

Corollary If $T: B \rightarrow B$ has the property that all $x^* \in B^*$ have bounded orbits under T^* then Theorem 2 holds without the condition of integrability on the norm.

Proof: The assumption is equivalent to the assumption that the norms $\|T^i\|$, $i=0,1,2,\dots$, are uniformly bounded. In this case we can assume w.l.o.g. that

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T is a contraction by considering if necessary the equivalent norm $\|x\|' = \sup\{\|T^i x\| : i=0,1,2,\dots\}$. If now T is a contraction and it accepts an invariant measure then the closed balls $B_a = \{x : \|x\| \leq a\}$ are m -invariant under T in the sense that $m(T^{-1}(B_a) \Delta B_a) = 0$. Indeed assume that for some $A \subset B_a$ with positive measure we have $m(T^{-1}A \cap B) = 0$. Then we have $m(T^{-i}A \cap A) = 0$ for all $i=1,2,\dots$, which contradicts the measure preserving property of T . It follows that we can consider the restriction of m to the sets B_a which give measures of integrable norm in the subspaces spanned by B_a , and we can apply Theorem 2. Q.E.D.

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§3. In this section we apply Theorem 2 to an example and we pose a problem related to the condition of norm integrability:

We consider a probability space (S, Σ, μ) and a non-singular transition probability $P(\cdot, E)$ with the property $P\mu(E) \leq c\mu(E)$ for some $c > 0$ and all $E \in \Sigma$, where $P\mu(E) = \int P(\cdot, E) d\mu(\cdot)$. We note that we always have a measure equivalent to μ for which this happens, e.g. the measure $\mu' = \sum_0^\infty P^n \mu / 2^n$ where $P^n \mu = P(P^{n-1} \mu)$. If this condition is satisfied then P induces in each complex $L_p(S, \Sigma, \mu)$, $1 \leq p < \infty$, a bounded linear operator denoted also by P , where $Pf(\cdot) = \int f(s) dP(\cdot, ds)$. Then every essentially bounded $f \in L_p$ has bounded orbit under P in L_p , because P is a contraction in L_∞ . We can therefore apply Theorem 2 to the adjoint $T = P^*: L_q \rightarrow L_q$, for $1/p + 1/q = 1$. For simplicity we will apply Theorem 2 to the case where the transition probability is induced by a nonsingular point transformation $h: S \rightarrow S$ in the sense that $P(\cdot, E) = 1_E(h(\cdot))$ where 1_E is the characteristic function. We note however that solutions obtained would be the same for the general case of a transition probability. For convenience we assume also that h is invertible as a m.p.t. and ergodic. The operator $T = P^*$ is now given by $Tf = f(h^{-1}(\cdot))\varphi(\cdot)$ where the R-N derivative $\varphi(\cdot) = d\mu h^{-1} / d\mu$ is essentially bounded by assumption. Assume now that $T = P^*$ accepts an invariant m of integrable norm whose support spans BCL_q for a fixed q , $1 \leq q < \infty$ where B is also assumed separable if $q = \infty$. Then B is spanned by $\{f_i\} \subset L_q$, where

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$$f_i(h^{-1}(\cdot))\varphi(\cdot) = c_i f_i(\cdot), \quad |c_i| = 1, \quad i=0,1,2,\dots$$

In particular we have $|f_i|(h^{-1}(\cdot))\varphi(\cdot) = |f_i|(\cdot)$. We set $\chi(\cdot) = |f_0|(\cdot)$ and $d\mu' = \chi(\cdot)d\mu$. By nonsingularity of h we have $\varphi(\cdot) > 0$ μ -a.e. and then by the ergodicity of h we have $\chi(\cdot) > 0$ μ -a.e. Assuming also $\chi(\cdot)$ properly normalized we have that μ' is a probability measure equivalent to μ and clearly invariant under h . We also have that \bar{f}_i/χ are eigenfunctions of h having eigenvalues c_i . It is now clear how we can construct all solutions (B,T) for our example. We start with an ergodic invertible m.p.t. $h: (S, \Sigma, \mu') \rightarrow (S, \Sigma, \mu')$ and we choose a positive function $\chi(\cdot)$ such that

$$(*) \quad \int \chi^{-1}(\cdot) d\mu' = 1 \text{ and } \chi(h(\cdot))/\chi(\cdot) = \varphi(\cdot) \text{ is ess. bd.}$$

We then set $d\mu = \chi^{-1}d\mu'$. If $\{g_i\}$ are the eigenfunctions of h we set $f_i = g_i\chi$. These are the eigenfunctions of $T=P^*$. The solutions (B,T) are now given by the invariant subspaces of the space spanned by $\{f_i\}$ in L_q . Naturally even if the collection $\{g_i\}$ is nonempty, solutions exist iff $\{f_i\} \subset L_q$, which since $|g_i|(\cdot) = \text{constant a.e.}$ is equivalent to the condition $\chi \in L_q(\mu)$, where $\chi(\cdot)$ satisfies (*). This condition is always satisfied if $q=1$ because of (*) while for $q>1$ it becomes

$$(**) \quad \begin{cases} \int \chi^{q-1} d\mu' < \infty & \text{for } q>1 \\ \chi \text{ is ess. bd.} & \text{for } q=\infty \end{cases}$$

Thus the solution depends on the eigenvalues of h and also on the integrability properties (**) of functions $\chi(\cdot)$ satisfying (*).

Remark. The interest in the conditions above stems mainly from the following observation. Calling an ergodic

m.p.t. h of type A if the condition $\chi(h(\cdot))/\chi(\cdot)$ ess. bd. implies $\chi d\mu' < \infty$, where $\chi(\cdot) > 0$, we have: "If h is of type A then any solution $X(\cdot): S \rightarrow B$ of the eigenoperator equation $X(h(\cdot)) = TX(\cdot)$ is integrable". While we can construct ergodic m.p.t. that are not of type A, e.g. cartesian product of a Bernoulli shift with irrational rotation on the circle, we do not know if any type A transformations exist. An equivalent condition is the following: "For $A \in \mathcal{E}$ with $\mu'(A) > 0$ we construct the sets $A_n = \{s: s, \dots, h^{n-1}(s) \notin A, h^n(s) \in A\}$. Then h is of type A iff $\mu'(A_n) \rightarrow 0$ faster than any power". We do not know whether this is true even for irrational rotations on the unit circle. At any rate for ergodic m.p.t. of type A that are defined by linear operators we can use the constructions in the proof of Theorem 2, because the norm function is necessarily integrable.

R E F E R E N C E S

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