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INFINITELY DIVISIBLE MEASURES ON THE CONE OF AN ORDERED LOCALLY CONVEX VECTOR SPACES

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There are at least two reasons for the study of probability measures on the cone of an ordered vector space. First of all there are some applications to the theory of random measures and especially to the theory of point processes : let T be a locally compact second countable space, $\mathcal{K}(T)$ the space of all continuous functions on T with compact support (with the usual inductive limit topology), $\mathcal{M}(T)$ the space of all Radon measures on T and $\mathcal{M}_+(T)$ the cone of all positive Radon measures. If one provides $\mathcal{M}(T)$ with the weak topology $\sigma(\mathcal{M}(T), \mathcal{K}(T))$, the Borel field of $\mathcal{M}(T)$ is exactly the σ-algebra usually considered in the theory of random measures ([3], [4]). So a random measure on T as a measurable map X from some probability space $(\Omega, \mathfrak{A}, P)$ into $\mathcal{M}_+(T)$ can be viewed as a probability measure of the cone $\mathcal{M}_+(T)$ of $\mathcal{M}(T)$. Moreover one can prove, that a sequence of random measures converges in distribution (see for ex. [3]) if and only if the associated sequence of measures on $\mathcal{M}_+(T)$ converges weakly. The second reason is after my opinion the connection with the structure of ordered locally convex spaces, especially Banach lattices, similar to the situation, when one studies probability measures on not necessarily ordered Banach spaces.

In this paper I treat only one topic which is of interest in the theory of probability measures on vector spaces, namely the description of the structure of those probability measures on a cone which are infinitely divisible.

1. Preliminaries

For the moment let E be an arbitrary real locally convex space with the only additional property, that every compact subset of E is contained in a compact convex circled one (this is the case for example if E is complete).

Let $\mathcal{M}(E)$, $\mathcal{M}^b(E)$, $\mathcal{P}(E)$ denote the set of all positive Radon measures, all positive and bounded Radon measures and all probability measures respectively. $\mathcal{J}(E)$ denotes the set of all infinitely divisible probability measures.

Définition

(i) $\mu \in \mathcal{J}(E)$ is called a Poisson measure :

$$\exists\ G \in \mathcal{M}^b(E) : \mu = e(G) := e^{-||G||} \left(\sum_{k \geq 0} \frac{G^k}{k!} \right)$$

(ii) $\mu \in \mathcal{J}(E)$ is called a Gaussian measure :

$$\mu = \lambda * \nu \ , \ \lambda \in \mathcal{P}(E) \ , \ \nu \text{ Poisson measure} \Rightarrow \nu = \varepsilon_0$$

(Dirac measure)

(iii) $\mu \in \mathcal{J}(E)$ is called a generalized Poisson measure :

$$\mu = \lim e(G_i) * \varepsilon_{x_i} \quad \text{with}$$

(a) $(e(G_i) * \varepsilon_{x_i})$ is an uniformly tight net and

(b) the net (G_i) is increasing.

One has : $F := \sup G_i$ uniquely describes μ in the sense that if $(e(G_i) * \varepsilon_{y_i})$ is uniformly tight then every limit point of this net is of the form $\nu = \mu * \varepsilon_x$ for some $x \in E$ (see [1] for this and the following).

We therefore use the notation $\mu = \tilde{e}(F)$ and call F Lévy measure.

F has the property : $\exists\ K \subset E$ (compact) : $F(K^c) < \infty$.

We have the following result (s. [1], p. 290) :

Every $\mu \in \mathfrak{J}(E)$ is of the form $\mu = \rho * \tilde{e}(F)$ with a symmetric Gaussian measure ρ.

With the aid of this result one can compute the Fourier transform of μ :

$$\hat{\mu}(x') = \exp\left[i\,\langle a,x'\rangle - Q(x') + \int_{E\setminus\{0\}} \left(e^{i\langle x,x'\rangle} - 1 - i\langle x,x'\rangle 1_K\right) dF(x)\right]$$

with $a \in E$, Q the positive quadratic form belonging to ρ ([1], p. 289), F Lévy measure of μ with $F(K^c) < \infty$.

2. The Laplace transform of infinitely divisible measures on a cone

From now on we demand that E has the following additional properties :

(i) E is an ordered locally convex space with positive cone C.

(ii) C is a normal cone, i.e. there is a generating family $\mathcal{P}$ of semi-norms on E such that $p(x) \le p(x+y)$ for all $x, y \in C$ and all $p \in \mathcal{P}$.

(iii) Every order bounded, directed net in C converges. (See [5] and [6]).

Remark : Every Banach lattice with order continuous norm ([6], p. 92), for example the L^p-spaces $(1 \le p < \infty)$, fullfills the above conditions.

Let $\mathfrak{J}(C)$ denote the set of all infinitely divisible probability measures concentrated on C. We have the following

Theorem

Every $\mu \in \mathfrak{J}(C)$ has the following Laplace transform

$$L\mu(x') = \exp\left[-\,\langle a,x'\rangle + \int_{C\setminus\{0\}} \left(e^{-\langle x,x'\rangle} - 1\right) dF(x)\right] \quad \text{for all } x' \in C'$$

where $a \in C$ and the Lévy measure F (also concentrated on C) has the property that for every neighborhood U the following condition holds :

$$\sup \int |\langle x,\ x'\rangle|\ dF(x) < \infty.$$

Proof :

First of all one can easily prove with the aid of the Hahn Banach theorem that with μ also the Lévy measure F of μ must be concentrated on C. Furthermore μ cannot have any non trivial Gaussian factor ρ. For suppose $\mu = \rho * \nu$ with $\rho \neq \varepsilon_0$ and symmetric. Because of

$$1 = \mu(C) = \int \rho(C - x)\, d\nu(x)$$

there is a $b \in E$ with $\rho(C-b) = 1$. Symmetry implies $\rho(-C+b) = 1$, therefore $\rho([-b,b]) = 1$.

But now : $\exists\ x' \in C'$ (dual Cone) : $\pi_{x'}(\rho) \neq \varepsilon_0$ (where $\pi_{x'}$ denotes the canonical projection : $\pi_{x'}(x) = \langle x,x'\rangle\ \forall\ x \in E$). Therefore, $\pi_{x'}(\rho)(\pi_{x'}([-b,b])) = \pi_{x'}(\rho)([-\langle b,x'\rangle, \langle b,x'\rangle]) = 1$, which is impossible.

Now we have $\mu = \tilde{e}(F)$ with $F(K^c) < \infty$ for some compact, convex circled set $K \subset E$. Let $\{U_i\}$ be a neighborhood base, and set $G_i := \mathrm{Rest}_{K\cap U_i^c} F$. Then we have for all i $\mu = \lambda_i * e(G_i) * e(\mathrm{Rest}_{K^c} F)$. Intoducing centering elements $a_i \in C$, defined by $\langle a_i,x'\rangle = \int \langle x,x'\rangle\, dG_i(x)$, we get

$$\mu = \nu_i * \varepsilon_{-a_i} * e(G_i) * e(\mathrm{Rest}_{K^c}F).$$

One can prove (see [1], p. 294), that the net (ν_i) converges to a point measure ε_z with $z \in C$ and the net $(\varepsilon_{-a_i} * e(G_i))$ is uniformly tight and converges to a certain probability measure λ. Now it is easy to show $z \geq a_i$ for all i, i.e. (a_i) is an order bounded net, which converges by assumption about E against a certain element b of C. Therefore in the case of an infinitely divisible measure on C centering is not necessary, and with $a = z-b$ one computes the Laplace transform as asserted. The condition about F follows now easily from the normality of C :
$\forall\ i\ \exists\ j \geq i : U_i^o \subset U_j^o \cap C' - U_j^o \cap C'$. Hence

$$\sup_{x'\in U_i^o} |\langle a_i,x'\rangle| \leq \sup_{x'\in U_j^o\cap C'-U_j^o\cap C'} |\langle a_i,x'\rangle| \leq 2 \sup_{x'\in U_j^o\cap C'} \langle a_i,x'\rangle \leq 2 \sup_{x'\in U_j^o\cap C'} \langle z,x'\rangle < \infty.$$

Because of the $\langle a_i,x'\rangle = \int_{K\cap U_i^c} \langle x,x'\rangle\, dF(x)$ one gets the final assertion.

Remarks :

1. It is not clear, wether or not the above formula for the Laplace transform of an infinitely divisible measure on C is still valid in spaces, in which not every order bounded directed net in C converges. For example in C[0,1] it might happen that the centering elements only converge in the bidual of C[0,1].

2. If E is a Hilbert space (not necessarily ordered) one knows that a measure F is a Levy measure if and only if F fullfills the condition $\int \inf(1_E, ||x||^2)dF(x) < \infty$. For probability measures on R_+ one has the stronger property $\int \inf(1,t)dF(t) < \infty$. If E is an (AL)-space, one can also prove $\int \inf(1_E, ||x||)dF(x) < \infty$ for every Lévy measure on the positive cone of E whereas this result is not correct for ℓ^2. The question is now : if one has $\int \inf(1_E, ||x||)dF(x) < \infty$ for some measure F on C, is then F a Lévy measure ? Moreover : for which spaces is this valid ?

3. The Laplace transform of stable measures on a cone

If E is an arbitrary locally convex space, a probability measure μ is called stable, if μ satisfies the property

$$\forall\, s_1, s_2 > 0 \ \exists\, a \in E \ \exists\, t > 0 : H_{s_1}(\mu) * H_{s_2}(\mu) = H_t(\mu) * \varepsilon_a ,$$

where for every s > 0, $H_s : E \to E$ denotes the mapping defined by $H_s(x) = s\,x$ for all $x \in E$.

Let μ be a stable measure with Lévy measure F. Let K be a compact convex, circled subset of E with $F(K^c) < \infty$, and let E_K denote the Banach space with unit ball K and sphere $S_K := \{x \in E_K : ||x|| = 1\}$. Consider the following diagramm:

$$E \xleftarrow[i]{} E_K \setminus \{0\} \xrightarrow[\varphi]{} R_+^\times \times S_K ,$$

where i denotes the inclusion map and φ is defined by $\varphi(a,x) = ax$ for all $a > 0$, $x \in S_K$. One can show (see [2], p. 156 for the idea of the proof), that there exist a measure $H \in \mathcal{M}^b(S_K)$ and a measure $\nu \in \mathcal{M}^b(\mathbb{R}_+)$ with density $t \to \frac{1}{t^{1+\alpha}}$ (for some $\alpha \in \,]0, 2[$) with respect to Lebesgue measure, such that $F = (i \circ \varphi)(\nu \otimes H)$.

Let now E be an ordered locally convex space with the three properties of the last section. Then one gets the Laplace transform

$$L\mu(x') = \exp \left[-\langle a,x'\rangle + \int_{S_K \cap C} \left(\int_0^\infty (e^{-t\langle z,x'\rangle} - 1) \frac{1}{t^{1+\alpha}}\, dt\right) dH(z)\right]$$

(for all $x' \in C'$). Furthermore one has for all x' C'

$$\infty > \int_K \langle x,x'\rangle dF(x) = \int_{S_K} \langle z,x'\rangle \left(\int_0^1 t^{-\alpha} dt\right) dH(z).$$

Since the function $t \to t^{-\alpha}$ is integrable over $[0, 1]$ only for $\alpha \in \,]0, 1[$ we have the following

<u>Theorem</u>

For every stable measure μ on the positive cone C of E, there exist an $\alpha \in \,]0,1[$, an $a \in C$ and a measure $H \in \mathcal{M}^b(S_K)$, such that the Laplace transform of μ is of the form

$$L\mu(x') = \exp \left[-\langle a,x'\rangle + A \int_{S_K \cap C} \langle z,x'\rangle^\alpha \, dH(z)\right] \quad \text{for all } x' \in C'$$

where the constant $A > 0$ only depends on α.

References

[1] Dettweiler, E. : Grenzwertsätze für Wahrscheinlichkeitsmasze auf Badrikianschen Räumen. Z. Wahrscheinlichkeitstheorie verw. Gebiete 34, 285-311 (1976)

[2] Dettweiler, E. : Stabile Masze auf Badrikianschen Räumen. Math. Z. 146, 149-166 (1976)

[3] Kallenberg, O : Lectures on random measures. University of North Carolina at Chapel Hill, Institute of Statistics Mineo Series n° 963 (1974)

[4] Neveu, J. : Processus ponctuels. Lectures. Ecole d'été de Saint-Flour (1976)

[5] Schaefer, H.H. : Topological Vector Spaces. New-York London Macmillan (1966)

[6] Schaefer,H.H. : Banach Lattices and Positive Operators. Berlin-Heidelberg-New-York, Springer (1974).