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THE THEORY OF BOOLEAN ALGEBRAS WITH A DISTINGUISHED SUBALGEBRA IS UNDECIDABLE

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§ 0. INTRODUCTION

We prove the following theorems:

Theorem 1** Let $T_1$ and $T_2$ be theories in the language $L = \{ \cup, \cap, -, 0, 1 \}$ such that there are infinite Boolean algebras (hereafter denoted by BA) $B_1, B_2$ such that $B_i \models T_i$ for $i = 1, 2$, let $P$ be a unary predicate and $S = T_1 \cup T_2(P)$, where $T_2(P)$ is the relativization of $T_2$ to $P$, then $S$ is undecidable.

Theorem 2: The theory of 1-dimensional cylindric algebras (denoted by CA1) is undecidable. Theorems 1 and 2 answer a question of Henkin and Monk in [2] Problem 7; there they also point out that the decidability problems of theorems 1 and 2 are closely related, this relation is formulated in the following proposition:

Proposition: (a) Let $<B, c>$ be a CA1 where $B$ is a BA and $c$ a unary operation on $B$ then $A = \{ b \mid b \in B$ and $c(b) = b \}$ is a subalgebra of $B$, and for every $b \in B$ $c(b)$ is the minimum of the set $\{ a \mid b \leq a \in A \}$.

(b) Let $B$ be a BA and $A$ be a subalgebra of $B$ suppose that for every $b \in B$ $a_b = \min(\{ a \mid b \leq a \in A \})$ exists; define $c(b) = a_b$, then $<B, c>$ is a CA1.

Let $T_C$ be the theory of CA1's and $T_B$ be the theory of BA's with a distinguished subalgebra $P$, with the additional axiom that for every $b$ there is a minimal $a_b$ such that $P(a_b)$ and $b \leq a_b$, then certainly $T_C$ and $T_B$ are bi-interpretable.

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** R. McKenzie proved independently at about the same time, that the theory of Boolean algebras with a distinguished subalgebra is undecidable. The method of his proof is different from ours.
The classical result about the decidability of the theory of BA's appears in Tarski's \[5\], and in Ershov \[1\]. Ershov in \[1\] also proved that the theory of BA's with a distinguished maximal ideal is decidable, Rabin \[4\] proved the decidability of the theory of countable BA's with quantification over ideals.

Henkin proved that the equational theory of CA\(_2\)'s is decidable and Tarski proved the undecidability of the equational theory of CA\(_n\)'s for \(n \geq 4\).

In our construction we interpret the theory of two equivalence relations in a model \(<B, U, \cap, \cdot, 0, 1, A\rangle\) but neither B nor A are complete BA's. We do not know the answer to the following question:

Let \(K = \{<B, U, \cap, \cdot, 0, 1, A> \mid B \text{ is a BA, } A \text{ is a subalgebra of } B, A \text{ and } B \text{ are complete}\}\). Is \(\text{Th}(K)\) decidable?

We also do not know whether an analogue of theorem 1 for \(T_B\) holds. For instance let \(S\) be \(T_B\) together with the axioms that say that both the universe and \(P\) are atomic BA's is \(S\) decidable?
§ 1. THE CONSTRUCTION

∪, ∩, ·, 0, 1 denote the operations and constants of a BA and ≤ denotes its partial order. A, B, C denote BA’s; At(B), A ã (B), As(B) denote the set of atoms of B, the set of non-zero, non-maximal atomless elements of B and the set of non-zero, non-maximal atomic elements of B respectively. Let I(B) be the ideal generated by At(B) ∪ As(B), B ⊕ = B/I(B) and if b ∈ B b(0) = b/I(B). If D ⊆ B c δ (D) denotes the subalgebra of B generated by D. B × C denotes the direct product of B and C. \( \prod_{j \in J} B_j \) denotes the direct product of \( \{ B_j \mid j \in J \} \), and we assume that for every \( j_1 \neq j_2 \) \( B_{j_1} \cap B_{j_2} = \{ 0 \} \), so we can identify the element c of \( B_{j_0} \) with the element \( f_c = \prod_{j \in J} B_j \) where \( f_c(j) = 0 \) if \( j \neq j_0 \) and \( f_c(j_0) = c \). We denote by \( I_B \) the maximal element of B.

Let \( B_T \) be the BA of finite and cofinite subsets of \( \omega \) and \( B_L \) the countable atomless BA. Let \( F_1 \) be the non-principal ultrafilter of \( B_T \) and \( F_2 \) be an ultrafilter in \( B_L \); let \( B_M \) be the following subalgebra of \( B_T \times B_L : B_M = \{ (a,b) \mid a \in F_1 \text{ iff } b \in F_2 \} \); notice that \( B_M(0,1) \). For every i let \( B_i \cong B_M \), \( B_i > = \prod_{i \in \omega} B_i \) and \( B_i < = \sum_{i \in \omega} B_i \). We denote \( I_B \) by \( I_i \).

Lemma 3: Let \( E_0 \) and \( E_1 \) be equivalence relations on \( \omega \) then there is a model \( M = \langle B_T, \cup, \cap, \cdot, 0, 1, A \rangle \models T_B \) such that \( \langle \omega, E_0, E_1 \rangle \) is explicitly interpretable in M.

Proof: We denote by \( i/E_0 \) the \( E_0 \)-equivalence class of i and by \( \omega /E_0 \) the set of \( E_0 \)-equivalence classes. For every \( i \in \omega \) let

\[
\{ b^i_{e, \sigma, j} \mid e \in \{ 0,1 \}, \sigma \in \omega /E_0, j \in \omega \} \subseteq A \cup \{ B_i \}
\]

be such that

\[<e, \sigma, j > \neq <e', \sigma', j'> \Rightarrow b^i_{e, \sigma, j} \cap b^i_{e', \sigma', j'} = 0 \text{ and for every } b \in A \cup \{ B_i \} \]

\[1 \leq \langle <e, \sigma, j > \mid b \cap b^i_{e, \sigma, j} \neq 0 \rangle \leq \aleph_0 \]

For every \( i \in \omega \) let

\[\{ a^i_{e, \sigma, j} \mid e \in \{ 0,1 \}, \sigma \in \omega /E_0, j \in \omega \}
\]

be a set of pairwise disjoint subsets of \( \text{At}(B_j) \) such that \( \text{At}(B_j) = \cup \{ a^i_{e, \sigma, j} \mid e \in \{ 0,1 \}, \sigma \in \omega /E_0, j \in \omega \} \) and

\[
\begin{aligned}
&1 \leq \langle a^i_{e, \sigma, j} \mid e \in \{ 0,1 \}, \sigma \in \omega /E_0, j \in \omega \rangle \leq \aleph_0.
&1 \leq \langle a^i_{e, \sigma, j} \mid e \in \{ 0,1 \}, \sigma \in \omega /E_0, j \in \omega \rangle \leq \aleph_0.
\end{aligned}
\]
For every $\varepsilon$, $\sigma$, and $j$ as above let $c_{\varepsilon, \sigma, j} \in B^>$ be
\[
\bigcup \{ b_i \mid c_{\varepsilon, \sigma, j} \}\bigcup \bigcup \{ a_i \mid c_{\varepsilon, \sigma, j} \}
\]
where $\bigcup D$ denotes the supremum of $D$ in $B^>$. Let $A = \{ c \mid \varepsilon \in \{0, 1\}, \sigma \in \omega, j \in \omega, B = c \omega \}$ and $M = \langle B, \omega, \cap, \cdot, 0, 1, A \rangle$. We show that $M \models T_B$. It suffices to show that $a(b) = \min\{ a \mid b \subseteq a \in A \}$ exists for elements $b \in B$ of the following forms:

- $b \in \text{At}(B_i) \cup \text{At}(B_i)$; $b \in B_i$ and $b(1) = 1$ for almost all $i \in \omega$; this follows from the fact that every $b \in B$ can be represented in the form $b = \bigcup_{i=1}^n (b_i \cap a_i)$ where each $b_i$ is of the above form and $a_i \in A$. In each of the above cases the existence of $a(b)$ is easily checked. Thus $M \models T_B$.

We now define formulas $\varphi_U(x,y), \varphi_{Eq}(x,y)$, $\varphi_{c\omega(x,y)} \in \{0, 1\}$ such that

- $M \models \varphi_U(a) \text{ iff for some } i \in \omega a(1)_i = 1$, $M \models \varphi_{Eq}(a,b) \text{ iff } a(1)_i = b(1)_i$.

and $M \models \varphi_{c\omega}(a_1, a_2) \text{ iff for some } i_1, i_2 \in \omega a(1)_{i_1} = 1 \text{ and } a(1)_{i_2} > a(1)_{i_1}$.

$\varphi_U(x)$ says that $x(1) \in \text{At}(B(1))$ and for no $y \in \text{At}(A)$ $x(1) = y(1)$.

$\varphi_{Eq}(x,y)$ says that $x(1) = y(1)$.

$\varphi_{c\omega}(x,y)$ says: $\varphi_U(x) \land \varphi_U(y)$ and there are $x_1, y_1$ such that $x(1) = x_1(1), y(1) = y_1(1)$ and for every $z \in \text{At}(A)$ $|\{ u \mid z \cap x_1 = u \in \text{At}(B)\}| = 1 \iff |\{ u \mid z \cap y_1 = u \in \text{At}(B)\}| = 1$. $z_1$ is defined similarly. The desired properties of $\varphi_U, \varphi_{Eq}$ and $\varphi_{c\omega}$ are easily checked, and the lemma is proved.

Since the theory of two equivalence relations is undecidable $T_B$ and $T_C$ are undecidable and theorem 2 is proved.

Theorem 1 easily follows from the following lemma.

**Lemma 4**: Let $E_1$, $E_2$ be equivalence relations on $\omega$ then there are models $M_i = \langle B_i, \omega, \cap, \cdot, 0, 1, A_i \rangle$ such that $\langle \omega, E_1, E_2 \rangle$ is explicitly interpretable in $M_i$ and $B_1$, $A_1$ are atomic, $B_2$, $A_2$ are atomless, $B_3$ is atomic $A_3$ is atomless, and $B_4$ is atomless $A_4$ is atomic.

**Proof**: Let $B_0$, $A_0$, $M_0$ denote $B$, $A$ and $M$ of lemma 3 respectively. For $i = 1, 2$ $M_i$ can easily be constructed so that $\langle B_i/H_i, \omega, \cap, \cdot, 0, 1, A_i/H_i \rangle \cong M_0$ where $H_i = \{ b \mid b \in B_i \}$ and for every $a \subseteq b \in A_i$. Since such an $H_i$ is definable in $M_i$ $M_0$ can be interpreted in $M_i$ $i = 1, 2$. 

For $i = 3$ a similar construction works. Let $B$ be an atomic saturated countable BA and $I$ a maximal non-principal ideal of $B$. Let $A$ be an atomless subalgebra of $B$ such that:
(a) for every $b \in B$ which contains infinitely many atoms there is a non-zero $a \in A$ such that $a \subseteq b$;
(b) for every $b \in A_3(B)$ there is an $a \in A$ such that $(a-b) \cup (b-a)$ contains only finitely many atoms of $B$. Let $J = 1 \cap A$. For every non-zero $a \in B_0$ let $F_a$ be an ultrafilter in $B$ which contains $a$, and $< B_a, A_a, I_a, J_a >$ a copy of $< B, A, I, J >$. Let $B^1 = \prod \{ B_a \mid 0 \neq a \in B_0 \}$ and let $B_3$ be the following subalgebra of $B^1$:

$$B_3 = c \times (\bigcup \{ I_a \mid 0 \neq a \in B_0 \} \cup \{ g_a \mid 0 \neq a \in B_0 \})$$

where $g_a(b) = 1_{B_a}$ if $a \in F_b$ and $g_a(b) = 0$ otherwise. Let $A_3 = c \times (\bigcup \{ I_a \mid 0 \neq a \in B_0 \} \cup \{ g_a \mid a \in A_0 \})$.

Certainly $B_3$ is atomic and $A_3$ is atomless. Let $I = \{ a \mid 0 \neq a \in A_3 \}$, then $< B^1_3, U, \cap, \cup, 0, 1, A_3^1 > \cong M_2$, so $M_2$ is interpretable in $M_3$ and thus $< \omega, E_1, E_2 >$ is interpretable in $M_3$ as desired.

In order to construct $M_4$ we assume that $B_1$ is a subalgebra of $P(\omega)$ and

$$\text{At}(B_1) = \{ \{ n \} \mid n \in \omega \}.$$ Let $B_i \cong B$ for every $i \in \omega$. $B_4$ is the following subalgebra of $\bigcup_{i \in \omega} B_i$:

$$B_4 = c \times \left( \bigcup_{i \in \omega} B_i \cup \{ f_a \mid a \in B_1 \} \right)$$

where $f_a(n) = 1_{B_a}$ if $n \in a$ and $f_a(n) = 0$ otherwise. Let $A_4 = c \times \{ f_a \mid a \in A_1 \}$ and $M_4 = < B_4, U, \cap, \cup, 0, 1, A_4 >$.

Certainly $B_4$ is atomless and $A_4$ is atomic. Let $B^1_4 = \{ b \mid b \in B_4 \}$ and for every $a \in \text{At}(A_4)$ either $b \supseteq a$ or $-b \supseteq a$, then $< B^1_4, U, \cap, \cup, 0, 1, A_4 > \cong M_1$ and $B^1_4$ is certainly definable in $M_4$, thus $< \omega, E_1, E_2 >$ is definable in $M_4$ and the lemma is proved.

We omit the proof of theorem 1 which follows easily from lemma 4, the fact that every countable BA can be embedded in e.g. $B_4$, and from [6] pp. 293-302.
REFERENCES


