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ON AN OSCILLATING RADOM WALK

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Let $\{Y_n, n \geq 0 ; Y_o = x\}$ be a Markov chain with values in R and such that

$$\Pr (Y_{n+1} - y \in A \mid Y_n = y) = \mu (A) \qquad \text{if} \quad y < 0 ;$$
$$= \nu (A) \qquad \text{if} \quad y > 0 ;$$
$$= \alpha \mu (A) + \beta \nu (A) \qquad \text{if} \quad y = 0.$$

Here, μ and ν denote given probability measures on R, while α and β are non negative constants, $\alpha + \beta = 1$. We are interested in recurrence properties and exact formulae for the process $\{Y_n\}$.

The special case $\mu = \nu$ is precisely the ordinary random walk governed by the measure μ . The special case $\nu(A) = \bar{\mu}(-A)$ will be called the antisymmetric case. In that case one may identify the points y and -y and thus obtain the process $Y_{n+1} = |Y_n - X_{n+1}|$; here, $\{X_n\}$ denotes an i.i.d. sequence of random variables with a common distribution μ . If moreover the measure μ is carried by $[0, + \infty)$ one may speak of a one-sided antisymmetric case.For an application to the construction of electrical cables, see [6] and [4] p. 208.

For the case where both μ and ν are carried by $\{-1, 0, +1\}$, this process was already treated by Bhat [1], [2] . The general process has applications in statistics, see [3] , and in information theory, see [5].

Consider the measure

$$Q\ (A) = \sum_{n=0}^{\infty} t^n \Pr\ (Y_n \in A),$$

where t is a fixed number, $|t| < 1$. It satisfies the identity

$$(Q^- + \alpha Q^\circ)\ (\delta_0 - t\mu) + (Q^+ + \beta Q^\circ)\ (\delta_0 - t\nu) = \delta_x$$

Here, Q^- denotes the restriction of Q to $(-\infty, 0)$. **Similarly,** Q° and Q^+. Further, δ_x denotes the probability measure supported by $\{x\}$. Finally, the above equation is to be interpreted in terms of the Banach algebra of all finite measures, where the multiplication is taken as the ordinary convolution. Let

$$L_\mu^+ = \sum_{n=1}^{\infty} \frac{t^n}{n} (\mu^n)^+,$$

similarly, L_μ° , L_ν^- , etc. Using that

$$\delta_0 - t\mu = \exp\ (- L_\mu^- - L_\mu^\circ - L_\mu^+)\ ,$$

one easily finds that

$$Q^\circ = \lambda_{-x}\ (\alpha\, e^{-L_\mu^\circ} + \beta\, e^{-L_\nu^0})^{-1}\ ,$$

where λ_j is defined by

$$\lambda = e^{L_\mu^+}\ e^{L_\nu^-} = \sum_{j=-\infty}^{+\infty} \lambda_j\ \delta_j$$

If μ and ν are absolutely continuous (relative to Lebesgue measure) and $x \neq 0$ then one further has that

$$Q^- = (\delta_x\ \lambda)^-\ e^{L_\mu^- - L_\nu^-}\ ;$$

$$Q^+ = (\delta_x\ \lambda)^+\ e^{L_\nu^+ - L_\mu^+}\ .$$

These formulae have many applications.

We discuss in detail the case where both μ and ν are supported by the set Z of all integers. For instance, in the one-sided antisymmetric case the state 0 is recurrent if and only if

$$\int_{-\varepsilon}^{+\varepsilon} |1 - \hat{\mu}(\theta)|^{-2} \, d\theta = +\infty .$$

Here, $\hat{\mu}(\theta)$ denotes the Fourier transform of μ . In the general case, the state 0 is recurrent if and only if

$$\sum_{h=1}^{\infty} C_\mu^+(h)\, C_\nu^-(h) = +\infty .$$

Here,

$$C_\mu^+(h) = \sum_{n=1}^{\infty} P_\mu (S_n = h \; ; \; S_m > 0 \quad \text{if} \quad 1 \leqslant m \leqslant n) \; ;$$

$$C_\nu^-(h) = \sum_{n=1}^{\infty} P_\nu (S_n = -h \; ; \; S_m < 0 \text{ if } \; 1 \leqslant m \leqslant n).$$

Here, the index μ in P_μ indicates that $(S_n = X_1 + \dots + X_n)$ is an ordinary random walk governed by the measure μ . The quantity $C_\mu^+(h)$ may also be interpreted as the renewal function associated with the random variable $Z_\mu^+ = S_N$ with $N = \inf \{n > 0 \; ; \; S_n > 0\}$, see [7]. Similarly, for $C_\nu^-(h)$.

Let m_μ and σ_μ^2 denote the mean (assumed finite) and variance associated with the measure μ ; similarly, m_ν and σ_ν^2 . Then the following conditions are each sufficient for the state 0 to be recurrent.

(i) $0 < m_\mu < \infty \quad ; \; -\infty < m_\nu \leqslant 0 \; ;$

(ii) $0 \leqslant m_\mu < \infty \quad ; \; -\infty < m_\nu < 0 \; ;$

(iii) $m_\mu = 0 \; ; \; m_\nu = 0 \; ;$ either $\sigma_\mu^2 < \infty$ or $\sigma_\nu^2 < \infty$.

Non-recurrent would be for instance the case where μ is supported by $[0,\infty)$ such that $\mu(\{n\}) \sim n^{-a-1}$, while ν is supported by $(-\infty, 0]$ such that $\nu(\{-n\}) \sim n^{-b-1}$, with a and b as positive constants such that $a + b < 1$. With a positive probability 0 will never be reached, though the process always moves in the direction of 0, occasionally making huge jumps across 0.

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