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THE RULE OF INDUCTION IN THE THREE VARIABLE ARITHMETIC BASED ON + AND ·

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I - INTRODUCTION

Shoenfield [1] has shown for the free variable system with constants 0, ' (successor), P (predecessor), + and axioms :

$$\begin{aligned} &A1 && x' \neq 0 \\ &A2 && P0 = 0 \\ &A3 && Px' = x \\ &A4 && x + 0 = x \\ &A5 && x + y' = (x + y)' \end{aligned}$$

that the rule of induction

$$A(0),\ A(x) \longrightarrow A(x') - A(x)$$

is equivalent to the four axioms

$$\begin{aligned} &B1 && x \neq 0 \longrightarrow x = (Px)' \\ &B2 && x + y = y + x \\ &B3 && (x + y) + z = x + (y + z) \\ &B4 && x + y = x + z \longrightarrow y = z. \end{aligned}$$

We do the same here for the system obtained from A1 – 5 by adding a new constant · and axioms

$$\begin{aligned} &A6 && x \cdot 0 = 0 \\ &A7 && x \cdot y' = x \cdot y + x \end{aligned}$$

We show that here the rule of induction is equivalent to B1 – 4 above plus

$$\begin{aligned} &B5 && x \cdot y = y \cdot x \\ &B6 && x \cdot (y \cdot z) = (x \cdot y) \cdot z \\ &B7 && x \cdot (y + z) = x \cdot y + x \cdot z \end{aligned}$$

and

$$(d = 2,3,\ldots)\ C'_d \quad dy = dy_1 \longrightarrow \bigvee_{i=0}^{d-1} (x + i)y = (x + i)y_1.$$

By constructing non standard models we show the non finite axiomatisability of this theory and the independence from it of various simple axioms, e.g. of $x^3 + y^3 \neq z^3 \vee xyz = 0$ which answers a question of Kreisel [1]. I am grateful to Kreisel for his infectious interest in such free variable systems and to M.D. Gladstone whose unpublished work was the starting point of this.

2 - RULES AND AXIOMS OF INDUCTION -

Although our main interest is in free variable systems we will work throughout in the full first order predicate calculus with identity. This will allow us to consider the relative strengths of the free variable rule of induction and various axioms containing quantifiers. The completeness theorem for the free variable calculus tells us that the theorems of the free variable system based on certain quantifier free axioms and the rule of induction above coincide with the open (i.e. quantifier free) theorems of the first order system with the same axioms and the restricted rule of induction

$$\text{RIO} \quad A(0),\ A(x) \longrightarrow A(x') \vdash A(x) \text{ for all } \textit{open}\ A$$

In this framework we can now state our result and Shoenfield's in stronger terms.

Consider the axiom schema of induction

$$\text{AIO} \quad A(0) \wedge (x)\,(A(x) \longrightarrow A(x')) \longrightarrow A(x) \text{ for all } \textit{open}\ A$$

This is clearly at least as strong as RIO ; in general it is not derivable from RIO - the usual proof of the axiom of induction from the rule of induction uses the rule on a quantified formula. However both in Shoenfield's case and ours it turns out that AIO is derivable from the other axioms and RIO for :

Theorem 1. A1 – 5, B1 – 4 *imply all instances of* AIO *arising* from formulae A(x) which contain only the constants $0, ', P, +, \cdot$

Theorem 2. A1 – 7, B1 – 7, C'_d $(d = 2,3,\ldots)$ *imply all instances of* AIO *arising from formulae* A(x) $0, ', P, +, \cdot$

Theorem 1 was proved by Shoenfield ; theorem 2 will be proved here. Since it is routine to show that B1 – 4 (B1 – 7, C'_d respectively) are provable from A1 – 5 (A1 – 7) by RIO applied to formulae in $0, ', P, +$ only ($0, ' P, +, \cdot$) these theorems are all that is needed to give the equivalences referred to above. Note that the fact that AIO is provable means that these equivalences are preserved under the addition of new axioms. (For a fuller discussion of the relation between RIO and axioms of induction see Shepherdson [2]). The formula A(x) in AIO is of course allowed to contain other free variables ; replacing these by constants standing for elements of a model we get the model theoretic version of theorem 2 which is what we shall prove :

Lemma 1. *If* M *is any model for* A1 – 7, B1 – 7, C'_d $(d = 2,3,\ldots)$, *and* A(x) *is any formula built up from the logical constants* $\neg, \vee, =$ *and the non logical constants* $0, ', P, +, \cdot$ *together with individual constants of* M, *then if* A(0), (x) (A(x) ⟶ A(x')) *are true in* M *so is* (x)A(x).

3 - OUTLINE OF THE PROOF -

We first observe that, being a model for A1 – 7, B1 – 7, M can in the familiar way be embedded in a commutative ring R (of formal differences). From now on we shall work mainly in R. We note that $', P$ can be eliminated, viz. replace x' by x + 1 (we write n for $0'\ldots$ (n times) $\ldots'$, so $1 = 0'$ is the unit element of R), replace B(Pt) by $(t = 0 \wedge B(0)) \vee (t \neq 0 \wedge B(t-1))$. In this way the atomic formulae of A(x) are reduced to equations between terms built up using $+, -, \cdot$ from x and elements of M. These can be further reduced, since R is a ring, to equations of the form f(x) = 0 where f is a polynomial with coefficients in R. Note that axioms C' can in R be expressed by

$$dy = 0 \longrightarrow \bigvee_{i=0}^{d-1} (x + i)y = 0.$$

[This is easy but not quite as obvious as it looks for we are asserting its truth for all x, y of R, not merely for all x of M and y of R.]

Now since A(0), (x) (A(x) ⟶ A(x + 1)) are supposed true in M, A(x) is certainly true for all natural x of M (numbers of the form $0'\ldots'$). So we have only to show that A(a) is true for all unnatural elements a of M. Axiom B1 implies that for such an a the model M contains elements

$$\ldots,\ a-2,\ a-1,\ a,\ a+1,\ a+2,\ \ldots$$

all distinct; we call such a set of elements (i.e an equivalence class under the relation $x-y$ = integer) a *comparison class*. We now briefly sketch Shoenfield's proof of theorem 1 since our argument is best regarded as an extension of it. In his case (+ only) the basic equations of $A(x)$ were linear equations $bx + c = 0$. Axioms A1 – 5, B1 – 4 are not sufficient to imply that a non identical equation of this kind has at most one solution but they do imply it has at most one solution in each comparison class. Thus, for sufficiently large n, the equation has the same truth value for n and for a-n (viz : T if it is an identity, F otherwise). This property is clearly preserved under $\neg$, $\vee$ so the whole formula $A(x)$ has it. But $A(n)$ is true for all natural n, hence $A(a-n)$ is true for sufficiently large n, and this together with the truth of $(x)\,(A(x) \longrightarrow A(x+1))$ shows that $A(a)$ is true.

Gladstone (unpublished) observed that if we add axioms which ensure that our ring R is an integral domain then Shoenfield's argument carries over to the case of multiplication and shows that the enlarged axiom system implies AIO. For in an integral domain a non identical polynomial equation of the n^{th} degree has at most n roots. We shall see that Gladstone's axioms have a non standard model containing elements t, $t\sqrt[3]{2}$ such that $t^3 + t^3 = (t\sqrt[3]{2})^3$ which answers Kreisel's question about the independence of this. However Gladstone's axioms themselves are not derivable by RIO from A1 – 7 so they do not give an answer to our problem of finding an equivalent open axiom system without induction. From the system A1 – 7, B1 – 7, $C_d^!$ $(d = 2, 3, \ldots)$ which we claim is the answer, we cannot prove that R is an integral domain. We cannot even prove that every non identical polynomial equation has only a finite number of solutions in each comparison class. Instead we define for each positive integer d the ideal I_d of R formed by all elements which annihilate all elements annihilated by d :

$$z \in I_d \longleftrightarrow (y)\ (dy = 0 \longrightarrow zy = 0)$$

(the variables now range over R). We show

Lemma 2. For each positive integer d *each element of* R *is congruent modulo* I_d *to one of* $0, 1, \cdots, d-1$

Lemma 3. If, in R, *a polynomial equation of the* n^{th} *degree has more than* n *distinct roots in some comparison class then there is a positive integer* d *such that the set of its roots is the union of certain residue classes modulo* I_d.

The proofs of these are given in the next two sections. We proceed to show how to obtain Lemma 1 from them. First

Lemma 4. For each open formula $A(x)$ *of* M *with* x *as the only free variable there exists a positive integer* d *such that*

$$x \equiv y(I_d) \longrightarrow (A(x) \longleftrightarrow A(y))$$

for 'almost all' x, y in R, *i.e.*

for all $x, y \in R - E$ *where the exceptional set* E *contains at most a finite number of elements in each comparison class. Proof of lemma 4.* Take first the case where $A(x)$ is a non identical polynomial equation $f(x) = 0$. Let Z be its set of roots. Then $A(x)$ is false for all $x \in R - Z$ so if Z has only a finite number of elements in each comparison class the lemma is satisfied with $d = 1$ (as it clearly is if $f(x) = 0$ is identically true). And if Z has an infinity of elements in any comparison class we can take the d given by lemma 3 and E as the null set. Now if the result is true for $A(x)$ it is clearly true for $\neg A(x)$. If true for $A_1(x)$ with $d = d_1$ and $E = E_1$ and for $A_2(x)$ with $d = d_2$ and $E = E_2$ it is true for $A_1(x) \vee A_2(x)$ with $d = d_1 d_2$, $E = E_1 \cup E_2$. For $d_1, d_2 \mid d$ so $I_d \subset I_{d_1}, I_{d_2}$ and hence $x \equiv y(I_d) \longrightarrow x \equiv y(I_{d_1}) \wedge x \equiv y(I_{d_2})$. Hence it is true for all open formulae $A(x)$.

Proof of lemma 1 from lemmas 4, 2. Let a be any unnatural element of M. Since $a, a-1, a-2, \cdots$ are distinct elements of the same comparison class at least one of them, $a-n$ say, does not belong to E. By lemma 2 there exists i, $0 \leqslant i \leqslant d-1$ such that $a - n \equiv i\ (I_d)$. Now i may belong to E but at least one of the infinity of comparable elements $i, i+d, i+2d, \cdots$, say $i + kd$, does not belong to E. Now $i + kd \equiv i\,(I_d)$ so $a - n \equiv i + kd\,(I_d)$ and since neither of $a-n$, $i + kd \in E$ lemma 4 gives $A(a-n) \longleftrightarrow A(i + kd)$. But $i + kd$ is a natural element so as above $A(i + kd)$ is true. Hence $A(a-n)$ is true and since $(x)(A(x) \longrightarrow A(x+1))$ is true this yields as above the truth of $A(a)$.

Before giving the proof of lemma 2 we observe that it is not necessary if all we want is to give a set of open sentences equivalent to A1 – 7 plus RIO. For examination of the proofs just given and the proof of lemma 3 which follows later will show that we do in fact deal only with a finite number of elements of I_d at a time so that instead of I_d we could use each time an ideal $I_{x_1, \ldots, x_n}$ consisting of all elements of R which annihilate all of $x_1, \cdots, x_n$, where $x_1, \cdots, x_n$ are certain

elements annihilated by d. If we then choose instead of C'_d the apparently stronger axioms ($d = 2,3,\cdots$; $n = 1,2,3\ \cdots$)

$$C'_{d.n}\ \Lambda_{j=1}^{n}\ dx_j = dy_j \longrightarrow \vee_{i=0}^{d-1}\ \Lambda_{j=1}^{n}\ (x + i)\,x_1 = (x + i)\,y_1$$

which are easily seen to be provable from A1 – 7 by RIO, we have already the result corresponding to lemma 2 and can proceed straight to lemma 3. What we are really doing in the next section is showing that the $C'_{d.n}$ are not independent but all follow from the $C_{d.1}$.

4 - PROOF OF LEMMA 2 -

We first prove in 2 steps that,

$$E_d : \textit{If } x_1 \textit{ is of order } d \textit{ and } dx_2 = 0 \textit{ and } zx_1 = 0 \textit{ then } zx_2 = 0,$$

holds for all d. Here when we say x_1 is of order d we mean that d is the least positive integer which annihilates x_1.

Lemma 5. E_d *holds for* d *of the form* p^n (p *prime)*

Proof. By induction on n. If $n = 0$ both x_1, x_2 are 0 and the result is trivial. Suppose now $n > 0$. Since $dx_1 = dx_2 = 0$, $d(x_1 + x_2) = 0$ also. Hence, by C'_d, for some i,j, $0 \leqslant i,j, < d$,

$$(z + i)\,x_2 = 0 = (z + j)\,(x_1 + x_2)$$

Subtracting and using $zx_1 = 0$ we get

$$jx_1 + (j - i)x_2 = 0. \tag{1}$$

If $j = 0$ then $z(x_1 + x_2) = 0$ from which the desired result $zx_2 = 0$ follows ; if $j = i$ then $jx_1 = 0$ so x_1 is of order $< d$, contrary to hypothesis. So we may assume both j, $j - i \neq 0$. Let p^k ($k < n$) be the highest power of p which divides $j - i$ so that $j - i = p^k b_2$ where $p \nmid b_2$. Multiplying (1) by p^{n-k} we get

$$0 = p^{n-k}\,(jx + p^k b_2 x_2) = p^{n-k} jx_1$$

since $p^n x_2 = 0$. Hence $p^n \mid p^{n-k} j$ i.e. $p^k \mid j$ i.e. $j = p^k a_1$ and (1) takes the form

$$p^k(a_1 x_1 + b_2 x_2) = 0. \tag{2}$$

Now $p^{n-k}x_1$ is of order p^k and $z(p^{n-k}x_1) = p^{n-k}zx_1 = 0$ so by the induction hypothesis (2) gives

$$z(a_1 x_1 + b_2 x_2) = 0.$$

Since $zx_1 = 0$ this gives $zb_2x_2 = 0$. But $p \nmid b_2$ so $(p^n, b_2) = 1$ and there exist a, b such that

$$a\,b_2 + b\,p^n = 1.$$

Thus

$$zx_2 = z(ab_2 + bp^n)x_2 = a(zb_2x_2) + bz(p^n x_2) = 0,$$

which completes the proof.

Lemma 6. If d_1, d_2 *are coprime* E_{d_1} , $E_{d_2} \longrightarrow E_{d_1 d_2}$

Proof : If x_1 is of order d_1d_2 then d_1x_1 is of order d_2 ; also if $d_1d_2x_2 = 0$ then $d_2(d_1x_2) = 0$. Now if $zx_1 = 0$ we have $z(d_1x_1) = 0$ so, by E_{d_2}, $zd_1x_2 = 0$; similarly by E_{d_1}, $zd_2x_2 = 0$. Hence $z(ad_1 + bd_2)x_2 = 0$ for all a, b. Since d_1, d_2 are coprime we can choose a, b so that $ad_1 + bd_2 = 1$ which gives $zx_2 = 0$.

Lemmas 5, 6 tell us that E_d is true for all d. Note that E_d can be written in the equivalent form : if x_1 is of order d then

$$z \in I_d \longleftrightarrow zx_1 = 0,$$

so if d is such that there exists an element x_1 of order d, C'_d applied to x_1 gives the result of

lemma 2 at once, viz $(z + i)x_1 = 0$ for some i, $0 \leqslant i < d$, i.e. $(z + i) \in I_d$ i.e. $z \equiv -i\ (I_d)$ i.e. $z \equiv$ one of $0, 1, \cdots, d-1 (I_d)$, for $d \equiv 0 (I_d)$. If there is no element of order d but there are non zero elements annihilated by d (if there aren't the result is trivially true) and $d = p_1^{\alpha_1} \cdots p_k^{\alpha_1}$ where $p_1, \cdots, p_k$ are distinct primes, then if $p_1^{\beta_1}, \cdots, p_k^{\beta_k}$ are the highest powers of $p_1, \cdots, p_k$ for which there exist non-zero elements $x_1, \cdots, x_k$ of these orders which are annihilated by d, it is easily seen that the element $x = x_1 + x_2 + \cdots + x_k$ is of order $d_1 = p_1^{\beta_1} \cdots p_k^{\beta_k}$ and that $dy = 0 \longleftrightarrow d_1 y = 0$, hence $I_d = I_{d_1}$ from which the result follows as above since $d_1 < d$.

5 - PROOF OF LEMMA 3 -

Let

$$f(x) = a_o + a_1 x + \cdots + a_n x^n.$$

Consider the Lagrange interpolation formula in the form (i,j,r,s run from 0 to n) :

$$\prod_{r<s} (\alpha_r - \alpha_s) f(x) = \sum_i (-1)^i f(\alpha_i) \prod_{j \neq i} (x - \alpha_j) \prod_{r<s, r,s \neq i} (\alpha_r - \alpha_s).$$

This is an equation between polynomials in x, $\alpha_o, \cdots, \alpha_n, a_o, \cdots, a_n$ with integer coefficients. By the usual proof it is true for all elements x, $\alpha_o, \cdots, \alpha_n$, $a_o, \cdots, a_n$ of any field, in particular of any infinite field. Hence it is a polynomial identity. In particular the polynomials L_k and R_k which are the coefficients of x^k on the LHS and RHS are equal ($k = 0, \cdots, n$). Hence for all elements $\alpha_o, \cdots, \alpha_n, a_o, \cdots, a_n$ from any commutative ring $L_k = R_k$, i.e. the formula is an identity between polynomials in x valid in any commutative ring.

Now apply this to the ring R supposing that $\alpha_o, \cdots, \alpha_n$ are n + 1 distinct roots of $f(x) = 0$ from the same comparison class, i.e. such that all differences $\alpha_r - \alpha_s$ are integers. Then $d = \prod_{r<s} (\alpha_r - \alpha_s)$ is an integer and $df(x) = 0$ an identity between polynomials in x with coefficients in R, i.e.

$$d\,a_o = d\,a_1 = \cdots = d\,a_n = 0.$$

(A result which could equally well be reached via the other familiar proof that an equation of the n^{th} degree has at most n roots in a field, viz using the Vandermonde determinant).

Now if $f(x_o) = 0$ and $x \equiv x_o (I_d)$ then $x^k \equiv x_o^k (I_d)$ ($k = 0, \cdots, n$) so, since $da_k = 0$, $(x^k - x_o^k)\, a_k = 0$ i.e. $a_k x^k = a_k x_o^k$ and so $f(x) = f(x_o)$ is also = 0, which completes the proof of lemma 3.

6 - SOME INDEPENDENCE RESULTS

It is easy to characterise the models for A1 – 7, B1 – 7 :

Lemma 7. If R *is a commutative ring with unit and* M *a subset of* R *which is closed under* +, ·, x – 1 *(for* $x \neq 0$*), contains* 0,1 *but not* – 1, *then if',* P *are defined on* M *by* $x' = x + 1$, $Px = x - 1$ $(x \neq 0)$ $P0 = 0$ *we get a model for* A1 – 7, B1 – 7. *All models may be obtained in this way.*

Now take for R the ring I[u,v] of polynomials in two indeterminates u,v with integer coefficients, modulo the ideal (p(u-v)) where p is a prime. Take the subset M consisting of 0 and those elements which are images of elements of I[u,v] with the property that all coefficients of the terms of highest degree (i.e. all terms $au^m v^n$ which are not dominated by terms $a_1 u^{m_1} v^{n_1}$ with $m \geqslant m_1$, $n \geqslant n_1$ and at least one of these inequalities strict) are positive. M clearly satisfies all the conditions of lemma 7 except possibly the last, i.e. that it does not contain –1. This means that no polynomial f(u,v) with positive leading terms should be $\equiv -1 (p(u-v))$ i.e. that $p(u-v) \mid f(u,v) + 1$. This is so for if $p(u-v) \mid f(u,v) + 1$ then putting u = v we should have $f(u,u) + 1 = 0$ which is clearly impossible for such a polynomial. So M is a model for A1 – 7, B1 – 7. It also satisfies C_d' for all d such that $p \nmid d$, for if $p \nmid d$ then $dy = dy_1 \longrightarrow y = y_1$ (viz : in the ring I[u,v] if $p(u-v) \mid d(y - y_1)$ then $p(u-v) \mid (y - y_1)$). But it does not satisfy C_p', for $pu = pv$, but for no integer i is $(u + i)u = (u + i)v$ for this would mean $p(u-v) \mid (u+i)(u-v)$ in I[u,v]. Thus :

Theorem 3. For p *prime* C_p' *is independent of* A1 – 7, B1 – 7 $C_2', \cdots, C_{p-1}'$. *Hence* A1–7, B1–7, $C_d' (d = 2,3 \cdots)$ *is not finitely axiomatisable.* The C_d' are however not independent. It is easy to check that if $d_1 \mid d_2$ then $C_{d_2}' \longrightarrow C_{d_1}'$; that if d_1, d_2 are coprime $C_{d_1}' \wedge C_{d_2}' \longrightarrow C_{d_1 d_2}'$. And a slightly more complicated

version of the above model shows that if p is prime C'_{p^n} is not a consequence of $\{C'_d | p^n \nmid d\}$. So a basis for the C'_d must contain these for a set of d's which contains multiples of all prime powers ; no such set is independent. The stronger axioms

$$C_d \quad dy = dy_1 \longrightarrow y = y_1 \qquad (d = 2,3,\ldots)$$

are provable by double induction, A(x,0), A(0,y), $A(x,y) \longrightarrow A(x',y') \vdash A(x,y)$ applied to open formulae or by single induction applied to quantified formulae. But they are not provable using only RIO ; in fact no finite subset of them will yield the rest using only RIO, for

Theorem 4. If p *is prime* C_d *is independent of A1-7, B1-7,*

$$C'_d \ (d = 2,3,\ldots) \ C_1, \cdots, C_{p-1}$$

So by theorem 2 it is not provable from them using RIO. For a proof use the above argument with the ideal (p(u-v), u(u-v), v(u-v)). This gives for all p a model which satisfies C'_d for all d, and fails to satisfy C_p. If p is prime it satisfies all of $C_1, \cdots, C_{p-1}$.

It follows from this that

Theorem 5. The axiom

$$C^* \quad zy = zy_1 \longrightarrow y = y_1 \vee z = 0$$

is independent of A1-7, B1-7 $C_d (d = 2, 3, \ldots)$.

For C^* implies all of the C_d. A direct independance proof can be given as above using the ideal (u(u-v), v(u-v)). Since $C_d \longrightarrow C'_d$ the system A1-7, B1-7 C_d $(d = 2,3,\ldots)$ is closed under RIO so C^* is not provable from these by RIO, although it is provable by a double induction. The stronger axiom

$$C^+ \quad z_1y_1 + z_2y_2 = z_1y_2 + z_2y_1 \longrightarrow z_1 = z_2 \wedge y_1 = y_2$$

which implies that the ring R obtained from a model is an integral domain, is also provable by double induction from A1 – 7, but is not provable by RIO from A1 – 7, B1 – 7, C^* (use polynomials in y_1, y_2, z_1, z_2 modulo the ideal $((y_1 - y_2)(z_1 - z_2))$. Shoenfield showed that all true open sentences in ', P, + were provable from A1 – 5, B1 – 5, C_d $(d = 2,3,\ldots)$ and

$$D_{m,n} \quad mx + n \neq my \qquad (0 < n < m).$$

We have already shown that the C_d are not provable from A1–7, B1 – 7 by RIO and now observe that even if we add all the C_d, in fact even C^+ we still cannot prove the $D_{m,n}$ by RIO (although they are again provable by double induction). For we get a model for A1 – 7, B1 – 7, C^+ by taking the ring R[t] of polynomials in t with rational coefficients and the elements of M to be 0 and those polynomials with leading coefficient a positive integer. This does not satisfy any of the $D_{m,n}$ for $mt + n = m(t + n/m)$. In this model not only are some elements e.g. $2t + 1$, both even and odd, but some e.g. t, are neither. So one would hardly expect a relatively deep result like $x^3 + y^3 \neq z^3 \wedge xyz = 0$ or the irrationality of $\sqrt{2}$ to be provable from A1 – 7 by RIO. To get a model which shows their unprovability from A1 – 7, C^+ we need only take polynomials in t with real coefficients, the elements of M, being 0 and those polynomials with positive leading coefficient. (In Shepherdson [1] we have shown they are unprovable by RIO even in a system containing $\dot{-}$, <, [x/n] $(n = 2,3,\ldots)$).

Once $\dot{-}$, with suitable axioms is added, the rule of double induction mentioned above becomes derivable from RIO. On the other hand as we have seen, in the system based on 0, ', P, +, · this rule of double induction is definitely stronger than RIO. It would be interesting to know exactly how strong it is, i.e. to find a simple set of open axioms whose consequences coincide with those obtainable from A1 – 7 by the use of this form of double (or triple, ...) induction.

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