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CONSTRUCTION OF DIFFUSIONS

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1 - INTRODUCTION -

The generator $\mathcal{G}$ of one-dimensional classical diffusions is given by a second order differential operator :

$$\mathcal{G}u(\xi) = \frac{a(\xi)^2}{2} \frac{d^2u}{d\xi^2} + b(\xi) \frac{du}{d\xi}, \tag{1}$$

as one finds in the systematic discussions by A. N. Kolmogorov [1]. W. Feller [2] extended the concept of classical diffusions, introducing a topologically invariant definition of general diffusions and determined their generator $\mathcal{G}$ in the form :

$$\mathcal{G}u(\xi) = D_m D_s u(\xi). \tag{2}$$

In order to construct Kolmogorov's diffusions, we can use the method of stochastic differential equations [3]. In fact, solving the stochastic differential equation :

$$dx_t = a(x_t)d\beta_t + b(x_t)dt, \tag{3}$$

where β_t is the standard Brownian motion, we can construct the paths of the diffusions with the generator (1).

However, this method does not apply to Feller's diffusions. To construct the paths of Feller's diffusions the stochastic time substitution is a powerful tool. This was discussed by K. Ito and H. P. McKean, Jr. [4] using Lévy-Trotter's local time of Brownian motions [5]. The time substitution is availble to general diffusions, as V. A. Volkonski [6], H. P. McKean, JR. and H. Tanaka [7] and R. K. Blumenthal, R. K. Getoor and H. P. McKean, Jr. [8] discussed.

A third method is to construct Feller's diffusion as a projective limit of processes of simpler type. F. Knight [10] constructed the Brownian motion as a projective limit of random walks whose time and space scales get smaller in a certain way. Using the same idea we shall construct Feller's diffusions as a projective limit of semi-Markov processes with polygonal paths. This construction is the aim of our paper and will be discussed in Section 4. We shall show the background of our method in Section 2 and prepare in Section 3 some properties of the solutions of $\alpha u - D_m D_s u = 0$ which will be useful in Section 4.

2 - THE POLYGONAL SEMI-MARKOV PROCESSES DERIVED FROM FELLER'S DIFFUSION -

Consider a strong Markov process $\mathfrak{M}$ with the state space $[0,1]$ and with continuous paths. We shall use the following notations :

w : continuous path

$x_t(w)$: the value of w at t

W : the space of all continuous paths

$\mathbf{B}$: the Borel algebra generated by all cylindrical subsets of W

P_a : the probability law of the path of $\mathfrak{M}$ starting at a

σ_a : the first passage time for a.

We shall assume :

$$P_a(\sigma_0 < \infty) > 0, \quad P_a(\sigma_1 < \infty) > 0 \qquad a \in (0,1) \tag{1}$$

$$E_a(\tau_{01}) < \infty \qquad \tau_{01} = \min(\sigma_0, \sigma_1) \tag{2}$$

and :

$$P_0(x_t = 0) = 1, \qquad P_1(x_t = 1) = 1, \tag{3}$$

and call $\mathfrak{M}$ Feller's diffusion in this paper, though Feller discussed other types of diffusions.

Introducing the scales and speed measure dm as

$$s(a) = P_a(\sigma_1 < \sigma_0) \tag{4}$$

$$dm(a) = -d\, D_s E_a(\tau_{01}), \tag{5}$$

we can express the generator $\mathfrak{G}$ of $\mathfrak{M}$ as

$$\mathfrak{G}u(a) = D_m D_s u(a) \qquad a \in (0,1) \tag{6}$$

$$\mathfrak{G}u(0) = \mathfrak{G}u(1) = 0. \tag{7}$$

Given Feller's diffusion $\mathfrak{M}$ mentioned above and given a set of division points of $[0,1]$

$$\delta : 0 = a_0 < a_1 < \ldots < a_N = 1, \tag{8}$$

we shall define a semi-Markov process with polygonal paths. Take any path w of $\mathfrak{M}$ starting at a point in δ, and introduce

$$T_1(w) = \min(\sigma_{a_{k-1}}, \sigma_{a_{k+1}}) \qquad \text{if } x_0(w) = a_k \quad 0 < k < N \tag{9}$$

$$= \infty \qquad \text{if } x_0(w) = a_0 \quad \text{or} \quad a_N$$

$$T_2(w) = T_1(w^+_{T_1}) \qquad \text{if } T_1(w) < \infty \tag{10}$$

$$T_3(w) = T_1(w^+_{T_1+T_2}) \qquad \text{if } T_1(w) + T_2(w) < \infty \tag{11}$$

and so on, where w^+_s is the *shifted path* defined as

$$x_t(w^+_s) = x_{t+s}(w).$$

Connecting $(0, x_0(w))$, $(T_1(w), x_{T_1}(w))$, $(T_1 + T_2, x_{T_1+T_2}(w)), \ldots,$ we shall get a polygonal path which will be denoted with $\pi_\delta w$. Here we shall define

$$x_t(\pi_\delta w) = x_T(w) \qquad \text{if } T = T_1 + T_2 + \ldots + T_n < t < \infty = T_{n+1} \tag{12}$$

$\mathfrak{M}_\delta \equiv [x_t(\pi_\delta w),\ w \in W(P_a),\ a \in \delta]$ gives a semi-Markov polygonal process. It is *semi-Markov* in the sense that it starts afresh not at every Markov time but at every Markov time of the form $T_1,\ T_1 + T_2,\ T_1 + T_2 + T_3, \ldots$

It is clear by the definition that

$$|x_t(\pi_\delta w) - x_t(w)| < \|\delta\| = \max_i |S(a_i) - S(a_{i-1})| \tag{13}$$

and that the probability law governing $\mathfrak{M}_\delta$ is determined completely by :

$$u_{k\alpha} = E_{a_k}(e^{-\alpha T_1},\ x(T_1) = a_{k+1}) \tag{14}$$

$$v_{k\alpha} = E_{a_k}(e^{-\alpha T_1},\ x(T_1) = a_{k-1}) \qquad (k = 1.2, \ldots N-1) \tag{15}$$

But $u_{k\alpha}$ is the value at a_k of the solution of

$$\alpha u - D_m D_s u = 0, \quad u(a_{k-1}) = 0, \quad u(a_{k+1}) = 1, \tag{16}$$

while $v_{k\alpha}$ is the value at a_k of the solution of

$$\alpha v - D_m D_s v = 0, \quad v(a_{k-1}) = 1, \quad v(a_{k+1}) = 0. \tag{17}$$

Such observation is the background of our construction which will be carried out in Section 4.

3 - THE EQUATION $\alpha u - D_m D_s u = 0$ -

Consider the equation :

$$\alpha u(\xi) - D_m D_s u(\xi) = 0 \qquad b < \xi < c, \ \alpha \geq 0, \tag{1}$$

and its solutions $u_\alpha = u_\alpha(\xi)$ and $v_\alpha = v_\alpha(\xi)$ with the following boundary conditions

$$u_\alpha(b) = 0, \quad u_\alpha(c) = 1 \tag{2}$$

$$v_\alpha(b) = 1, \quad v_\alpha(c) = 0. \tag{3}$$

u_α is an increasing solution of (1), while v_α is a decreasing solution of (1), and u_α and v_α constitute a fundamental system of solutions of (1).

Define the Green function $G^\circ_\alpha(\xi, \eta)$ as

$$G^\circ_\alpha(\xi,\eta) = G^\circ_\alpha(\eta, \xi) = u_\alpha(\xi) v_\alpha(\eta)/ [u_\alpha, v_\alpha] \qquad b \leq \xi \leq \eta \leq c, \tag{4}$$

where $[u_\alpha, v_\alpha]$ is the Wronskian (constant)

$$[u_\alpha, v_\alpha] = v_\alpha D_s u_\alpha - u_\alpha D_s v_\alpha, \tag{5}$$

and the Green operator G°_α as

$$(G^\circ_\alpha f)(\xi) = \int_b^c G^\circ_\alpha(\xi, \eta) f(\eta)\, dm(\eta). \tag{6}$$

As McKean proved, we have

$$\partial^n u_\alpha / \partial \alpha^n = (-)^n \, n! \, (G^\circ_\alpha)^n \, u_\alpha \qquad (\alpha \geq 0) \tag{7}$$

and so u_α can be expressed as

$$u_\alpha(\xi) = \int_0^\infty e^{-\alpha t} \mu_\xi(dt), \tag{8}$$

where $\mu_\xi(dt)$ is a bounded measure on $[0, \infty)$. Similarly we have

$$\partial^n v_\alpha / \partial \alpha^n = (-)^n \, n! \, (G^\circ_\alpha)^n \, v_\alpha \tag{9}$$

$$v_\alpha(\xi) = \int_0^\infty e^{-\alpha t} \nu_\xi(dt) \tag{10}$$

Setting

$$\lambda_\alpha(\xi) = u_\alpha(\xi) + v_\alpha(\xi) \tag{11}$$

$$\vartheta_\xi(dt) = \mu_\xi(dt) + \nu_\xi(dt), \tag{12}$$

we have

$$\alpha \lambda_\alpha - D_m D_s \lambda_\alpha = 0 \qquad \text{in } (b,c) \tag{13}$$

$$\lambda_\alpha(b) = \lambda_\alpha(c) = 1 \tag{14}$$

$$\lambda_\alpha(\xi) = \int_0^\infty e^{-\alpha t} \vartheta_\xi(dt). \tag{15}$$

Since $\lambda_o(\xi) \equiv 1$, ϑ_ξ is a probability measure on $[0,\infty)$.

Let $G(\xi,\eta)$ denote $G_o^o(\xi,\eta)$. Then

$$G(\xi,\eta) = G(\eta,\xi) \equiv (s(c) - s(\eta))(s(\xi) - s(b))/(s(c) - s(b)) \qquad (b \le \xi \le \eta \le c) \tag{16}$$

Introducing the integral operator

$$Gf(\xi) = \int_b^c g(\xi,\eta) f(\eta) dm(\eta), \tag{17}$$

we have

$$\int_o^\infty t^n \vartheta_\xi(dt) = (-)^n \partial^n \lambda_\alpha / \partial \alpha^n \Big|_{\alpha=o} = n!\,(G^n 1)(\xi), \tag{18}$$

so that we have

$$\int_o^\infty t^n \vartheta_\xi(dt) \le n!\,\gamma^{n-1}(G1)(\xi), \qquad \gamma = (s(c)-s(b))m(b,c). \tag{19}$$

It follows from this inequality that

$$\begin{aligned} \int_o^\infty e^{\alpha t}(dt) &= 1 + \alpha(G1)(\xi) + \alpha^2(G^2 1)(\xi) + \dots \\ &\le 1 + \alpha(G1)(\xi)\,[1 + \alpha\gamma + (\alpha\gamma)^2 + \dots] \end{aligned} \tag{20}$$

and

$$\begin{aligned} \int_o^\infty e^{-t}\vartheta_\xi(dt) &= 1 - (G1)(\xi) + (G^2 1)(\xi) - \dots \\ &\ge 1 - (G1)(\xi)\,[1 + \gamma + \gamma^2 + \dots], \end{aligned} \tag{21}$$

so that

$$\int_o^\infty e^{\alpha t}\vartheta_\xi(dt) - 1 \le 3\alpha\left(1 - \int_o^\infty e^{-t}\vartheta_\xi(dt)\right) \text{ if } \gamma,\ \alpha\gamma < 1/3. \tag{22}$$

Using this estimate and noticing

$$(e^{\alpha\epsilon} - 1)\int_\epsilon^\infty \vartheta(dt) \le \int_o^\infty e^{\alpha t}\vartheta_\xi(dt) - 1 \tag{23}$$

and

$$\lim_{\alpha\uparrow\infty} \alpha/(e^{\alpha\epsilon} - 1) = 0, \tag{24}$$

we can immediately prove the following

LEMMA 1 - For any ε, $\eta > 0$, we can determine ζ **depending only on** ε **and** η and independent of b, c, and ξ such that

$$(s(c) - s(b))\,m(b,c) < \zeta \tag{25}$$

implies

$$\int_\epsilon^\infty \vartheta_\xi(dt) < \eta\left(1 - \int_o^\infty e^{-t}\vartheta_\xi(dt)\right). \tag{26}$$

4 - CONSTRUCTION OF DIFFUSIONS -

(i) *Construction of* $\mathfrak{M}_\delta$ *and its properties.*

Consider a set δ of division points of $[0,1]$

$$\delta = (0 = a_o < a_1 < \dots < a_N = 1), \tag{1}$$

the equation :

$$\alpha u(\xi) - D_m D_s u(\xi) = 0 \qquad \xi \in (a_{k-1}, a_{k+1}), \tag{2}$$

its solutions $u^k(\xi)$ and $v^k(\xi)$ and the measures μ_ξ^k, ν_ξ^k and ϑ_ξ^k introduced in Section 3. Let μ_k, ν_k and ϑ_k denote respectively μ_ξ^k, ν_ξ^k and ϑ_ξ^k evaluated at a_k.

Starting at a_k, $k = 1,2,\ldots,N-1$, we shall construct a polygonal path w_δ by connecting $(0, a_k)$, (T_1, Y_1), $(T_1 + T_2, Y_2), \ldots$, where $(T_1, Y_1, T_2, Y_2, \ldots)$ is governed by the following probability law :

$$P_{a_k}^{\delta}(T_1 \in dt,\ Y_1 = a_{k\pm1}) = \mu_k(dt) \quad \text{or} \quad \nu_k(dt) \tag{3}$$

$$P_{a_k}^{\delta}(T_{n+1} \in dt,\ Y_{n+1} = a_{j_n\pm1} / T_1 = t_1, \ldots\ T_n = t_n,\ Y_1 = a_{j_1}, \ldots,\ Y_n = a_{j_n}) = \mu_{j_n}(dt) \quad \text{or} \quad \nu_{j_n}(dt) \tag{4}$$

If $Y_n = 0$ or 1, then we define $T_{n+1} = \infty$ and $Y_{n+1} = Y_n$ by convention.

Such a measure $P_{a_k}^{\delta}$ on the space W_δ of all such paths W_δ can be easily constructed by means of Kolmogorov's extension theorem. We shall use $x_t(w_\delta)$ or $x(t, w_\delta)$ to indicate the value of w_δ at t, and the process thus obtained is denoted by $\mathfrak{M}_\delta = (W_\delta, P_a^{\delta}, a \in \delta)$. $\mathfrak{M}_\delta$ is semi-Markov in the sense explained in Section 2, as is clear by the definition.

LEMMA 2 -

$$P_{a_k}^{\delta}(x(T_1 + \ldots + T_n) = 0 \quad \text{or } 1 \text{ for some } n) = 1. \tag{5}$$

Proof. It is clear by the construction that the above probability p_k satisfies

$$p_k = u_o^k(a_k)\, p_{k+1} + v_o^k(a_k)\, p_{k-1} \qquad k = 1,2,\ldots, N-1. \tag{6}$$

Since $u_o^k(a_k), v_o^k(a_k) > 0$ and their sum is 1, (6) implies that p_k is increasing or decreasing. Therefore $p_k \geq p_o = 1$ or $p_k \geq p_1 = 1$, so that $p_k \equiv 1$.

Now we shall introduce

$$\mathbf{m} = \mathbf{m}(\varepsilon, w_\delta) = \min(n : T_n > \varepsilon). \tag{7}$$

Then we get

LEMMA 3 - Let ζ denote the ζ determined in Lemma 1. If

$$\|\delta\| = \max_k |s(a_k) - s(a_{k-1})| < \zeta/2m(0,1), \tag{8}$$

then :

$$E_{a_k}^{\delta}(e^{-(T_1+\ldots+T_{m-1})}) < \eta. \tag{9}$$

Proof. Using Lemma 1, (8) implies

$$P_{a_k}^{\delta}(T_1 > \varepsilon) \leq \eta\,(1 - E_{a_k}^{\delta}(e^{-T_1})). \tag{10}$$

Noticing the semi-Markov property of $\mathfrak{M}$, we have

$$\begin{aligned}
&E_{a_k}^{\delta}(e^{-(T_1+\ldots+T_{m}-1)}) \\
&= P_{a_k}^{\delta}(T_1 > \varepsilon) + E_{a_k}^{\delta}(e^{-T_1}, T_1 \leq \varepsilon, T_2 > \varepsilon) + E_{a_k}^{\delta}(e^{-(T_1+T_2)},\ T_1, T_2 \leqslant \varepsilon, T_3 > \varepsilon) + \ldots \\
&\leq P_{a_k}^{\delta}(T_1 > \varepsilon) + E_{a_k}^{\delta}(e^{-T_1}, P_{x(T_1)}^{\delta}(T_1 > \varepsilon)) + E_{a_k}^{\delta}(e^{-(T_1+T_2)} P_{x(T_1+T_2)}^{\delta}(T_1 > \varepsilon)) + \ldots \\
&\leq \eta\,[(1 - E_{a_k}^{\delta}(e^{-T_1})) + E_{a_k}^{\delta}(e^{-T_1}(1 - E_{x(T_1)}^{\delta}(e^{-T_1}))) + E_{a_k}^{\delta}(e^{-(T_1+T_2)}(1 - E_{x(T_1+T_2)}^{\delta}(e^{-T_1})))\ \ldots] \\
&= \eta\,[(1 - E_{a_k}^{\delta}(e^{-T_1})) + (E_{a_k}^{\delta}(e^{-T_1}) - E_{a_k}^{\delta}(e^{-T_1-T_2})) + (E_{a_k}^{\delta}(e^{-T_1-T_2})) - E_{a_k}^{\delta}(e^{-T_1-T_2-T_3})) + \ldots] = \eta.
\end{aligned}$$

We shall introduce a random variable T(t) as

$$T(t) = T_1 + T_2 + \ldots + T_n \qquad \text{if} \quad T_1 + \ldots + T_n \leq t < T_1 + \ldots + T_{n+1} \tag{12}$$

and prove

LEMMA 4 - If (8) is satisfied, then

$$P^{\delta}_{a_k}(T(t) - t > \varepsilon) \leq \eta\ e^{t} \tag{13}$$

Proof. It is clear by the definition and Lemma 3 that

$$P^{\delta}_{a_k}(T(t) - t > \varepsilon)$$
$$\leq P^{\delta}_{a_k}(T_1 + T_2 + \dots + T_{m-1} \leq t)$$
$$\leq E^{\delta}_{a_k}(e^{-(T_1+T_2+\dots+T_{m-1})})\ e^{t} \leq \eta\ e^{t}$$

Let us take three points $b, a, c \in \delta$ such that $b \leq a \leq c$. Let τ be the smaller of σ_b and σ_c. τ is the minimum of $T_1 + T_2 + \dots + T_n$ such that $x(T_1 + \dots + T_n) = b$ or c. It follows from Lemma 2 that :

$$P^{\delta}_{a}(T_1 + \dots + T_n) < \infty) = 1,$$

and we have

LEMMA 5 -

$$E^{\delta}_{a}(e^{-\alpha\tau}, x(\tau) = b) = u_\alpha(a) \tag{14.a}$$

$$E^{\delta}_{a}(e^{-\alpha\tau}, x(\tau) = c) = v_\alpha(a) \tag{14.b}$$

$$E^{\delta}_{a}(e^{-\alpha\tau}) = \lambda_\alpha(a), \tag{14.c}$$

where u_α, v_α, and λ_α are defined in Section 3.

Proof. Suppose that $b = a_q$, $a = a_k$, and $c = a_r$, $q \leq k \leq r$.

Then we see by the definition that $\tilde{u}_k \equiv E^{\delta}_{a_k}(e^{-\alpha\tau},\ x(\tau) = b)$ satisfies

$$\tilde{u}_k = u^k(a_k)\,\tilde{u}_{k+1} + v^k(a_k)\,\tilde{u}_{k-1}, \quad \tilde{u}_q = 0, \quad \tilde{u}_r = 1$$

and it follows from the property of the solutions of linear homogeneous second order differential equations that $\hat{u}_k = u_\alpha(a_k)$ satisfies the same difference equation and the same boundary conditions. The uniqueness of the solution of such difference equation with fixed boundary conditions implies $\tilde{u}_k = \hat{u}_k$ which shows (14.a). Similarly for (14.b) and (14.c).

Setting $\alpha = 0$ in (14.a) we have

LEMMA 6 - $P^{\delta}_{a}(\sigma_b < \sigma_c) = (s(a) - s(b))/(s(c) - s(b))$.

Differentiating both sides of (14.c) in α and setting $\alpha = 0$, we have

LEMMA 7 -

$$E^{\delta}_{a}(\tau_{bc}) = \int_b^c G_{bc}(a, \eta)\,dm(\eta) \qquad \tau_{bc} = \min(\tau_b, \tau_c)$$

where

$$G_{bc}(\xi, \eta) = G_{bc}(\eta, \xi) = (s(c) - s(\eta))\,(s(\xi) - s(b))/(s(c) - s(b)) \quad (b \leq \xi \leq \eta \leq c \tag{15}$$

Noticing the semi-Markov property of $\mathfrak{M}_\delta$ we have

LEMMA 8 - If $g^{\delta}(a) = E^{\delta}_{a}\left(\int_0^\infty e^{-\alpha t} f(x_t)dt\right)$, then

$$g^{\delta}(a) = E_a\left(\int_0^\tau e^{-\alpha t} f(x_t)\,dt\right) + u_\alpha(a)g(c) + v_\alpha(a)g(b)$$

where $b < a < c$, $b, a, c \in \delta$ and $u_\alpha(a)$ and $v_\alpha(a)$ were defined in Section 3.

(ii) *Projection* $\pi_{\delta\Delta}$ *from* $\mathfrak{M}_\Delta$ *onto* $\mathfrak{M}_\delta$ *in case* $\Delta \supset \delta$.

Let δ and Δ be two sets of division points of $[0,1]$ such that $\Delta \supset \delta$, and $\mathfrak{M}_\delta = (W_\delta, P_a^\delta, a \in \delta)$ and $\mathfrak{M}_\Delta = (W_\Delta, P_b^\Delta, b \in \Delta)$ be the corresponding semi-Markov processes defined above.

Define a projection $\pi_{\delta\Delta}$ which carries $w_\Delta (\in W_\Delta)$ starting at $a \in \delta$ to $\pi_{\delta\Delta} w_\Delta (\in W_\delta)$ starting at a just as we defined π_δ in Section 2.

We shall now prove.

LEMMA 9 - $P_a^\delta = P_a^\Delta \pi_{\delta\Delta}^{-1}$. $\quad a \in \delta$

Proof. It is enought to prove that $\tilde{\mathfrak{M}}_\delta = (W_\delta, P_a^\Delta \pi_{\Delta\delta}^{-1}, a \in \delta)$ is the same as $\mathfrak{M}_\delta$. Since $\tilde{\mathfrak{M}}_\delta$ is a semi-Markov process by the definition, it is enough to show that

$$E_{a_k}^\delta(e^{-\alpha\tau_k}, x(\tau_k) = a_{k+1}) = E_{a_k}^\Delta(e^{-\alpha\tau_k}, x(\tau_k) = a_{k+1}) \tag{16.a}$$

$$E_{a_k}^\delta(e^{-\alpha\tau_k}, x(\tau_k) = a_{k-1}) = E_{a_k}^\delta(e^{-\alpha\tau_k}, x(\tau_k) = a_{k-1}) \tag{16.b}$$

for $\tau_k = \min(\sigma_{a_{k+1}}, \sigma_{a_{k-1}})$. But (16.a) is clear, because both sides are the solution evaluated at a_k of the equation :

$$\alpha u - D_m D_s u = 0 \qquad \text{in } (a_{k-1}, a_{k+1})$$

$$u(a_{k-1}) = 0, \qquad u(a_{k+1}) = 1$$

by Lemma 5. Similarly for (16.b).

(iii) *The projective limit of* $\mathfrak{M}_\delta$ *for* $\| \delta \| \longrightarrow 0$.

We shall define Feller's diffusion $\mathfrak{M} = (W, P_a, a \in [0,1])$ as the projective limit of $\mathfrak{M}_\delta$ for $\| \delta \| \longrightarrow 0$. For each $a \in [0,1]$, we shall define P_a as follows.

Consider the class C_a of all sets of division points containing a and P_a^δ (defined in (i)) for $\delta \in C_a$. As we proved in (i), we have

$$P_a^\delta = P_a^\Delta \pi_{\delta\Delta}^{-1} \qquad \delta, \Delta \in C_a, \quad \delta \subset \Delta. \tag{17}$$

Applying Bochner's theorem [11], a generalized version of Kolmogorov's extension theorem, we can construct a probability measure space $\Omega(P)$ on which a system of stochastic processes $y_t^\delta(\omega)$, $t \geq 0$ depending on $\delta \in C_a$, is defined such that each $y_t^\delta(\omega)$ is the version of $x_t(w_\delta)$, $w_\delta \in W\, P_a^\delta)$, and that, if $a \in \delta \subset \Delta$, then

$$\pi_{\delta\Delta}\, y_\cdot^\Delta(\omega) = y_\cdot^\delta(\omega), \tag{18}$$

so that

$$|s(y_t^\Delta(\omega)) - s(y_t^\delta(\omega))| < 2\varepsilon \quad \text{if } \| \delta \|, \| \Delta \| < \varepsilon . \tag{19}$$

Since $s(\xi)$ is continuous in $\xi \in [0,1]$ and one to one, $y_t^\delta(\omega)$ converges uniformly in (t,ω), as $\| \delta \| \longrightarrow 0$. Let $y_t(\omega)$ denote the limit. Since $y_t^\delta(\omega)$ is continuous in t, its uniform limit $y_t(\omega)$ is also continuous in t.

It is easily seen that

$$\pi_\delta\, y_\cdot(\omega) = y_\cdot(\omega), \tag{20}$$

where π_δ is the mapping defined in Section 2.

Let P_a be the probability law the stochastic process $y_t(\omega)$, $t \geq 0$, $\omega \in \Omega(P)$, yields on the space W of continuous paths. It is clear that

$$P_a^\delta = P_a \pi_\delta^{-1} \qquad a \in \delta. \tag{21}$$

Now it remains to prove that $\mathfrak{M} = (W, P_a, a \in [0,1])$ is Feller's process we wished to construct.

For the proof we shall start with

LEMMA 10 -

$$g(a) = E_a\left(\int_0^\infty e^{-\alpha t} f(x_t)dt\right) \tag{22}$$

is continuous if f is continuous.

Proof. It follows from Lemma 7 and Lemma 8,

$$\begin{aligned} &|g^\delta(a) - u_\alpha(a)g^\delta(c) - v_\alpha(a)g^\delta(b)| \\ &\leq E_a\left(\int_0^\tau e^{-\alpha t} f(x_t)\,dt\right) \\ &\leq \|f\| \; E_a^\delta(\tau) \\ &= \|f\|\int_b^c G_{(b,c)}(a,\xi)\,dm(\xi) \qquad \|f\| = \max_\xi|f(\xi)| \end{aligned} \tag{23}$$

But as far as $\delta \ni \xi$,

$$\begin{aligned} g^\delta(\xi) &= E_\xi^\delta\left(\int_0^\infty e^{-\alpha t} f(x_t)dt\right) \\ &= E_\xi\left(\int_0^\infty e^{-\alpha t} f(x_t(\pi_\delta w))\,dt\right) \\ &\longrightarrow E_\xi\left(\int_0^\infty e^{-\alpha t} f(x_t(w))\,dt\right) = g(a) \qquad \text{as } \|\delta\| \longrightarrow 0. \end{aligned}$$

Letting $\|\delta\| \longrightarrow 0$ in (23) under the condition that $\delta \ni a,b,c,$ we have

$$\begin{aligned} &|g(a) - u_\alpha(a)g(c) - v_\alpha(a)g(b)| \\ &\leq \|f\|\int_b^c G_{bc}(a,\xi)\,dm(\xi) \\ &\leq \|f\|\,(s(c) - s(b))\,m(b,c) \\ &\leq \|f\|\,m(0,1)\,(s(c) - s(b)), \end{aligned}$$

from which we can see the continuity of g, using the fact that $u_\alpha(\xi)$ and $v_\alpha(\xi)$ are continuous in $\xi \in [b,c]$ and that $s(\xi)$ is continuous in $\xi \in [0,1]$.

Now we shall prove the Markov property of $\mathfrak{M}$.

Let $f(\xi)$ and $F(\xi_1, \xi_2, \ldots, \xi_n)$ be continuous. We shall prove that

$$E_a(f(x_{t_n+s})\,F(x_{t_1}, x_{t_2}, \ldots, x_{t_n})) = E_a[E_{x(t_n)}(f(x_s))\,F(x_{t_1}, x_{t_2}, \ldots, x_{t_n})] \qquad (0 \leq t_1 < t_2 < \ldots < t_n+s) \tag{24}$$

Since both sides are continuous in s, it is enough to prove that

$$\int_0^\infty e^{-\alpha s}\, E_a(f(x_{t_n+s})\,F(x_{t_1}, x_{t_2}, \ldots, x_{t_n}))\,ds = E_a(g(x_{t_n})\,F(x_{t_1}, x_{t_2}, \ldots x_{t_n})), \tag{25}$$

where g is a continuous function introduced in Lemma 10.

Let $T_\delta(t)$ be the minimum of u such that $x(u) \in \delta$ and that $u \geq t$. Then it is clear that :

$$T_\delta(t,w) = T_\delta(t, \pi_\delta w). \tag{26}$$

Using Lemma 4, we have :

$$\lim_{\substack{\delta \ni a \\ \|\delta\| \to 0}} P_a^\delta(T_\delta(t) - t > \varepsilon) = 0 \qquad (\varepsilon > 0)$$

and so

$$\lim_{\substack{\delta \ni a \\ \|\delta\| \to 0}} P_a(T_\delta(t, \pi_\delta w) - t > \varepsilon) = 0.$$

Using (26), we get

$$\lim_{\substack{\delta \ni a \\ \|\delta\| \to 0}} P_a(T_\delta(t,w) - t > \varepsilon) = 0.$$

Therefore we can take a sequence $\delta_m (\ni a)$, $m = 1, 2, \ldots$, such that

$$P_a(\lim_{m \to \infty} T_{\delta_m}(t,w) = t) = 1,$$

for each t, so that

$$P_a(\lim_{m \to \infty} T_{\delta_m}(t_i, w) = t_i,\ i = 1, 2, \ldots, n) = 1.$$

Write T_m, P_a^m and π_m respectively for T_{δ_m}, $P_a^{\delta_m}$ and π_{δ_m} and notice that

$$|s(x_t(\pi_\delta w)) - s(x_t(w))| < 2 \|\delta\|. \tag{27}$$

Since $f(\xi)$ and $g(\xi)$ are uniformly continuous in ξ and so in $s(\xi)$ and $F(\xi_1, \xi_2, \ldots, \xi_n)$ is uniformly continuous in $(\xi_1, \ldots, \xi_n)$ and so in $(s(\xi_1), \ldots, s(\xi_n))$, we get :

$$\int_0^\infty e^{-\alpha s} E_a(f(x_{t_n+s})\ F(x_{t_1}, \ldots, x_{t_n}))\, ds$$

$$= \lim_{m \to \infty} \int_0^\infty e^{-\alpha s}\ E_a[f(x_{T_m(t_n)+s}) F(x_{T_m(t_1)}, \ldots, x_{T_m(t_n)})]\, ds$$

$$= \lim_{m \to \infty} \int_0^\infty e^{-\alpha s}\ E_a[f(x_{T_m(t_n, \pi_m w)+s}\ (\pi_m w))\ F(x_{T_m(t_1, \pi_m w)}\ (\pi_m w), \ldots, x_{T_m(t_n, \pi_m w)}\ (\pi_m w))]\ ds$$

$$= \lim_{m \to \infty} \int_0^\infty e^{-\alpha s}\ E_a^m[f(x_{T_m(t_n)+s})\ F(x_{T_m(t_1)}, \ldots, x_{T_m(t_n)})]\ ds$$

$$= \lim_{m \to \infty} \int_0^\infty e^{-\alpha s}\ E_a^m[E^m_{x(T_m(t_n))}\ (f(x_s)) F(x_{T_m(t_1)}, \ldots, x_{T_m(t_n)})]\ ds$$

$$= \lim_{m \to \infty} \int_0^\infty e^{-\alpha s}\ E_a[E_{x(T_m(t_n))}\ (f(x_s)) F(x_{T_m(t_1)}, \ldots, x_{T_m(t_n)})]\ ds$$

$$= E_a(g(x_{t_n})\ F(x_{t_1}, \ldots, x_{t_n})).$$

We used the semi-Markov property of $\mathfrak{M}_{\delta_n}$, (26), (27) and the continuity of g and F in the last four steps.

Thus we have proved that $\mathfrak{M}$ is Markov, and therefore $\mathfrak{M}$ is also strong Markov by virtue of Lemma 10.

To identify our process with Feller's diffusion, it is enough to observe :

$$P_a(\sigma_1 < \sigma_0) = P_a(\sigma_1(\pi_\delta w) < \sigma_0(\pi_\delta w)) \quad (a \in \delta)$$

$$= P_a^\delta(\sigma_1 < \sigma_0)$$

$$E_a(\tau_{01}) = E_a(\tau_{01}(\pi_\delta w)) = E_a^\delta(\tau_{01}) \quad (\tau_{01} = \min(\sigma_0, \sigma_1)),$$

so that

$$P_a(\sigma_1 < \sigma_0) = s(a)$$

$$E_a(\tau_{01}) = \int_0^1 G_{01}(a, \xi) dm(\xi)$$

and so $-dD_s E_a(\tau_{01}) = dm(a)$.

Thus we have proved

THEOREM - $\mathfrak{M}$, defined above, is Feller's diffusion with the generator $D_m D_s$ and with sticking boundaries.

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DISCUSSION

M. NEVEU - Que sait-on sur la représentation explicite du générateur infinitésimal d'un processus de diffusion pluri-dimensionnel ?

M. ITO - mentionne la formule générale de représentation de Dynkin et cite les travaux concernant les processus de diffusion ayant les mêmes probabilités d'absorption que le mouvement brownien.