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Abstract

In this paper, $K$ denotes a complete, non-trivially valued, non-archimedean field. The entries of sequences, series and infinite matrices are in $K$. In the present paper, we prove the Silvermann-Toeplitz theorem for double sequences and series in $K$ and apply it to Nörlund means for double sequences and series in $K$.

Throughout the present paper, $K$ denotes a complete, non-trivially valued, non-archimedean field. The entries of sequences, series and infinite matrices are in $K$. In this paper, we prove the Silvermann-Toeplitz theorem for double sequences and series in $K$ (see Theorem 2, proved in the sequel). We then introduce Nörlund means for double sequences and series in $K$ and apply Silvermann-Toeplitz theorem for these means.

For analysis in the classical case a general reference is [2] while for analysis in non-archimedean fields a general reference is [1].

For a given infinite matrix $A = (a_{n,k})$ and a sequence $\{x_k\}$, the sequence $\{y_n\}$ is defined as follows:

$$y_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \quad n = 1, 2, \ldots,$$

it being assumed that the series on the right converge. If $\lim_{n \to \infty} y_n = s$ whenever $\lim_{k \to \infty} x_k = s$, we say that $A$ is regular. The criterion for $A$ to be regular in terms of the entries of the matrix $A$ are well-known (see [4], [6]).

**Theorem 1.** $A = (a_{n,k})$ is regular if and only if

(i) $\sup_{n,k} |a_{n,k}| < \infty$;

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and

(iii) \( \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} = 1. \)

In the sequel, the following definitions are needed.

**Definition 1.** Let \( \{x_{m,n}\} \) be a double sequence in \( K \) and \( x \in K \). We say that \( \lim_{m+n \to \infty} x_{m,n} = x \) if for each \( \epsilon > 0 \), the set \( \{(m, n) \in \mathbb{N}^2 : |x - x_{m,n}| \geq \epsilon\} \) is finite. In such a case we say that \( x \) is the limit of \( \{x_{m,n}\} \).

**Definition 2.** Let \( \{x_{m,n}\} \) be a double sequence in \( K \) and \( s \in K \). We say that

\[
s = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{m,n},
\]

if

\[
s = \lim_{m+n \to \infty} s_{m,n},
\]

where

\[
s_{m,n} = \sum_{k=1}^{m} \sum_{l=1}^{n} x_{k,l}, \quad m, n = 1, 2, \ldots.
\]

**Remark.** If \( \lim_{m+n \to \infty} x_{m,n} = x \), then the sequence \( \{x_{m,n}\} \) is automatically bounded.

It is easy to prove the following results.

**Lemma 1.** \( \lim_{m+n \to \infty} x_{m,n} = x \) if and only if

(i) \( \lim_{n \to \infty} x_{m,n} = x, \quad m = 1, 2, \ldots, \)

(ii) \( \lim_{m \to \infty} x_{m,n} = x, \quad n = 1, 2, \ldots, \)

and

(iii) for each \( \epsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that \( |x - x_{m,n}| < \epsilon \), for all \( m, n \geq N \), which we write as \( \lim_{m,n \to \infty} x_{m,n} = x. \)
Lemma 2. \( \lim_{m+n \to \infty} s_{m,n} \) exists if and only if
\[
\lim_{m+n \to \infty} x_{m,n} = 0. \tag{1}
\]

Given the matrix \( A = (a_{m,n,k,l}) \), we define
\[
y_{m,n} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{m,n,k,l} x_{k,l}, \quad m, n = 1, 2, \ldots \tag{2}
\]
assuming that the series on the right converge.

Necessary and sufficient conditions for \( A = (a_{m,n,k,l}) \) to be regular for the class of all double sequences and series in the classical case have been found by Kojima [3]. It has been found that convergence and boundedness play a vital role for double sequences and series, a role analogous to that of convergence for simple sequences and series. Robison [8] proved Silvermann-Toeplitz theorem for such a class of bounded and convergent double sequences in the classical case. We prove here its analogue in a complete, non-trivially valued, non-archimedean field.

In this context, the following definition is needed.

**Definition 3.** If whenever \( \{x_{m,n}\} \) is a convergent sequence, \( \{y_{m,n}\} \) converges to the same value, then the matrix \( A = (a_{m,n,k,l}) \) is said to be regular.

**Theorem 2.** In order that whenever a sequence \( \{x_{m,n}\} \) has a limit \( x \), \( \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{m,n,k,l} x_{k,l} \) shall converge and \( \lim_{m+n \to \infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{m,n,k,l} x_{k,l} = x \), i.e., for \( A = (a_{m,n,k,l}) \) to be regular it is necessary and sufficient that

(a) \( \lim_{m+n \to \infty} a_{m,n,k,l} = 0, \quad k, l = 1, 2, \ldots \);

(b) \( \lim_{m+n \to \infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{m,n,k,l} = 1; \)

(c) \( \lim_{m+n \to \infty} \sup_{k \geq 1} |a_{m,n,k,l}| = 0, \quad l = 1, 2, \ldots \);

(d) \( \lim_{m+n \to \infty} \sup_{l \geq 1} |a_{m,n,k,l}| = 0, \quad k = 1, 2, \ldots \);

and
(e) \( \sup_{m,n,k,l} |a_{m,n,k,l}| < \infty. \)

**Proof.** Proof of necessity.

Define the sequence \( \{x_{k,l}\} \) as follows: For any fixed \( p, q \), let

\[
x_{k,l} = \begin{cases} 
1, & \text{when } k = p, \ l = q; \\
0, & \text{otherwise}.
\end{cases}
\]  

Then

\[
y_{m,n} = a_{m,n,p,q}.
\]

Since \( \{x_{k,l}\} \) has limit 0, it follows that (a) is necessary.

Define the sequence \( \{x_{k,l}\} \) where \( x_{k,l} = 1, \ k, \ l = 1, 2, \ldots \).

Now,

\[
y_{m,n} = \sum_{k=1, l=1}^{\infty, \infty} a_{m,n,k,l}, \ m, n = 1, 2, \ldots.
\]

This shows that \( \sum_{k=1, l=1}^{\infty, \infty} a_{m,n,k,l} \) converges for \( m, n = 1, 2, \ldots \). (4)

Since \( \{x_{k,l}\} \) has limit 1, it follows that

\[
\lim_{m+n \to \infty} \sum_{k=1, l=1}^{\infty, \infty} a_{m,n,k,l} = 1,
\]

so that (b) is necessary.

We now show that \( \lim_{m+n \to \infty} \sup_{k \geq 1} |a_{m,n,k,l}| = 0 \) for all \( l \in \mathbb{N} \). Suppose not.

Then there exists \( l_0 \in \mathbb{N} \) such that \( \lim_{m+n \to \infty} \sup_{k \geq 1} |a_{m,n,k,l_0}| = 0 \) does not hold.

So, there exists an \( \epsilon > 0 \), such that

\[
\left\{ (m, n) : \sup_{k \geq 1} |a_{m,n,k,l_0}| > \epsilon \right\} \text{ is infinite.} \] (5)

Let us choose \( m_1 = n_1 = r_1 = 1 \). Choose \( m_2, n_2 \in \mathbb{N} \) such that \( m_2 + n_2 > m_1 + n_1 \) and

\[
\sup_{1 \leq k \leq r_1} |a_{m_2,n_2,k,l_0}| < \frac{\epsilon}{8}, \text{ using (a)};
\]

and

\[
\sup_{k \geq 1} |a_{m_2,n_2,k,l_0}| > \epsilon, \text{ using (5)}.
\]
Then choose $r_2 \in \mathbb{N}$ such that $r_2 > r_1$ and
\[
\sup_{k > r_2} |a_{m_2,n_2,k,l_0}| < \frac{\epsilon}{8}, \text{ using (b)}. \tag{5}
\]
Inductively choose $m_p + n_p > m_{p-1} + n_{p-1}$ such that
\[
\sup_{1 \leq k \leq r_{p-1}} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8}; \tag{6}
\]
\[
\sup_{k \geq 1} |a_{m_p,n_p,k,l_0}| > \epsilon; \tag{7}
\]
and then choose $r_p > r_{p-1}$ such that
\[
\sup_{k > r_p} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8}. \tag{8}
\]
In view of (6), (7), (8), we have
\[
\sup_{r_{p-1} < k \leq r_p} |a_{m_p,n_p,k,l_0}| > \epsilon - \frac{\epsilon}{8} - \frac{\epsilon}{8} = \frac{3\epsilon}{4}. \tag{9}
\]
We can now find $k_p \in \mathbb{N}, \ r_{p-1} < k_p \leq r_p$ such that
\[
|a_{m_p,n_p,k_p,l_0}| > \frac{3\epsilon}{4}. \tag{9}
\]
Define the sequence $\{x_{k,l}\}$ as follows:
\[
x_{k,l} = \begin{cases} 0, & l \neq l_0; \\ 1, & \text{if } l = l_0, \ k = k_p, \ p = 1, 2, \ldots. \end{cases}
\]
We note that $\lim_{k+l \to \infty} x_{k,l} = 0$. Now, in view of (6),
\[
\left| \sum_{k=1}^{r_{p-1}} a_{m_p,n_p,k_0,l_0} x_{k,l_0} \right| \leq \sup_{1 \leq k \leq r_{p-1}} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8}; \tag{10}
\]
Using (8), we have,
\[
\left| \sum_{k=r_p+1}^{\infty} a_{m_p,n_p,k_0,l_0} x_{k,l_0} \right| \leq \sup_{k > r_p} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8}; \tag{11}
\]
and using (9), we get,

\[
\left| \sum_{k=r_p-1+1}^{r_p} a_{m_p,n_p,k,l_0} x_{k,l_0} \right| = \left| a_{m_p,n_p,k_p,l_0} \right| > \frac{3\epsilon}{4}.
\]

(12)

Thus

\[
\left| y_{m_p,n_p} \right| = \left| \sum_{k=1}^{\infty} a_{m_p,n_p,k,l_0} x_{k,l_0} \right|
\]

\[
\geq \left| \sum_{k=r_p-1+1}^{r_p} a_{m_p,n_p,k,l_0} x_{k,l_0} \right| - \left| \sum_{k=1}^{r_p-1} a_{m_p,n_p,k,l_0} x_{k,l_0} \right| - \left| \sum_{k=r_p+1}^{\infty} a_{m_p,n_p,k,l_0} x_{k,l_0} \right|
\]

\[
\geq \left| a_{m_p,n_p,k_p,l_0} \right| - \sup_{1 \leq k \leq r_p-1} \left| a_{m_p,n_p,k,l_0} \right| - \sup_{k \geq r_p} \left| a_{m_p,n_p,k,l_0} \right|
\]

\[
> \frac{3\epsilon}{4} - \frac{\epsilon}{8} - \frac{\epsilon}{8}, \text{ using (10), (11) and (12)}
\]

\[
\geq \frac{\epsilon}{2}, \quad p = 1, 2, \ldots
\]

Consequently \( \lim_{m+n \to \infty} y_{m,n} = 0 \) does not hold, which is a contradiction. Thus (c) is necessary. The necessity of (d) follows in a similar fashion.

To establish (e), we shall suppose that (e) does not hold and arrive at a contradiction. Since \( K \) is non-trivially valued, there exists \( \pi \in K \) such that \( 0 < \rho = |\pi| < 1 \). Choose \( m_1 = n_1 = 1 \). Using (a), (b), choose \( m_2 + n_2 > m_1 + n_1 \) such that

\[
\sup_{1 \leq k+l \leq m_1+n_1} |a_{m_2,n_2,k,l}| < 2, \text{ using (a)};
\]

\[
\sup_{k+l \geq 1} |a_{m_2,n_2,k,l}| > \left( \frac{2}{\rho} \right)^6;
\]

and

\[
\sup_{k+l > m_1+n_1} |a_{m_2,n_2,k,l}| < 2^2, \text{ using (b) and Lemma 1, Lemma 2}.
\]

It now follows that

\[
\sup_{k+l > m_2+n_2} |a_{m_2,n_2,k,l}| < 2^2.
\]
Choose \( m_3 + n_3 > m_2 + n_2 \) such that

\[
\sup_{1 \leq k + l \leq m_2 + n_2} |a_{m_3, n_3, k, l}| < 2^2,
\]

\[
\sup_{k + l \geq 1} |a_{m_3, n_3, k, l}| > \left( \frac{2}{\rho} \right)^8,
\]

and

\[
\sup_{k + l > m_3 + n_3} |a_{m_3, n_3, k, l}| < 2^4.
\]

Inductively, choose \( m_p + n_p > m_{p-1} + n_{p-1} \), such that

\[
\sup_{1 \leq k + l \leq m_{p-1} + n_{p-1}} |a_{m_p, n_p, k, l}| < 2^{p-1}, \quad (13)
\]

\[
\sup_{k + l \geq 1} |a_{m_p, n_p, k, l}| > \left( \frac{2}{\rho} \right)^{2p+2}, \quad (14)
\]

and

\[
\sup_{k + l > m_p + n_p} |a_{m_p, n_p, k, l}| < 2^{2p-2}. \quad (15)
\]

Using (13), (14), (15), we have,

\[
\sup_{m_{p-1} + n_{p-1} < k + l \leq m_p + n_p} |a_{m_p, n_p, k, l}| > \left( \frac{2}{\rho} \right)^{2p+2} - 2^{2p-2} - 2^{p-1},
\]

\[
\geq \left( \frac{2}{\rho} \right)^{2p+2} - \left( \frac{2}{\rho} \right)^{2p-2} - \left( \frac{2}{\rho} \right)^{p-1}, \quad \text{since } \frac{1}{\rho} > 1
\]

\[
= \left( \frac{2}{\rho} \right)^{p-1} \left[ \left( \frac{2}{\rho} \right)^{p+3} - \left( \frac{2}{\rho} \right)^{p-1} - 1 \right]
\]

\[
\geq \left( \frac{2}{\rho} \right)^{p-1} \left[ \left( \frac{2}{\rho} \right)^{p+1} - \left( \frac{2}{\rho} \right)^{p-1} - \left( \frac{2}{\rho} \right)^{p-1} \right], \quad \text{since } \left( \frac{2}{\rho} \right)^{p-1} \geq 1
\]

\[
= \left( \frac{2}{\rho} \right)^{p-1} \left[ \left( \frac{2}{\rho} \right)^4 \left( \frac{2}{\rho} \right)^{p-1} - 2 \left( \frac{2}{\rho} \right)^{p-1} \right],
\]

\[
> \left( \frac{2}{\rho} \right)^{p-1} \left[ \left( \frac{2}{\rho} \right)^4 \left( \frac{2}{\rho} \right)^{p-1} - \left( \frac{2}{\rho} \right) \left( \frac{2}{\rho} \right)^{p-1} \right], \quad \text{since } \frac{2}{\rho} > 2
\]
\[ \left( \frac{2}{\rho} \right)^{2p-1} \left[ \left( \frac{2}{\rho} \right)^3 - 1 \right] > \left( \frac{2}{\rho} \right)^{2p-1} [2^3 - 1], \text{ since } \frac{2}{\rho} > 2 \]
\[ = 7 \left( \frac{2}{\rho} \right)^{2p-1} \]
\[ > 4 \left( \frac{2}{\rho} \right)^{2p-1} \]
\[ = \frac{2^{2p+1}}{\rho^{2p-1}} \]
\[ > \frac{2^{2p+1}}{\rho^p}, \text{ since } \frac{1}{\rho} > 1. \] (16)

Thus there exist \( k_p \) and \( l_p, \) \( m_{p-1} + n_{p-1} < k_p + l_p \leq m_p + n_p \) such that
\[ |a_{m_p,n_p,k_p,l_p}| > \frac{2^{2p+1}}{\rho^p}. \] (17)

Now, define the sequence \( \{x_{k,l}\} \) as follows:

\[ x_{k,l} = \begin{cases} \pi^p, & \text{if } k = k_p, l = l_p, p = 1, 2, \ldots; \\ 0, & \text{otherwise}. \end{cases} \]

We note that \( \lim_{k+l \to \infty} x_{k,l} = 0. \) Now,

\[ |y_{m_p,n_p}| = \left| \sum_{k,l=1}^{\infty} a_{m_p,n_p,k,l} x_{k,l} \right| \]
\[ > \left| \sum_{k+l=(m_{p-1}+n_{p-1})+1}^{m_p+n_p} a_{m_p,n_p,k,l} x_{k,l} \right| \]
\[ - \left| \sum_{k+l=1}^{m_{p-1}+n_{p-1}} a_{m_p,n_p,k,l} x_{k,l} \right| \]
\[ - \left| \sum_{k+l=(m_{p}+n_{p})+1}^{\infty} a_{m_p,n_p,k,l} x_{k,l} \right| \]
\[ |a_{m,p,n_p,k_p,l_p}| \times |x_{k_p,l_p}| - \sup_{1 \leq k+l \leq m_{p-1}+n_{p-1}} |a_{m,p,n_p,k,l}| - \sup_{m_p+n_p<k+l<\infty} |a_{m_p,n_p,k,l}| \]

\[ > \frac{2^{2p+1}}{\rho_p} - 2^{2p-2} - 2^{p-1}, \text{ using (13), (15) and (17)} \]

\[ = 2^{2p+1} - 2^{2p-2} - 2^{p-1} \]

\[ = 2^{2p-2}(7) - 2^{p-1} \]

\[ = 2^{p-1}[7 \cdot 2^{p-1} - 1] \]

\[ \geq 2^{p-1}[7 \cdot 2^{p-1} - 2^{p-2}] \]

\[ = 2^{p-1}[2^{p-2}(14 - 1)] \]

\[ = 2^{p-1}[13 \cdot 2^{p-2}] \]

\[ = 13 \cdot 2^{2p-3} \]

i.e., \[ |y_{m,n,p}| > 13 \cdot 2^{2p-3}, \quad p = 1, 2, \ldots, \]

i.e., \( \lim_{m+n \to \infty} y_{m,n} = 0 \) does not hold, which is a contradiction. Thus (e) is necessary.

Proof of Sufficiency.

Let \( \lim_{m+n \to \infty} x_{m,n} = x \). Then

\[ y_{m,n} - x = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}x_{k,l} - x. \]

From (b) we have

\[ \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} + r_{m,n} = 1, \]

where

\[ \lim_{m+n \to \infty} r_{m,n} = 0. \quad (18) \]

Hence,

\[ y_{m,n} - x = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - x) + r_{m,n}x. \]

Given \( \epsilon > 0 \), we can choose sufficiently large \( p \) and \( q \) such that

\[ \sup_{k+l>p+q} |x_{k,l} - x| < \frac{\epsilon}{5H}. \quad (19) \]
where $H = \sup_{m,n,k,l \geq 1} |a_{m,n,k,l}|$. Observe that $H > 0$ (from (b)).

Let $L = \sup_{k,l \geq 1} |x_{k,l} - x|$. We now choose $N \in \mathbb{N}$ such that whenever $m + n \geq N$, the following are satisfied:

1. $\sup_{1 \leq k + l \leq p + q} |a_{m,n,k,l}| < \frac{\epsilon}{5pqL}$, using (a); (20)

2. $\sup_{k \geq 1} |a_{m,n,k,l}| < \frac{\epsilon}{5qL}$, $l = 1, 2, \ldots, q$, using (c); (21)

3. $\sup_{l \geq 1} |a_{m,n,k,l}| < \frac{\epsilon}{5pL}$, $k = 1, 2, \ldots, p$, using (d); (22)

and

4. $|r_{m,n}| < \frac{\epsilon}{5|x|}$, from the equation (18). (23)

Whenever $m + n \geq N$, we thus have,

$$|y_{m,n} - x| = \left| \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{m,n,k,l}(x_{k,l} - x) + r_{m,n}x \right|$$

$$\leq \left| \sum_{k=1}^{p} \sum_{l=1}^{q} a_{m,n,k,l}(x_{k,l} - x) \right| + \left| \sum_{k=1}^{p} \sum_{l=q+1}^{\infty} a_{m,n,k,l}(x_{k,l} - x) \right|$$

$$+ \left| \sum_{k=p+1}^{\infty} \sum_{l=1}^{q} a_{m,n,k,l}(x_{k,l} - x) \right| + \left| \sum_{k=p+1}^{\infty} \sum_{l=q+1}^{\infty} a_{m,n,k,l}(x_{k,l} - x) \right|$$

$$+ |r_{m,n}| \left| x \right|$$

$$< \frac{\epsilon}{5pqL} Lpq + \frac{\epsilon}{5pL} Lp + \frac{\epsilon}{5qL} Lq + \frac{\epsilon}{5H} H + \frac{\epsilon}{5|x|} |x|$$

$$= \epsilon, \quad \text{using (19), (20), (21), (22) and (23).}$$

Thus

$$\lim_{m+n \to \infty} y_{m,n} = x,$$

which completes the proof of the theorem.

Nörlund means for simple sequences and series in complete, non-trivially valued, non-archimedean fields were introduced by Srinivasan [9] and studied
Silvermann-Toeplitz theorem for double sequences and series

later in detail by Natarajan (for instance, see [7]). Nörlund means for double sequences and series in classical analysis were introduced by Moore [5]. We now define Nörlund means for double sequences and series in complete, non-trivially valued, non-archimedean fields and apply Theorem 2 for these means.

**Definition 4.** Given a doubly infinite set of elements $p_{m,n} \in K$, $m,n = 0,1,2,\ldots$, where $p_{0,0} \neq 0$, $|p_{i,j}| < |p_{0,0}|$, $(i,j) \neq (0,0)$, $i,j = 0,1,2,\ldots$, let

$$P_{m,n} = \sum_{i,j=0}^{m,n} p_{i,j}, \quad m,n = 0,1,2,\ldots.$$

Given any double sequence $\{s_{m,n}\}$ we define

$$\sigma_{m,n} = (N,p_{m,n})(s_{m,n}) = \frac{S_{m,n}}{P_{m,n}} = \frac{\sum_{i,j=0}^{m,n} p_{m-i,n-j} s_{i,j}}{P_{m,n}}, \quad m,n = 0,1,2,\ldots.$$

If $\lim_{m+n \to \infty} \sigma_{m,n} = \sigma$, we say that the double sequence $\{s_{m,n}\}$ is summable $(N,p_{m,n})$ to the value $\sigma$, written as

$$s_{m,n} \to \sigma(N,p_{m,n}).$$

Any double series $\sum_{m,n} u_{m,n}$ is said to be summable $(N,p_{m,n})$ to the value $\sigma$ if the double sequence $\{s_{m,n}\}$, where

$$s_{m,n} = \sum_{i,j=0}^{m,n} u_{i,j}, \quad m,n = 0,1,2,\ldots,$$

is summable $(N,p_{m,n})$ to the value $\sigma$.

**Definition 5.** Given the Nörlund means $(N,p_{m,n}),(N,q_{m,n})$, we say that they are consistent if

$$s_{m,n} \to \sigma(N,p_{m,n}) \text{ and } s_{m,n} \to \sigma'(N,q_{m,n}) \Rightarrow \sigma = \sigma'.$$

We say that $(N,p_{m,n})$ is included in $(N,q_{m,n})$, written as

$$(N,p_{m,n}) \subseteq (N,q_{m,n}),$$
if 

\[ s_{m,n} \rightarrow \sigma(N, p_{m,n}) \Rightarrow s_{m,n} \rightarrow \sigma(N, q_{m,n}). \]

The two methods \((N, p_{m,n}), (N, q_{m,n})\) are said to be equivalent if

\( (N, p_{m,n}) \subseteq (N, q_{m,n}) \) and \((N, q_{m,n}) \subseteq (N, p_{m,n}). \)

In view of Theorem 2, it is easy to prove the following result. 

**Theorem 3.** The necessary and sufficient conditions for the regularity of the Nörlund means \((N, p_{m,n})\) are:

\[
\lim_{m+n \to \infty} \sup_{0 \leq j \leq n} |p_{m-i,n-j}| = 0, \quad 0 \leq i \leq m; \quad \text{(24)}
\]

\[
\lim_{m+n \to \infty} \sup_{0 \leq i \leq m} |p_{m-i,n-j}| = 0, \quad 0 \leq j \leq n. \quad \text{(25)}
\]

In the sequel let \((N, p_{m,n}), (N, q_{m,n})\) be two regular Nörlund methods such that each row and each column of the infinite matrices \((p_{m,n}), (q_{m,n})\) is a regular Nörlund mean for simple sequences.

**Theorem 4.** Any two such regular Nörlund methods are consistent.

**Proof.** Given two Nörlund methods \((N, p_{m,n}), (N, q_{m,n})\), where each row and each column of the infinite matrices \((p_{m,n}), (q_{m,n})\) is a regular Nörlund mean for simple sequences, we define a third method \((N, r_{m,n})\) by the equation

\[
r_{m,n} = \sum_{i,j=0}^{m,n} p_{i,j} q_{m-i,n-j}, \quad m, n = 0, 1, 2, \ldots.
\]

We then readily obtain, for \( s = \{s_{m,n}\}, \)

\[
(N, r_{m,n})(s) = \sum_{i,j=0}^{m,n} \gamma_{m,n,i,j}(N, q_{i,j})(s),
\]

where

\[
\gamma_{m,n,i,j} = p_{m-i,n-j} q_{i,j} / \sum_{\mu,\nu=0}^{m,n} p_{m-\mu,n-\nu} Q_{\mu,\nu}.
\]

Since \((N, p_{m,n})\) and \((N, q_{m,n})\) are regular, we have,

\[
\lim_{m+n \to \infty} \sup_{0 \leq j \leq n} |p_{m-i,n-j}| = 0 = \lim_{m+n \to \infty} \sup_{0 \leq i \leq m} |p_{m-i,n-j}|.
\]
It now follows that
\[
\lim_{m+n \to \infty} \sup_{0 \leq j \leq n} \gamma_{m,n,i,j} = 0 = \lim_{m+n \to \infty} \sup_{0 \leq i \leq m} \gamma_{m,n,i,j}.
\]
Consequently \((N, r_{m,n})\) is regular. The regularity of this transformation enables us to infer that
\[
s_{m,n} \to \sigma'(N, q_{m,n}) \Rightarrow s_{m,n} \to \sigma'(N, r_{m,n}).
\]
Similarly we can show that
\[
s_{m,n} \to \sigma(N, p_{m,n}) \Rightarrow s_{m,n} \to \sigma(N, r_{m,n}).
\]
These imply that the two Nörlund methods \((N, p_{m,n})\) and \((N, q_{m,n})\) are consistent, completing the proof of the theorem.

If \((N, p_{m,n})\), \((N, q_{m,n})\) are regular, in view of conditions (24), (25), we have,
\[
P(x,y) = \sum P_{m,n} x^m y^n,
Q(x,y) = \sum Q_{m,n} x^m y^n,
p(x,y) = \sum p_{m,n} x^m y^n,
q(x,y) = \sum q_{m,n} x^m y^n,
\]
all converge for \(|x|, |y| < 1\). The series
\[
k(x,y) = \sum k_{m,n} x^m y^n = \frac{q(x,y)}{p(x,y)} = \frac{Q(x,y)}{P(x,y)},
l(x,y) = \sum l_{m,n} x^m y^n = \frac{p(x,y)}{q(x,y)} = \frac{P(x,y)}{Q(x,y)}
\]
are convergent for \(|x|, |y| < 1\) and further
\[
\sum_{i,j=0}^{m,n} k_{i,j} p_{m-i,n-j} = q_{m,n}; \quad \sum_{i,j=0}^{m,n} k_{i,j} p_{m-i,n-j} = Q_{m,n}, \quad (26)
\]
\[
\sum_{i,j=0}^{m,n} l_{i,j} q_{m-i,n-j} = p_{m,n}; \quad \sum_{i,j=0}^{m,n} l_{i,j} q_{m-i,n-j} = P_{m,n}. \quad (27)
\]
Theorem 5. If \((N, p_{m,n}), (N, q_{m,n})\) are regular, then \((N, p_{m,n}) \subseteq (N, q_{m,n})\) if and only if \(\lim_{m+n \to \infty} k_{m,n} = 0\).

Proof. Let \(s(x, y) = \sum s_{m,n}x^my^n\). Then for \(|x|, |y| < 1\), we have,

\[
\sum Q_{m,n}(N, q_{m,n})(s)x^my^n = \sum \left( \sum_{i,j=0}^{m,n} q_{m-i,n-j} s_{i,j} \right) x^my^n = q(x,y)s(x,y);
\]

similarly
\[
\sum P_{m,n}(N, p_{m,n})(s)x^my^n = p(x,y)s(x,y);
\]

Thus
\[
\sum Q_{m,n}(N, q_{m,n})(s)x^my^n = \sum k_{m,n}x^my^n \sum P_{m,n}(N, p_{m,n})(s)x^my^n
\]

which implies that
\[
Q_{m,n}(N, q_{m,n})(s) = \sum_{i,j=0}^{m,n} k_{m-i,n-j} P_{i,j}(N, p_{i,j})(s).
\]

Hence,
\[
(N, q_{m,n})(s) = \sum_{i,j=0}^{m,n} c_{m,n,i,j}(N, p_{i,j})(s), \quad (28)
\]

where
\[
c_{m,n,i,j} = k_{m-i,n-j} P_{i,j}/Q_{m,n}.
\]

If \((N, p_{m,n}) \subseteq (N, q_{m,n}), (c_{m,n,i,j})\) is regular and so, by Theorem 2 (a),
\[
\lim_{m+n \to \infty} c_{m,n,0,0} = 0,
\]
i.e.,
\[
\lim_{m+n \to \infty} \frac{|k_{m,n}| |P_{0,0}|}{|q_{0,0}|} = 0,
\]

which implies that \(\lim_{m+n \to \infty} k_{m,n} = 0\).

Conversely, if \(\lim_{m+n \to \infty} k_{m,n} = 0\), we can easily verify that \((c_{m,n,i,j})\) is regular. Consequently, using (28), it follows that \((N, p_{m,n}) \subseteq (N, q_{m,n})\). This completes the proof of the theorem.
Theorem 6, stated below, is an immediate consequence of Theorem 5.

**Theorem 6.** If $(N, p_{m,n})$ and $(N, q_{m,n})$ are regular Nörlund methods, then they are equivalent if and only if \( \lim_{m+n \to \infty} k_{m,n} = 0 \) and \( \lim_{m+n \to \infty} l_{m,n} = 0 \).

**Remark.** For the analogue of Theorem 6 in the classical case, see [5], Theorem III. Theorem 5, Theorem 6, in the case of regular Nörlund means for simple sequences, were established earlier by Natarajan (see [7], Theorem 3, Theorem 4).

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