

SERGEY V. LUDKOVSKY

Quasi-invariant measures on non-archimedean groups and semigroups of loops and paths, their representations. II

Annales mathématiques Blaise Pascal, tome 7, n° 2 (2000), p. 55-80

http://www.numdam.org/item?id=AMBP_2000__7_2_55_0

© Annales mathématiques Blaise Pascal, 2000, tous droits réservés.

L'accès aux archives de la revue « Annales mathématiques Blaise Pascal » (<http://math.univ-bpclermont.fr/ambp/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Quasi-invariant measures on non-Archimedean groups and semigroups of loops and paths, their representations. II.

Sergey V. Ludkovsky

permanent address: Theoretical Department, Institute of General Physics,
Str. Vavilov 38, Moscow, 117942, Russia.

Abstract

Loop groups G as families of mappings of one non-Archimedean Banach manifold M into another N with marked points over the same locally compact field K of characteristic $\text{char}(K) = 0$ are considered. Quasi-invariant measures on them are constructed. Then measures are used to investigate irreducible representations of such groups.

1 Introduction.

In the first part results on loop semigroups were exposed. This part is devoted to loop and path groups, quasi-invariant measures on them and their unitary representations. Results from Part I are used below (see also Introduction of Part I).

Irreducible components of strongly continuous unitary representations of Abelian locally compact groups are one-dimensional by Theorem 22.17 [10]. In general commutative non-locally compact groups may have infinite-dimensional irreducible strongly continuous unitary representations, for example, infinite-dimensional Banach spaces over \mathbb{R} considered as additive groups (see §2.4 in [1] and §4.5 [9]).

*Mathematics subject classification (1991 Revision) 43A05, 43A65 and 46S10.

In §3 for the investigation of a representation's irreducibility the pseudo-differentiability and some other specific properties of the constructed quasi-invariant measures are used. Besides continuous characters separating points of the loop group (see Theorem 3.3), strongly continuous infinite-dimensional irreducible unitary representations are constructed in §3.2.

The path groups and semigroups are investigated in §4.

In the real case there are known H -groups defined with the help of homotopies [18]. A composition on the H -group is defined relative to classes of homotopic mappings. In the non-Archimedean case homotopies are meaningless. A space of mappings $C(\xi, (M, s_0) \rightarrow (N, y_0))$ from one manifold M into another N preserving marked points (see I. §2.6) is supplied with the composition operation of families of mappings using loop semigroups. It is called a loop O -semigroup, since compositions are defined relative to certain equivalence classes, which are closures of families of certain orbits relative to the action of the diffeomorphism group of M preserving s_0 . From it a loop O -group is defined with the help of the Grothendieck construction. O -groups are considered in §5.

In §6 the notation is summarized.

2 Loop groups.

2.1. Note and Definition. For a commutative monoid $\Omega_\xi(M, N)$ with the unity and the cancellation property (see Theorem I.2.7 and Condition I.2.7.(5)) there exists a commutative group $L_\xi(M, N)$ equal to the Grothendieck group. This group is the quotient group F/B , where F is a free Abelian group generated by $\Omega_\xi(M, N)$ and B is a closed subgroup of F generated by elements $[f + g] - [f] - [g]$, f and $g \in \Omega_\xi(M, N)$, $[f]$ denotes an element of F corresponding to f . In view of §9 [12] and [17] the natural mapping

$$(1) \gamma : \Omega_\xi(M, N) \rightarrow L_\xi(M, N)$$

is injective. We supply F with a topology inherited from the Tychonoff product topology of $\Omega_\xi(M, N)^{\mathbb{Z}}$, where each element z of F is

$$(2) z = \sum_f n_{f,z} [f],$$

$n_{f,z} \in \mathbb{Z}$ for each $f \in \Omega_\xi(M, N)$,

$$(3) \sum_f |n_{f,z}| < \infty.$$

In particular $[nf] - n[f] \in B$, where $1f = f$, $nf = f \circ (n-1)f$ for each $1 < n \in \mathbb{N}$, $f+g := f \circ g$. We call $L_\xi(M, N)$ the loop group.

2.2. Proposition. *The space $L_\xi(M, N)$ from §2.1 is the complete separable Abelian Hausdorff topological group; it is non-discrete, perfect and has the cardinality c .*

Proof follows from §1.2.7 and §2.1, since in view of Formulas 2.1.(1-3) for each $f \in L_\xi(M, N)$ there are $g_j \in \Omega_\xi(M, N)$ such that $f = f_1 - f_2$, where $\gamma(g_j) = f_j$ for each $j \in \{1, 2\}$. Therefore, γ is the topological embedding such that $\gamma(f+g) = \gamma(f) + \gamma(g)$, $\gamma(e) = e$.

2.3. Theorem. *Let $G = L_\xi(M, N)$ be the same group as in §2.1, $\xi = (t, s)$ or $\xi = t$ with $0 \leq t \in \mathbb{R}$, $s_0 \in \mathbb{N}_0$.*

(1) *If $At'(\tilde{M})$ has $\text{card}(\Lambda'_{\tilde{M}}) \geq 2$, then G is isomorphic with $G_1 = L_\xi(\tilde{M}, N)$, where $\tilde{M} = U'_1 \cup U'_2$ (see §1.2.5). Moreover, $T_\eta G$ is the Banach space for each $\eta \in G$ and G is ultrametrizable.*

(2) *If $1 \leq t+s$, then G is an analytic manifold and for it the mapping $\tilde{E} : TG \rightarrow G$ is defined, where TG is the neighbourhood of G in TG such that $\tilde{E}_\eta(V) = \tilde{e}xp_{\eta(s)} \circ V_\eta$ from some neighbourhood \tilde{V}_η of the zero section in $T_\eta G \subset TG$ onto some neighbourhood $W_\eta \ni \eta \in G$, $\tilde{V}_\eta = \tilde{V}_e \circ \eta$, $W_\eta = W_e \circ \eta$, $\eta \in G$ and \tilde{E} belongs to the class $C(\infty)$ by V , \tilde{E} is the uniform isomorphism of uniform spaces \tilde{V} and W .*

(3) *There are atlases $\tilde{At}(TG)$ and $\tilde{At}(G)$ for which \tilde{E} is locally analytic. Moreover, G is not locally compact for each $0 \leq t$.*

Proof. The first statement follows immediately from Theorem 1.2.17 and §2.1. Therefore, to prove the second statement it is sufficient to consider the manifold M with a finite atlas $At(M)$.

Let $V_\eta \in T_\eta G$ for each $\eta \in G$, $V \in C_0(\xi, G \rightarrow TG)$, suppose also that $\tilde{\pi} \circ V_\eta = \eta$ be the natural projection such that $\tilde{\pi} : TG \rightarrow G$, then V is a vector field on G of class $C_0(\xi)$. The disjoint and analytic atlases $At(C_0(\xi, M \rightarrow N))$ and $At(C_0(\xi, M \rightarrow TN))$ induce disjoint clopen atlases in G and TG with the help of the corresponding equivalence relations and ultrametrics in these quotient spaces. These atlases are countable, since G and TG are separable. In view of Theorem 1.2.10 the space $T_\eta G$ is Banach and not locally compact, hence it is infinite-dimensional over K .

In view of Formulas I.2.6.2.(1-7) the multiplications

$$(1) R_f : G \rightarrow G, g \mapsto g \circ f = R_f(g) \text{ and}$$

$$(2) \alpha_h : C_0^0(\xi, (M, s_0) \rightarrow (N, y_0)) \rightarrow C_0^0(\xi, (M, s_0) \rightarrow (N, y_0)), \alpha_h(v) = v \circ h$$

for $f, g \in G$ and $h, v \in C_0^0(\xi, (M, s_0) \rightarrow (N, y_0))$ belong to the class $C(\infty)$.

Using Formulas (1,2) as in §I.2.10 we get, that the vector field V on G of class $C_0(\xi)$ has the form

$$(3) V_{\eta(x)} = v(\eta(x)),$$

where v is a vector field on N of the class $C_0(\xi)$, $\eta \in G$,

$$v(\langle f \rangle_{K, \xi}(x)) := \{v(g(x)) : g \in \langle f \rangle_{K, \xi}\}.$$

Since $\bar{exp} : \tilde{T}N \rightarrow N$ is analytic on the corresponding charts (see §I.2.8.). In view of Formulas I.2.8.(1-4) $\tilde{E}(V) = \bar{exp} \circ V$ has the necessary properties, where \bar{exp} is considered on $At^n(N)$ with $\psi''_i(V''_i)$ being K -convex in the Banach space Y . Therefore, due to Formula (3) we have

$$(4) \tilde{E}_\eta : T_\eta G \supset \bar{V}_\eta \rightarrow W_\eta \subset G$$

are continuous and

$$(5) \tilde{E}_\eta(V) = \bar{exp}_{\eta(x)} v(\eta(x)),$$

where $x \in M$, consequently, \tilde{E} is of class $C(\infty)$.

2.4. Note. Let $\Omega_\xi^{\{k\}}(M, N)$ be the same submonoid as in §I.3.5 such that $c > 0$ and $c' > 0$. Then it generates the loop group $G' := L_\xi^{\{k\}}(M, N)$ as in §2.1 such that G' is the dense subgroup in $G = L_\xi(M, N)$.

2.5. Theorem. *On the group $G = L_\xi(M, N)$ from §2.1 and for each $b \in \mathbb{C}$ there exist probability quasi-invariant and pseudo-differentiable of order b measures μ with values in \mathbb{R} and K_q for each prime number q such that $q \neq p$ relative to a dense subgroup G' .*

Proof. In view of Theorem 2.3 it is sufficient to consider the case of M with the finite atlas $At'(M)$. Let the operator \tilde{A} be defined on $TC_0^0(\xi, (M, s_0) \rightarrow (N, y_0))$ by Formulas I.3.6.(3,4). The factorization by the equivalence relation \tilde{K}_ξ from §I.3.6 and the Grothendieck construction of §2.1 produces the following mapping \tilde{Y} from the corresponding neighbourhood of the zero section

in $TL_\xi(M, N)$ into a neighbourhood of the zero section either in $TL_{\xi'}(M, Y)$ for $\dim_K M < \infty$ or into $c_0(\{TL_{\xi'}(M_a, Y) : a \in \mathbb{N}\})$ for $\dim_K M = \aleph_0$.

Therefore they are continuously strongly differentiable with $(D\tilde{Y}(f))(v) = \tilde{Y}(f)(v)$, where f and $v \in V_N \subset T_e L_\xi(M, N)$, V_N is the corresponding neighbourhoods of zero sections for the element $e = \langle \omega_0 \rangle_{K, \xi}$. In view of the existence of the mapping \tilde{E} (see Formulas 2.3.(4,5)) for $\tilde{T}G$ there exists the local diffeomorphism

$$(1) \Upsilon : W_e \rightarrow V'_0$$

induced by \tilde{E} and \tilde{Y} , where W_e is a neighbourhood of e in G , V'_0 is a neighbourhood of zero either in the Banach subspace \tilde{H} of $T_e L_{\xi'}(M, Y)$ for $\dim_K M < \infty$ or in the Banach subspace \tilde{H} of $c_0(\{T_e L_{\xi'}(M_a, Y) : a \in \mathbb{N}\})$ for $\dim_K M = \aleph_0$.

Let now W'_e be a neighbourhood of e in G' such that $W'_e W_e = W_e$. It is possible, since the topology in G and G' is given by the corresponding ultrametrics and there exists W_e with $W_e W_e = W_e$, hence it is sufficient to take $W'_e \subset W_e$. For $g \in W_e$, $v = \tilde{E}^{-1}(g)$, $\phi \in W'_\xi$ the following operator

$$(2) S_\phi(v) := \Upsilon \circ L_\phi \circ \Upsilon^{-1}(v) - v$$

is defined for each $(\phi, v) \in W'_e \times V'_0$, where $L_\phi(g) := \phi \circ g$. Then $S_\phi(v) \in V''_0 \subset V'_0$, where V''_0 is an open neighbourhood of the zero section either in the Banach subspace \tilde{H}' of $T_e G'$ for $\dim_K M < \infty$ or in the Banach subspace \tilde{H}' of $c_0(\{T_e G'_a : a \in \mathbb{N}\})$ for $\dim_K M = \aleph_0$, where $G'_a = L_\xi^{(k)}(M_a, N)$. Moreover, $S_\phi(v)$ is the $C(\infty)$ -mapping by ϕ and v . As in §1.3.6 a quasi-invariant and pseudo-differentiable of order b measure ν on $V'_0 \subset \tilde{H}$ exists relative to $\phi \in W'_e$, where

$$(3) \nu(dx) = \bigotimes_{j=1}^{\infty} \nu_{l(j)}(dx^j)$$

and Conditions 1.3.6.(13,14,17-20) are satisfied.

More general classes of quasi-invariant and pseudo-differentiable of order b measures ν with values in $[0, \infty)$ or in K_q exist on V'_0 relative to the action of $\phi \in W'_e$, $(\phi, v) \mapsto v + S_\phi(v)$, where $v \in V'_0$.

In view of Formulas (1-3) the measure ν induces a measure $\tilde{\mu}$ on W_e with the help of Υ such that

$$(4) \tilde{\mu}(A) = \nu(\Upsilon(A)) \text{ for each } A \in Bf(W_e),$$

since $\|\nu\|(V'_0) > 0$. The groups G and G' are separable and ultrametrizable, hence there are locally finite coverings $\{\phi_i \circ W_i : i \in \mathbb{N}\}$ of G and $\{\phi_i \circ W'_i : i \in \mathbb{N}\}$ of G' with $\phi_i \in G'$ such that W_i are open subsets in W_e and W'_i are open subsets in W'_e , that is,

$$\bigcup_{i=1}^{\infty} \phi_i \circ W_i = G \text{ and } \bigcup_{i=1}^{\infty} \phi_i \circ W'_i = G',$$

where $\phi_1 = e$, $W_1 = W_e$ and $W'_1 = W'_e$ [6]. Then $\tilde{\mu}$ can be extended onto G by the following formula

$$(5) \quad \mu(A) := \left(\sum_{i=1}^{\infty} \tilde{\mu}((\phi_i^{-1} \circ A) \cap W_i) r^i \right) / \left(\sum_{i=1}^{\infty} \tilde{\mu}(W_i) r^i \right)$$

for each $A \in Bf(G)$, where $0 < r < 1$ for real $\tilde{\mu}$ or $r = q$ for $\tilde{\mu}$ with values in K_q . In view of Formulas (4, 5) this μ is the desired measure, which is quasi-invariant and pseudo-differentiable of order b relative to the subgroup $G'' = G'$ (see also §§1.3.2-4).

3 Representations of loop groups.

3.1. Let μ be a real non-negative quasi-invariant relative to G' measure on $(G, Bf(G))$ as in Theorem 2.5. Assume also that $H := L^2(G, \mu, \mathbb{C})$ is the standard Hilbert space of equivalence classes of functions $f : G \rightarrow \mathbb{C}$ for which absolute values $|f|$ are square-integrable by μ . Suppose that $U(H)$ is the unitary group on H in a topology induced from a Banach space $L(H \rightarrow H)$ of continuous linear operators supplied with the operator norm.

Theorem. *There exists a strongly continuous injective homomorphism $T : G' \rightarrow U(H)$.*

Proof. Let f and h be in H , their scalar product is given by the standard formula

$$(1) \quad (f, h) := \int_G \bar{h}(g) f(g) \mu(dg),$$

where f and $h : G \rightarrow \mathbb{C}$, \bar{h} denotes the complex conjugated function h . There exists the regular representation

$$(2) \quad \mathbb{T} : G' \rightarrow U(H)$$

defined by the following formula:

$$(3) \quad T_z f(g) := [\rho(z, g)]^{1/2} f(z^{-1}g),$$

where

$$(4) \quad \rho(z, g) = \mu_z(dg)/\mu(dg), \quad \mu_z(S) := \mu(z^{-1}S)$$

for each $S \in Bf(G)$, $z \in G'$. For each fixed z the quasi-invariance factor $\rho(z, g)$ is continuous by g , hence $T_z f(g)$ is measurable, if $f(g)$ is measurable (relative to $Af(G, \mu)$ and $Bf(C)$). Therefore,

$$(5) \quad (T_z f(g), T_z h(g)) = \int_G \bar{h}(z^{-1}g) f(z^{-1}g) \rho(z, g) \mu(dg) = (f, h),$$

consequently, T_z is the unitary operator for each $z \in G'$. From

$$(6) \quad \rho(z'z, g) = \rho(z, (z')^{-1}g) \rho(z', g) = [\mu_{z'z}(dg)/\mu_{z'}(dg)][\mu_{z'}(dg)/\mu(dg)]$$

it follows that

$$(7) \quad T_{z'} T_z = T_{z'z}, \quad T_{Id} = I \text{ and } T_{z^{-1}} = T_z^{-1},$$

where I is the unit operator on H .

The embedding of $T_e G'$ into $T_e G$ is the compact operator. The measure μ on G is induced by the measure on $c_0(\omega_0, K)$, where ω_0 is the first countable ordinal. In view of Theorems 3.12 and 3.28 [13] for each $\delta > 0$ and $\{f_1, \dots, f_n\} \subset H$ there exists a compact subset B in G such that

$$(8) \quad \sum_{i=1}^n \int_{G \setminus B} |f_i(g)|^2 \mu(dg) < \delta^2.$$

Therefore, there exists an open neighbourhood W' of e in G' and an open neighbourhood S of e in G such that $\rho(z, g)$ is continuous and bounded on $W' \times W' \circ S$, where $S \subset W' \circ S \subset G$. In view of Formulas (5-8), Theorems 2.3 and 2.5 and the Hölder inequality we have

$$\lim_{j \rightarrow \infty} \sum_{i=1}^n \|(T_{z_j} - I)f_i\|_H = 0$$

for each sequence $\{z_j : j \in \mathbf{N}\}$ converging to e in G' . Indeed, for each $v > 0$ and a continuous function $f : G \rightarrow \mathbf{C}$ with $\|f\|_H = 1$ there is an open

neighbourhood V of id in G' (in the topology of G'), such that $|\rho(z, g) - 1| < v$ for each $z \in V$ and each $g \in F$ for some open F in G , $id \in F$ with

$$\mu_z^f(G \setminus F) < v \text{ for each } z \in V, \text{ where } \mu^f(dg) := |f(g)|\mu(dg)$$

and $f \in \{f_1, \dots, f_n\}$, $n \in \mathbb{N}$. At first this can be done analogously for the corresponding Banach space from which μ was induced.

In H continuous functions $f(g)$ are dense, hence for each $0 < v < 1$ there exists V'' such that

$$\int_G |f(g) - f(zg)(\rho(z, g))^{1/2}|^2 \mu(dg) < 4v$$

for each finite family $\{f_j\}$ with $\|f_j\|_H = 1$ and $z \in V' = V \cap V''$, where V'' is an open neighbourhood of id in G' such that $\|f(g) - f(zg)\|_H < v$ for each $z \in V''$, consequently T is strongly continuous (that is, T is continuous relative to the strong topology on $U(H)$ induced from $L(H \rightarrow H)$, see its definition in [8]).

Moreover, T is injective, since for each $g \neq id$ there is $f \in C^0(G, \mathbb{C}) \cap H$, such that $f(id) = 0$, $f(g) = 1$, and $\|f\|_H > 0$, so $T_f \neq I$.

Note. In general T is not continuous relative to the norm topology on $U(H)$, since for each $z \neq id \in G'$ and each $1/2 > v > 0$ there is $f \in H$ with $\|f\|_H = 1$, such that $\|f - T_z f\|_H > v$, when $\text{supp}(f)$ is sufficiently small with $(z \circ \text{supp}(f)) \cap \text{supp}(f) = \emptyset$.

3.2. Theorem. *Let G be a loop group with a real probability quasi-invariant measure μ relative to a dense subgroup G' as in Theorem 2.5. Then μ may be chosen such that the associated regular unitary representation (see §3.1) of G' is irreducible.*

Proof. Let ν on $c_0(\omega_0, K)$ be of the same type as in §3.23 or §3.30 [13] or it is given by Formulas I.3.6.(13-20). For example, ν is generated by a weak distribution such that

$$(1) \nu_j(dx^j) := c_j \exp(-|x^j/\xi^j|^\gamma) \nu(dx^j),$$

where $c_j > 0$, $\nu_j(K) = 1$, ν is the Haar non-negative measure on K ,

$$(2) \lim_{j \rightarrow \infty} \xi^j = 0,$$

$0 \neq \xi^j \in K$, $\gamma > 0$ is fixed with

$$(3) \sum_{j=1}^{\infty} |\xi^j|^{-\gamma} p^{-h(t_j, m_j)} < \infty$$

(see about $k(i, m)$ in §I.3.5). Let a ν -measurable function $f : c_0(\omega_0, K) \rightarrow C$ be such that $\nu(\{x \in c_0(\omega_0, K) : f(x+y) \neq f(x)\}) = 0$ for each $y \in sp_K(e_j : j \in N) =: X_0$ with $f \in L^1(c_0(\omega_0, K), \nu, C)$. Let also $P_k : c_0(\omega_0, K) \rightarrow L(k)$ be projectors such that $P_k(x) = x_k$ for each $x = (\sum_{j \in N} x^j e_j)$, where $x_k := \sum_{j=1}^k x^j e_j$ and $L(k) := sp_K(e_1, \dots, e_k)$. Then analogously to the proof of Proposition II.3.1 [4] in view of Fubini theorem there exists a sequence of cylindrical functions

$$(4) f_k(x) = f_k(x_k) = \int_{c_0(\omega_0, K) \ominus L(k)} f(P_k x + (I - P_k)y) \nu_{I-P_k}(dy)$$

which converges to f in $L^1(c_0(\omega_0, K), \nu, C)$, where $\nu = \nu_{L(k)} \otimes \nu_{I-P_k}$, ν_{I-P_k} is the measure on $c_0(\omega_0, K) \ominus L(k)$. Each cylindrical function f_k is ν -almost everywhere constant on $c_0(\omega_0, K)$, since $L(k) \subset X_0$ for each $k \in N$, consequently, f is ν -almost everywhere constant on $c_0(\omega_0, K)$. Let Υ be the local diffeomorphism from Formula 2.5.(1). In view of Theorems 5.13 and 5.16 [16] these Banach spaces are topologically K -linearly isomorphic with $c_0(\omega_0, K)$. From the construction of G' and μ with the help of Υ and ν as in §2.5 it follows that if a function $f \in L^1(G, \mu, C)$ satisfies the following condition $f^h(g) = f(g) \pmod{\mu}$ by $g \in G$ for each $h \in G'$, then $f(x) = \text{const} \pmod{\mu}$, where $f^h(g) := f(hg)$, $g \in G$.

Let $f(g) = \chi_U(g)$ be the characteristic function of a subset U , $U \subset G$, $U \in Af(G, \mu)$, then $f(hg) = 1 \Leftrightarrow g \in h^{-1}U$. If $f^h(g) = f(g)$ is true by $g \in G$ μ -almost everywhere, then

$$(5) \mu(\{g \in G : f^h(g) \neq f(g)\}) = 0,$$

that is $\mu((h^{-1}U) \Delta U) = 0$, consequently, the measure μ satisfies the condition (P) from §VIII.19.5 [8], where $A \Delta B := (A \setminus B) \cup (B \setminus A)$ for each $A, B \subset G$. For each subset $E \subset G$ the outer measure $\mu^*(E) \leq 1$, since $\mu(G) = 1$ and μ is non-negative [2], consequently, there exists $F \in Bf(G)$ such that $F \supset E$ and $\mu(F) = \mu^*(E)$. This F may be interpreted as the least upper bound in $Bf(G)$ relative to the latter equality. In view of Proposition VIII.19.5 [8] the measure μ is ergodic, that is for each $U \in Af(G, \mu)$ and $F \in Af(G, \mu)$ with $\mu(U) \times \mu(F) \neq 0$ there exists $h \in G'$ such that $\mu((h \circ E) \cap F) \neq 0$.

From Theorem I.1.2 [4] it follows that $(G, Bf(G))$ is a Radon space, since G is separable and complete. Therefore, a class of compact subsets approximates from below each measure $|f(g)|\mu(dg)$, where $f \in L^2(G, \mu, C)$. Due to

Egorov Theorem 2.3.7 [7] for each $\epsilon > 0$ and for each sequence $f_n(g)$ converging to $f(g)$ for μ -almost every $g \in G$, when $n \rightarrow \infty$, there exists a compact subset K in G such that $\mu(G \setminus K) < \epsilon$ and $f_n(g)$ converges on K uniformly by $g \in K$, when $n \rightarrow \infty$. Hence in view of the Stone-Weierstrass Theorem A.8 [8] an algebra $V(Q)$ of finite pointwise products of functions from the following space

$$(6) \text{ } sp_C\{\psi(g) := \rho^{1/2}(h, g) : h \in G'\} =: Q$$

is dense in H , since $\rho(e, g) = 1$ for each $g \in G$ and $L_h : G \rightarrow G$ are diffeomorphisms of the manifold G , where $L_h(g) := hg$.

For each $m \in \mathbb{N}$ there are locally analytic curves $S(\zeta, \phi_j)$ in G' with analytic restrictions $S(\zeta, \phi_j)|_{B(K, 0, 1)}$, where $j = 1, \dots, m$ and $\zeta \in K$ is a parameter, such that

$$S(0, \phi_j) = e \text{ and } (\partial S(\zeta, \phi_j)/\partial \zeta)|_{\zeta=0} \text{ are linearly independent in } T_e G'$$

for $j = 1, \dots, m$, since G' is the infinite-dimensional group, which is complete relative to its own uniformity. In accordance with §2.5 there exists infinitely pseudo-differentiable μ on G (that is, of order l for each $l \in \mathbb{N}$) relative to $S(\zeta, \phi_j)$ for each j . If two real non-negative quasi-invariant relative to G' measures μ and λ on G are equivalent, then the corresponding regular representations T^μ and T^λ are equivalent, since the mapping

$$f(g) \mapsto (\mu(dg)/\lambda(dg))^{1/2} f(g)$$

establishes an isomorphism of $L^2(G, \mu, \mathbb{C})$ with $L^2(G, \lambda, \mathbb{C})$, where $f \in L^2(G, \mu, \mathbb{C})$. Then the following condition $\det(\Psi(g)) = 0$ defines an analytic submanifold G_Ψ in G of codimension over K no less than one:

$$(7) \text{ } \text{codim}_K G_\Psi \geq 1,$$

where $\Psi(g)$ is a matrix function of the variable $g \in G$ with matrix elements

$$(8) \text{ } \Psi_{l,j}(g) := PD_c(l, \rho^{1/2}(S(\zeta, \phi_j), g))$$

for $l \geq 1$. If $f \in H$ is such that

$$(9) \text{ } (f(g), \rho^{1/2}(\phi, g))_H = 0$$

for each $\phi \in G' \cap W$, then

$$(10) PD_c(l, (f(g), \rho^{1/2}(S(\zeta, \phi_j), g)))_H = 0.$$

But $V(Q)$ is dense in H and in view of Formulas (6 – 10) this means that $f = 0$, since for each m there are $S(\zeta, \phi_j) \in G' \cap W$ such that $\det \Psi(g) \neq 0$ μ -almost everywhere on G . If $\|f\|_H > 0$, then $\mu(\text{supp}(f)) > 0$, consequently, $\mu((G' \text{supp}(f)) \cap W) = 1$, since $G'U = G$ for each open U in G and for each $\epsilon > 0$ there exists an open U such that $U \supset \text{supp}(f)$ and $\mu(U \setminus \text{supp}(f)) < \epsilon$.

Therefore, Q is dense in H . This means that the unit vector f_0 is cyclic, where $f_0 \in H$ and $f_0(g) = 1$ for each $g \in G$. The group G is Abelian, hence there exists a unitary operator $U : H \rightarrow H$ such that

$$(11) U^{-1}T_hU = F_h$$

are operators of multiplication on functions $F_h \in L^\infty(G, \mu, \mathbb{C})$ for each $h \in G'$, where

$$(12) F_h(g) = \exp(2\pi i f_h(g)),$$

$g \in G$, $f_h \in L^0(G, \mu, \mathbb{R})$, $L^0(G, \mu, \mathbb{R})$ is a Frechét space of classes of equivalent μ -measurable functions $f : G \rightarrow \mathbb{R}$, which is supplied with a metric

$$(13) d(f, v) := \int_G \min(1, |f(g) - v(g)|) \mu(dg),$$

$i = (-1)^{1/2}$ (see §IV.8 and Theorem X.2.1 and Theorem X.4.2 and Segal Theorem in §X.9 [5]). The following set $\{cl \text{ } sp_{\mathbb{C}}\{F_h : h \in G'\}\}$ is not contained in any ideal of the form $\{F : \text{supp}(F) \subset G \setminus A\}$ with $A \in Af(G, \mu)$ and $\mu(A) > 0$, since $|F_h(g)| = 1$ for each $(h, g) \in G' \times G$, where $cl(E)$ is taken in $L^\infty(G, \mu, \mathbb{C})$ for its subset E . Then $\{F_h : h \in G'\}$ is not contained in any set

$$\{F = \exp(2\pi i f) : f \in L^0(G, \mu, \mathbb{C}), \text{supp}(f) \subset G \setminus A\}$$

with $A \in Af(G, \mu)$ and $\mu(A) > 0$, since μ is ergodic relative to G' . From the construction of μ (see Formulas (1-3) and I.3.6.(13-17, 21-24)) it follows that for each $f_{1,j}$ and $f_{2,j} \in H$, $j = 1, \dots, n$, $n \in \mathbb{N}$ and each $\epsilon > 0$ there exists $h \in G'$ such that

$$|(T_h f_{1,j}, f_{2,j})_H| \leq \epsilon |(f_{1,j}, f_{2,j})_H|,$$

when $|(f_{1,j}, f_{2,j})_H| > 0$, hence

$$|(F_h U^{-1} f_{1,j}, U^{-1} f_{2,j})_H| \leq \epsilon |(U^{-1} f_{1,j}, U^{-1} f_{2,j})_H| = \epsilon |(f_{1,j}, f_{2,j})_H|,$$

since G is the Radon space by Theorem I.1.2 [4] and G is not locally compact. Therefore, for each $\tilde{f}_{1,j}$ and $\tilde{f}_{2,j} \in H$, $j = 1, \dots, n$, $n \in \mathbb{N}$ and $\epsilon > 0$ there exists $h \in G'$ for which $|(F_h \tilde{f}_{1,j}, \tilde{f}_{2,j})_H| \leq \epsilon |(\tilde{f}_{1,j}, \tilde{f}_{2,j})_H|$ for each $j = 1, \dots, n$, when $|(\tilde{f}_{1,j}, \tilde{f}_{2,j})_H| > 0$, since $UH = H$. This means that there is not any finite-dimensional G' -invariant subspace H' in H , that is, $F_h H' \subset H'$ for each $h \in G'$.

We suppose that λ is a probability Radon measure on G' such that λ has not any atoms and $\text{supp}(\lambda) = G'$. In view of the strong continuity of the regular representation there exists the S. Bochner integral $\int_G T_h f(g) \mu(dg)$ for each $f \in H$, which implies its existence in the weak (B. Pettis) sense. The measures μ and λ are non-negative and bounded, hence $H \subset L^1(G, \mu, \mathbb{C})$ and $L^2(G', \lambda, \mathbb{C}) \subset L^1(G', \lambda, \mathbb{C})$ due to the Cauchy inequality. Therefore, we can apply below Fubini theorem (see §II.16.3 [8]). Let $f \in H$, then there exists a countable orthonormal base $\{f^j : j \in \mathbb{N}\}$ in $H \ominus Cf$. Then for each $n \in \mathbb{N}$ the following set

$$B_n := \{q \in L^2(G', \lambda, \mathbb{C}) : (f^j, f)_H = \int_{G'} q(h)(f^j, T_h f_0)_H \lambda(dh) \text{ for } j = 0, \dots, n\}$$

is non-empty, since the unit vector f_0 is cyclic, where $f^0 := f$. There exists $\infty > R > \|f\|_H$ such that $B_n \cap B^R := B_n^R$ is non-empty and weakly compact for each $n \in \mathbb{N}$, since B^R is weakly compact, where

$$B^R := \{q \in L^2(G', \lambda, \mathbb{C}) : \|q\| \leq R\}$$

(see the Alaoglu-Bourbaki theorem in §(9.3.3) [15]). Therefore, B_n^R is a centered system of closed subsets of B^R , that is,

$$\bigcap_{n=1}^m B_n^R \neq \emptyset \text{ for each } m \in \mathbb{N},$$

hence it has a non-empty intersection, consequently, there exists $q \in L^2(G', \lambda, \mathbb{C})$ such that

$$(14) \quad f(g) = \int_{G'} q(h) T_h f_0(g) \lambda(dh)$$

for μ -almost each $g \in G$. If $F \in L^\infty(G, \mu, \mathbb{C})$, f_1 and $f_2 \in H$, then there exist q_1 and $q_2 \in L^2(G', \lambda, \mathbb{C})$ satisfying equation (14). Therefore,

$$(15) \quad (f_1, F f_2)_H = \int_G \int_{G'} \int_{G'} \bar{q}_1(h_1) q_2(h_2) F(g) \rho_\mu^{1/2}(h_1, g) \rho_\mu^{1/2}(h_2, g) \lambda(dh_1) \lambda(dh_2) \mu(dg).$$

Let

$$(16) \quad \xi(h) := \int_G \int_{G'} \int_{G'} \bar{q}_1(h_1) q_2(h_2) \rho_\mu^{1/2}(h_1, g) \rho_\mu^{1/2}(h h_2, g) \lambda(dh_1) \lambda(dh_2) \mu(dg).$$

Then there exists $\beta(h) \in L^2(G', \lambda, \mathbb{C})$ such that

$$(17) \quad \int_{G'} \beta(h) \xi(h) \lambda(dh) = (f_1, F f_2)_H =: c.$$

To prove this we consider two cases. If $c = 0$ it is sufficient to take β orthogonal to ξ in $L^2(G', \lambda, \mathbb{C})$. Each function $q \in L^2(G', \lambda, \mathbb{C})$ can be written as $q = q^1 - q^2 + i q^3 - i q^4$, where $q^j(h) \geq 0$ for each $h \in G'$ and $j = 1, \dots, 4$, hence we obtain the corresponding decomposition for ξ :

$$(18) \quad \xi = \sum_{j,k} b^{j,k} \xi^{j,k},$$

where $\xi^{j,k}$ corresponds to a pair (q_1^j, q_2^k) , where $b^{j,k} \in \{1, -1, i, -i\}$. If $c \neq 0$ we can choose (j_0, k_0) for which $\xi^{j_0, k_0} \neq 0$ and

$$(19) \quad \beta \text{ is orthogonal to others } \xi^{j,k} \text{ with } (j, k) \neq (j_0, k_0).$$

Otherwise, if $\xi^{j,k} = 0$ for each (j, k) , then $q_l^j(h) = 0$ for each (l, j) and λ -almost every $h \in G'$, since due to Formula (16):

$$\xi(0) = \int_G \mu(dg) \left(\int_{G'} \bar{q}_1(h_1) \rho_\mu^{1/2}(h_1, g) \lambda(dh_1) \right) \left(\int_{G'} q_2(h_2) \rho_\mu^{1/2}(h_2, g) \lambda(dh_2) \right) = 0$$

and this implies $c = 0$, which is the contradiction with the assumption $c \neq 0$. Hence due to Formula (18) there exists β satisfying Formula (17) and Condition (19).

Since $L^2(G', \lambda, \mathbb{C})$ is infinite-dimensional, then for each finite families

$$\{a_1, \dots, a_m\} \subset L^\infty(G, \mu, \mathbb{C}) \text{ and } \{f_1, \dots, f_m\} \subset H$$

there exists $\beta(h) \in L^2(G', \lambda, \mathbb{C})$, such that

$$\beta \text{ is orthogonal to } \int_G \bar{f}_s(g) [f_j(h^{-1}g) (\rho_\mu(h, g))^{1/2} - f_j(g)] \mu(dg)$$

for each $s, j = 1, \dots, m$. Hence each operator of multiplication on $a_j(g)$ belongs to A_G^n , since due to Formula (17) and Condition (19) there exists $\beta(h)$ such that

$$\begin{aligned} (f_s, a_j f_l) &= \int_G \int_{G'} \bar{f}_s(g) \beta(h) (\rho_\mu(h, g))^{1/2} f_l(h^{-1}g) \lambda(dh) \mu(dg) \\ &= \int_G \int_{G'} \bar{f}_s(g) \beta(h) (T_h f_l(g)) \lambda(dh) \mu(dg) \text{ and} \\ \int_G \bar{f}_s(g) a_j(g) f_l(g) \mu(dg) &= \int_G \int_{G'} \bar{f}_s(g) \beta(h) f_l(g) \lambda(dh) \mu(dg) = (f_s, a_j f_l). \end{aligned}$$

Hence A_G^n contains subalgebra of all operators of multiplication on functions from $L^\infty(G, \mu, \mathbb{C})$.

Let us remind the following. A Banach bundle B over a Hausdorff space G' is a bundle $\langle B, \pi \rangle$ over G' , together with operations and norms making each fiber B_h ($h \in G'$) into a Banach space such that

$BB(i)$ $x \mapsto \|x\|$ is continuous from B into \mathbb{R} ;

$BB(ii)$ the operation $+$ is continuous as a function from

$$\{(x, y) \in B \times B : \pi(x) = \pi(y)\} \text{ into } B;$$

$BB(iii)$ for each $\lambda \in \mathbb{C}$ the map $x \mapsto \lambda x$ is continuous from B into B ;

$BB(iv)$ if $h \in G'$ and $\{x_i\}$ is any net of elements of B such that $\|x_i\| \rightarrow 0$

and $\pi(x_i) \rightarrow h$ in G' , then $x_i \rightarrow 0_h$ in B ,

where $\pi : B \rightarrow G'$ is a bundle projection, $B_h := \pi^{-1}(h)$ is the fiber over h (see §II.13.4 [8]). If G' is a Hausdorff topological group, then a Banach algebraic bundle over G' is a Banach bundle $B = \langle B, \pi \rangle$ over G' together with a binary operation \bullet on B satisfying the following Conditions $AB(i - v)$:

$$AB(i) \pi(b \bullet c) = \pi(b)\pi(c) \text{ for } b \text{ and } c \in B;$$

$AB(ii)$ for each x and $y \in G'$ the product \bullet is bilinear from $B_x \times B_y$ into B_{xy} ;

$AB(iii)$ the product \bullet on B is associative;

$$AB(iv) \|b \bullet c\| \leq \|b\| \times \|c\| \text{ for each } b, c \in B;$$

$AB(v)$ the map \bullet is continuous from $B \times B$ into B

(see §VIII.2.2 [8]). With G' and a Banach algebra A the trivial Banach bundle $B = A \times G'$ is associative, in particular let $A = \mathbb{C}$ (see §VIII.2.7 [8]).

The regular representation T of G' gives rise to a canonical regular $L^2(G, \mu, \mathbb{C})$ -projection-valued measure \bar{P} :

$$(20) \quad \bar{P}(W)f := Ch_W f,$$

where $f \in L^2(G, \mu, \mathbb{C})$, $W \in Bf(G)$, Ch_W is the characteristic function of W . Therefore,

$$(21) \quad T_h \bar{P}(W) = \bar{P}(h \circ W) T_h$$

for each $h \in G'$ and $W \in Bf(G)$, since $\rho(h, h^{-1} \circ g)\rho(h, g) = 1 = \rho(e, g)$ for each $(h, g) \in G' \times G$,

$$Ch_W(h^{-1} \circ g) = Ch_{h \circ W}(g) \text{ and}$$

$$(22) \quad T_h(\bar{P}(W)f(g)) = \rho(h^{-1}, g)^{1/2} \bar{P}(h \circ W)f(h^{-1} \circ g).$$

Thus $\langle T, \bar{P} \rangle$ is a system of imprimitivity for G' over G , which is denoted T^μ , that is,

$SI(i)$ T is a unitary representation of G' ;

$SI(ii)$ \bar{P} is a regular $L^2(G, \mu, \mathbb{C})$ -projection-valued Borel measure on G and

$SI(iii)$ $T_h \bar{P}(W) = \bar{P}(h \circ W) T_h$ for all $h \in G'$ and $W \in Bf(G)$.

For each $F \in L^\infty(G, \mu, \mathbb{C})$ let α_F be the operator in $L(L^2(G, \mu, \mathbb{C}))$ consisting of multiplication by F :

$$\alpha_F(f) = Ff \text{ for each } f \in L^2(G, \mu, \mathbb{C}),$$

where $L(Z) := L(Z \rightarrow Z)$ (see §3.1). The map $F \mapsto \alpha_F$ is an isometric $*$ -isomorphism of $L^\infty(G, \mu, \mathbb{C})$ into $L(L^2(G, \mu, \mathbb{C}))$ (see §VIII.19.2[8]). Therefore, using the approach of this particular case given above we get, that Propositions VIII.19.2,5[8] are applicable in our situation.

If \bar{p} is a projection onto a closed H^μ -stable subspace of $L^2(G, \mu, \mathbb{C})$, then due to Formulas (20-22) \bar{p} commutes with all $\bar{P}(W)$. Hence \bar{p} commutes with α_F for each $F \in L^\infty(G, \mu, \mathbb{C})$, so by §VIII.19.2 [8] $\bar{p} = \bar{P}(V)$, where $V \in Bf(G)$. Also \bar{p} commutes with T_h for each $h \in G'$, consequently, $(h \circ V) \setminus V$ and $(h^{-1} \circ V) \setminus V$ are μ -null for each $h \in G'$, hence $\mu((h \circ V) \Delta V) = 0$

for all $h \in G'$. In view of the ergodicity of μ and Proposition VIII.19.5 [8] either $\mu(V) = 0$ or $\mu(G \setminus V) = 0$, hence either $\bar{p} = 0$ or $\bar{p} = I$.

3.3. Theorem. *On the loop group $G = L_\xi(M, N)$ from §2.1 there exists a family of continuous characters $\{\Xi\}$, which separate points of G .*

Proof. In view of Lemma I.2.17 it is sufficient to consider the case of the submanifold \tilde{M} having no more than two charts. Then \tilde{M} is clopen in $c_0(\alpha, K)$, where $\tilde{M} = \tilde{M} \setminus \{s_0\}$.

Let at first $\dim_K M < \aleph_0$. The Haar measure $\lambda_\alpha : Bf(K^\alpha) \rightarrow \mathbb{Q}_q$ with a prime number $q \neq p$ (see the Monna-Springer theorem in §8.4 [16]) induces the measure $\lambda_\alpha : Bf(\tilde{M}) \rightarrow \mathbb{Q}_q$, analogously for

$$(1) N_J := N \cap sp_K\{e_j : j \in J\}$$

for each $N \ni n \geq \alpha$ and $h \in L(N_J, \lambda_n, \mathbb{Q}_q)$ there corresponds a measure $\nu_{J,h}$ on $Bf(N_J)$ for which

$$(2) \nu_{J,h}(dy) = h(y)\lambda_n(dy)$$

and to $\nu_{J,h}$ there corresponds a differential form

$$(3) w_{J,h}(y) = h(y)dy^{j_1} \wedge \dots \wedge dy^{j_n},$$

where $y \in N_J$ and $J := \{j_1, \dots, j_n\}$. Hence there exists its pull back $(\pi_J \tilde{f})^* w$, where $\pi_J : c_0(\beta, K) \rightarrow sp_K\{e_j : j \in J\}$ is the projection for each $J \subset \beta$, $f \in C_0^0(\xi, \tilde{M} \rightarrow N)$, $\tilde{f} = P(l, s+1)f$, $l = [t] + 1$ (see §1.2.11 and Corollary I.2.16).

As usually, for a mapping $h : \tilde{M} \rightarrow N_J$ of class $C(1)$ and a tensor T of the type $(0, k)$ with components T_{i_1, \dots, i_k} defined for N_J we have:

$$(4) (h^*T)_{i_1, \dots, i_k}(x^1, \dots, x^\alpha) = \left[\sum_{i_1, \dots, i_k} T_{i_1, \dots, i_k} (\partial y^{i_1} / \partial x^{l_1}) \dots (\partial y^{i_k} / \partial x^{l_k}) \right] (y(x^1, \dots, x^\alpha))$$

such that h^*T is defined for \tilde{M} , where (x^1, \dots, x^α) are coordinates in \tilde{M} induced from K^α and $(y^1, \dots, y^n) = y$ are coordinates in N_J induced from K^n , $y^j = y^j(x^1, \dots, x^\alpha) = h^j(x^1, \dots, x^\alpha)$, x^j and $y^j \in K$.

Let now $\dim_K M = \dim_K N = \aleph_0$. Let λ be equivalent with a probability \mathbb{Q}_q -valued measure either on the entire $T_y N$ or on its Banach infinite-dimensional over K subspace P (see Formulas I.3.6.(13-20)). Each such λ

induces a family of probability measures ν on $Bf(N)$ or its cylinder subalgebra induced by the projection of $T_y N$ onto P , which may differ by their supports.

Let $T_y N =: L$ be an infinite-dimensional separable Banach space over K , so there exists a topological vector space $L^N := \prod_{j=1}^{\infty} L_j$, where $L_j = L$ for each $j \in N$ [15]. Consider a subspace Λ^{∞} of a space of continuous ∞ -multilinear functionals $\eta : L^N \rightarrow K$ such that

$$\eta(x+y) = \eta(x) + \eta(y), \quad \eta(\sigma x) = (-1)^{|\sigma|} \eta(x) \text{ and } \eta(x) = \lambda \eta(z)$$

for each $x, y \in L^N$, $\sigma \in S_{\infty}$ and $\lambda \in K$, where

$$x = \{x^j : x^j \in L, j \in N\} \in L^N, \quad z^j = x^j \text{ for each } j \neq k_0 \text{ and } \lambda z^{k_0} = x^{k_0},$$

S_{∞} is a group of all bijections $\sigma : N \rightarrow N$ such that $\text{card}\{j : \sigma(j) \neq j\} < \aleph_0$, $|\sigma| = 1$ for $\sigma = \sigma_1 \dots \sigma_n$ with odd $n \in N$ and pairwise transpositions $\sigma_l \neq I$, that is,

$$\sigma_l(j_1) = j_2, \quad \sigma_l(j_2) = j_1 \text{ and } \sigma_l|_{N \setminus \{j_1, j_2\}} = I$$

for the corresponding $j_1 \neq j_2$, $|\sigma| = 2$ for even n or $\sigma = I$. Then Λ^{∞} (or Λ^j) induces a vector bundle $\Lambda^{\infty} N$ (or $\Lambda^j N$) on a manifold N of ∞ -multilinear skew-symmetric mappings over $F(N)$ of $\Psi(N)^{\infty}$ (or $\Psi(N)^j$ respectively) into $F(N)$, where $\Psi(N)$ is a set of differentiable vector fields on N and $F(N)$ is an algebra of K -valued C^1 -functions on N . This $\Lambda^{\infty} N$ is the vector bundle of differential ∞ -forms on N . Then there exist a subfamily $\Lambda_C^{\infty} N$ of differential forms w on N induced by the family $\{\nu\}$.

Let $\Lambda^j N$ be the space of differential j -forms w on N such that $w = \sum_{|J|=j} w_J dx^J$, where $dx^J = dx^{j_1} \wedge \dots \wedge dx^{j_n}$ for a multi-index $J = (j_1, \dots, j_n)$, $n \in N$, $|J| = j_1 + \dots + j_n$, $0 \leq j_i \in Z$, $w_J : N \rightarrow K$ are C^{∞} -mappings, $B^k N := \bigoplus_{j=0}^k \Lambda^j N$. Here the manifold $B^k N$ is considered to be of classes of smoothness C^{∞} .

Let $\bar{B}^{\infty} N := (\bigoplus_{0 \leq j \in Z} \Lambda^j N) \oplus \Lambda_C^{\infty} N$ for $\dim_K N = \infty$ and $\bar{B}^k N = \bigoplus_{j=0}^k \Lambda^j N$ for each $k \in N$. We choose $w \in \bar{B}^k N$, where $k = \min(\dim_K N, \dim_K M)$. There exists its pull back $\tilde{f}_*^* w$ for each $f \in C_0(\xi, M \rightarrow N)$ (see for comparison the classical case in §§1.3.10, 1.4.8 and 1.4.15 in [11] and the non-Archimedean case in [3]), where

$$\tilde{f}_*^* := \sum_{a=1}^{\infty} \kappa_a \{A_a(f|_{M_a}) - A_{a-1}(f|_{M_{a-1}})\},$$

$|\kappa_a| \times \|A_a\| \leq 1$ and $\kappa_a \in K$ for each $a \in N$, $A_0 := 0$ (see Formula I.3.6.(1)). This series is correctly defined and converges due to Lemma I.2.4.2 and Formulas I.2.4.3.b.(1-4). When $f \neq 0$ there exists $\kappa := \{\kappa_a : a \in N\}$ such that $\tilde{f}_\kappa \neq 0$. Let $E_j : S_j \rightarrow P$ be a family of continuous linear operators from Banach spaces S_j into a Banach space P , then there exists a continuous linear operator

$E : c_0(\{S_j : j \in N\}) \rightarrow P$ such that

$$Ex = \sum_{j=1}^{\infty} E_j x^j,$$

where $x = \{x^j : x^j \in S_j, j \in N\} \in c_0(\{S_j : j \in N\})$. We take $w \in C_0(\infty, \tilde{M} \rightarrow B^k N)$, when $\dim_K M \leq \dim_K N$. When $N_0 > \dim_K M > \dim_K N$ we take $w \in C_0(\infty, \tilde{M} \rightarrow B^k(N^m))$, where $N^m = N_1 \times \dots \times N_m$ with $N_j = N$ for each $j = 1, \dots, m$ such that $N \ni m \geq \dim_K M / \dim_K N$. A mapping $F \in C_0(t, \tilde{M} \rightarrow N)$ generates a mapping $F^{\otimes m} := (F, \dots, F) : \tilde{M} \rightarrow N^m$ and the pull back $(F^{\otimes m})^*$ which is also denoted simply by F^* , where F^*w is a $C_0(t-1)$ -mapping, when $1 \leq t \in \mathbb{R}$, (F, \dots, F) is an m -tuple. When $N_0 = \dim_K M > \dim_K N$ we take instead of N or N^m a submanifold \tilde{N} of $N^\infty := \bigotimes_{j=1}^{\infty} N_j$ modelled on $c_0(\{S_j : j \in N\})$, where $S_j = T_y N$ for each j , that is, in accordance with our notation $\tilde{N} := c_0(N_j : j \in N)$. Therefore, there exists a pull back \tilde{f}^*w for ν and w either on N^s or on \tilde{N} instead of N in the corresponding cases of $\dim_K M$ and $\dim_K N$.

Moreover, to $(\pi_J \tilde{f}_\kappa)^*w$ a \mathbb{Q}_q -valued measure μ_w on \tilde{M} corresponds, since ν is the \mathbb{Q}_q -valued measure. When $\dim_K M < N_0$ we take \tilde{f} instead of \tilde{f}_κ . Then there exists a \mathbb{Q}_q -valued functional:

$$(5) \quad F_{J,w,\kappa}(f) := \int_{\tilde{M}} (\pi_J \tilde{f}_\kappa)^*w = \int_{\tilde{M}} (\pi_J \tilde{f}_\kappa \circ \psi)^*w$$

for each $f \in C_0^0(\xi, (\tilde{M}, s_0) \rightarrow (\tilde{N}, y_0))$ and $\psi \in G_0(\xi, \tilde{M})$, consequently, $F_{J,w,\kappa}$ is continuous and constant on each class $\langle f \rangle_{K,\xi}$, where either $\tilde{N} = N$ or $\tilde{N} = N^m$ or $\tilde{N} = \tilde{N}$ in the corresponding cases. If h is not locally constant then h^* is not zero operator, hence the family $\{F_{J,w,\kappa} : J, w, \kappa\}$ separates points in the loop semigroup, where κ is omitted in the case $\dim_K M < N_0$.

Let $\tilde{\Xi}_y : \mathbb{Q}_q \rightarrow S^1$ be a continuous character of \mathbb{Q}_q as the additive group (see §25.1 [10]), where $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle, x and

$y \in \mathbb{Q}_q$,

$$(6) \tilde{\Xi}_y(x) = \exp[2\pi i(\sum_{n=-\infty}^{\infty} (\sum_{s=n}^{\infty} y_{-s} q^{(n-s-1)}))],$$

$x = \sum_{n=-\infty}^{\infty} x_n q^n$, $x_n \in \{0, 1, \dots, q-1\}$. For a given x and y this sum in $[\ast]$ is finite, where y is fixed. In view of Formulas (1-6)

$$\Xi(g) := \tilde{\Xi}\left(\begin{pmatrix} + \\ - \end{pmatrix} F_{J,w,\kappa}(f)\right)$$

is a continuous character on $L_\xi(M, N) = L_\xi(\tilde{M}, N)$, where $F_{J,w,\kappa}(f)$ [or $-F_{J,w,\kappa}(f)$] corresponds to g [or $-g$ respectively], for g being the image of $\langle f \rangle_{K,\xi}$ relative to the embedding

$$\gamma : \Omega_\xi(\tilde{M}, N) \hookrightarrow L_\xi(\tilde{M}, N)$$

(see also §2.2).

3.4. Note. The loop groups and semigroups were considered above for analytic manifolds with disjoint clopen charts. Each metrizable manifold M on a Banach space X over a local field K is a disjoint union of clopen subsets diffeomorphic with balls in X , since the value group $\Gamma_K := \{|x|_K : 0 \neq x \in K\}$ is discrete in $(0, \infty)$ (see [14] and Lemma 7.3.6 [6]).

Suppose now that a new atlas $At'(M)$ is with open charts (U'_j, ϕ'_j) such that there are $U_j \cap U'_i \neq \emptyset$ for some $i \neq j$. Using spaces $C_0(\xi, \phi'_j(U'_j) \rightarrow Y)$ we can define $C_0(\xi, M \rightarrow N)$ correctly only if connecting mappings $\phi_i \circ \phi'^{-1}_j$ on $\phi'_j(U'_j \cap U'_i)$ are of class of smoothness not less than $C_0(\xi)$ for each $i \neq j$ with $U'_j \cap U'_i \neq \emptyset$. Here the atlases $At'(M)$ and $At'(N)$ need not be disjoint. The same condition need to be imposed on $\psi'_i \circ \psi'^{-1}_j$ for each $V'_j \cap V'_i \neq \emptyset$ for a new atlas $At'(N)$ of N with open charts (V'_j, ψ'_j) . This is also necessary for the definition of $G(\xi, M)$. Let $\phi : M \rightarrow M'$ be a diffeomorphism for $1 \leq \xi = t$ or $\xi = (t, s)$ with $0 \leq t$ and $1 \leq s$ (a homeomorphism for $0 \leq \xi = t < 1$) of class not less than $C_0(\xi)$ of two manifolds (may be one set with two different atlases), then $G(\xi, M)$ and $G(\xi, M')$ are diffeomorphic (or homeomorphic) topological groups with the diffeomorphism (the homeomorphism respectively)

$$g \mapsto \phi \circ g \circ \phi^{-1},$$

since $G(\xi, M)$ have a Banach manifold structure for $1 \leq t$ or $1 \leq s$, where $g \in G(\xi, M)$. If $\psi : N \rightarrow N'$ is a diffeomorphism (homeomorphism) of class

at least $C_0(\xi)$, then $C_0(\xi, M \rightarrow N)$ and $C_0(\xi, M' \rightarrow N')$ are diffeomorphic (homeomorphic) due to the following map

$$g \mapsto \psi \circ g \circ \phi^{-1},$$

where $g \in C_0(\xi, M \rightarrow N)$. If $\{f_n\}$ and $\{g_n\}$ are sequences in $C_0(\xi, (M, s_0) \rightarrow (N, y_0))$ converging to f and g respectively, $\{\eta_n\}$ is a sequence in $G_0(\xi, M)$ such that $g_n = f_n \circ \eta_n$ for each $n \in \mathbb{N}$, then

$$\psi \circ f_n \circ \phi^{-1} \circ \phi \circ \eta_n \circ \phi^{-1} = \psi \circ g_n \circ \phi^{-1}.$$

This gives a bijective correspondence between classes $\langle g \rangle_{K, \dagger}$ and $\langle \tilde{g} \rangle_{K, \dagger}$ in $C_0(\xi, (M, s_0) \rightarrow (N, y_0))$ and $C_0(\xi, (M', s'_0) \rightarrow (N', y'_0))$ respectively, where

$$\tilde{g} = \psi \circ g \circ \phi^{-1} \in C_0(\xi, (M', s'_0) \rightarrow (N', y'_0)),$$

$s'_0 = \phi(s_0)$, $y'_0 = \psi(y_0)$. Therefore, $\Omega_\xi(M, N)$ and $\Omega_\xi(M', N')$ are diffeomorphic (homeomorphic respectively) topological semigroups, consequently, $L_\xi(M, N)$ and $L_\xi(M', N')$ are diffeomorphic (homeomorphic) topological groups due to Theorems I.2.7, I.2.10, 2.3 and Proposition 2.2. This means independence of these semigroups and groups relative to a choice of equivalent atlases of manifolds.

4 Path groups.

4.1. Definition and Note. In view of Equations I.2.9.(1-3) each space N^ξ has the additive group structure, when $N = B(Y, 0, R)$, $0 < R \leq \infty$.

Therefore, the factorization by the equivalence relation $K_\xi \times id$ produce the monoid of paths $C_0^g(\xi, \bar{M} \rightarrow N)/(K_\xi \times id) =: S_\xi(M, N)$ in which compositions are defined not for all elements, where $y_1 id y_2$ if and only if $y_1 = y_2 \in N$. There exists a composition $f_1 f_2 = (g_1 g_2, y)$ if and only if $y_1 = y_2 = y$, where $f_i = (g_i, y_i)$, $g_i \in \Omega_\xi(M, N)$ and $y_i \in N^\xi$, $i \in \{1, 2\}$. The latter semigroup has elements e_y such that $f = e_y \circ f = f \circ e_y$ for each f , when their composition is defined, where $y \in N^\xi$, $f = (g, y)$, $g \in \Omega_\xi(M, N)$, $e_y = (e, y)$. If N^ξ is a monoid, then $S_\xi(M, N)$ can be supplied with the structure of a direct product of two monoids. Therefore, $P_\xi(M, N) := L_\xi(M, N) \times N^\xi$ is called the path group.

4.2. Theorem. *On the monoid $G = S_\xi(M, N)$ from §4.1, when $N = B(Y, 0, R)$ and N^ξ is supplied with the additive group structure, and each $b \in \mathbb{C}$ there are probability quasi-invariant and pseudo-differentiable of order b measures μ with values in \mathbb{R} and \mathbb{Q}_q for each prime number $q \neq p$ relative to a dense submonoid G' .*

Proof. In view of Formulas 2.9.(1-3) there is the following isomorphism $S_\xi(M, N) = \Omega_\xi(M, N) \times N^\xi$. Hence it is sufficient to construct $\mu = \mu_1 \times \mu_2$, where μ_2 is a quasi-invariant and pseudo-differentiable measure on N^ξ and μ_1 on $\Omega_\xi(M, N)$, since μ_1 was constructed in Theorem 1.3.6. The desired measure μ_2 on N^ξ exists due to Theorems 3.23, 3.27 and 4.3 [13].

4.3. Theorem. *On the path group $G = P_\xi(M, N)$ from §4.1, when $N = B(Y, 0, R)$ and N^ξ is supplied with the additive group structure, and each $b \in \mathbb{C}$ there are probability quasi-invariant and pseudo-differentiable of order b measures μ with values in \mathbb{R} and \mathbb{Q}_q for each prime number $q \neq p$ relative to a dense subgroup G' .*

Proof. Since $P_\xi(M, N) = L_\xi(M, N) \times N^\xi$, it is sufficient to construct $\mu = \mu_1 \times \mu_2$, where μ_2 is a quasi-invariant and pseudo-differentiable measure on N^ξ and μ_1 on $L_\xi(M, N)$, since μ_1 was constructed in Theorem 2.5 and μ_2 in §4.2.

4.4. Remark. Loop and path groups can be defined also for manifolds modelled on locally K -convex spaces.

In general for locally K -convex spaces X and Y a mapping $F : U \rightarrow Y$ is called of class $C(t)$ if the partial difference quotient $\Phi^v F$ has a bounded continuous extension $\bar{\Phi}^v F : U \times V^s \times S^s \rightarrow Y_{\Lambda_p}$ for each $0 \leq v \leq t$ and each derivative $F^{(k)}(x) : X^k \rightarrow Y$ is a continuous k -linear operator for each $x \in U$ and $0 < k \leq [t]$, where U and V are open neighbourhoods of 0 in X , $U + V \subset U$, $k \in \mathbb{N}_0$, Y_{Λ_p} is a locally Λ_p -convex space obtained from Y by extension of a scalar field from K to Λ_p , $s = [v] + \text{sign}\{v\}$. If F is of class $C(n)$ for each $n \in \mathbb{N}$ then it is called of class $C(\infty)$.

For $C(m)$ -manifolds M and N modelled on locally K -convex spaces X and Y with atlases $At(M) = \{(U_i, \phi_i) : i \in \Lambda_M\}$ and $At(N) = \{(V_i, \psi_i) : i \in \Lambda_N\}$ a mapping $F : M \rightarrow N$ is called of class $C(n)$ if $F_{i,j}$ are of class $C(n)$ for each i and j , where $F_{i,j} = \psi_i \circ F \circ \phi_j^{-1}$, $\phi_i \circ \phi_j^{-1}$ and $\psi_i \circ \psi_j^{-1}$ are of class $C(m)$, $\infty \geq m \geq n \geq 0$.

Then quite analogously to §1.2.6 and §2.1 loop and path semigroups and groups can be defined. For the construction of quasi-invariant measures in addition there can be used closed subspaces S of separable type over

K in dual spaces to nuclear locally K -convex spaces. From such spaces S quasi-invariant measures can be induced on containing them locally K -convex spaces Z with the help of the standard procedure based on algebras of cylindrical subsets with the subsequent extension onto the Borel σ -field. Then measures on groups can be constructed analogously to the considered above cases. If a group G is non-separable, then a non-zero Borel measure μ may be quasi-invariant relative to a subgroup G' which is not dense in G . Nevertheless, with the help of μ a regular representation of G' associated with μ can be induced.

5 Quasi-invariant measures on O -groups.

5.1. Definition. The space $C_0^0(\xi, (M, s_0) \rightarrow (N, y_0))$ is not a semigroup itself, but compositions are defined for the families $\langle f \rangle_{K_\xi}$, that is, relative to the equivalence relation K_ξ . Henceforth, let the topology of $\Omega_\xi(M, N)$ be defined relative to countable $At(M)$ as in §I.2.5 and §I.2.6. If F is the free Abelian group corresponding to $\Omega_\xi(M, N)$ from §2.1, then there exists a set \bar{W} generated by formal finite linear combinations over \mathbb{Z} of elements from $C_0^0(\xi, (M, s_0) \rightarrow (N, y_0))$ and a continuous extension \bar{K}_ξ of K_ξ onto $W_\xi(M, N)$ and a subset \bar{B} of \bar{W} generated by elements $[f + g] - [f] - [g]$ such that $W_\xi(M, N)/\bar{K}_\xi$ is isomorphic with $L_\xi(M, N)$, where

$$W_\xi(M, N) := \bar{W}/\bar{B},$$

f and $g \in C_0^0(\xi, (M, s_0) \rightarrow (N, y_0))$, $[f]$ is an element in \bar{W} corresponding to f , \bar{W} is in a topology inherited from the space $C_0^0(\xi, (M, s_0) \rightarrow (N, y_0))^{\mathbb{Z}}$ in the Tychonoff product topology. We call $W_\xi(M, N)$ an O -group. Clearly the composition in $C_0^0(\xi, (M, s_0) \rightarrow (N, y_0))$ induces the composition in $W_\xi(M, N)$. Then $W_\xi(M, N)$ is not the algebraic group, but associative compositions are defined for its elements due to the homomorphism χ^* given by Formulas 2.6.2.(5,6), hence $W_\xi(M, N)$ is the monoid without the unit element.

Let $\mu_h(A) := \mu(h \circ A)$ for each $A \in Bf(W_\xi(M, N))$ and $h \in W_\xi(M, N)$, then as in §§I.3.3 and I.3.4 we get the definition of quasi-invariant and pseudo-differentiable measures.

Let now $G^c := W_\xi^{(k)}(M, N)$ be generated by $C_{0, \{k\}}^0(\xi, (M, s_0) \rightarrow (N, 0))$ as in §I.3.5, then it is the dense O -subgroup in $W_\xi(M, N)$, where $c > 0$ and

$\epsilon' > 0$.

5.2. Theorem. *Let $G := W_\xi(M, N)$ be the O -group as in §5.1 and $At(M)$ be finite. Then there exist quasi-invariant and pseudo-differentiable measures μ on G with values in $[0, \infty)$ and in \mathbb{Q}_q (for each prime number q such that $q \neq p$) relative to a dense O -subgroup G' .*

Proof. In view of the definition of the space $C_0^0(\xi, M \rightarrow Y)$ the mapping \tilde{A} given by Formula I.3.6.(3) for $At(M)$ instead of $At'(M)$ is the isomorphism of $T_0 C_0^0(\xi, (M, s_0) \rightarrow (N, 0))$ onto the Banach subspace of \tilde{Z} for $\xi = (t, s)$, since $At(M)$ is finite and $\phi_j(U_j)$ are bounded in X (see §I.2.4.1). In view of the existence of the mapping $w_{\text{sep}}(V)$ given by Formulas I.2.8.(3,4) there exists the local diffeomorphism $\Upsilon : W_\epsilon \rightarrow V'_0$ induced by \tilde{A} and \tilde{K}_ξ , where W_ϵ is a neighbourhood of 0 in $W_\xi(M, N)$, V'_0 is a neighbourhood of zero either in the Banach subspace \tilde{H} of $T_0 W_{\xi'}(M, Y)$ for $\dim_K M < \infty$ or in the Banach subspace \tilde{H} of $c_0(\{T_0 W_{\xi'}(M_a, Y) : a \in N\})$ for $\dim_K M = \aleph_0$.

Let now W'_ϵ be a neighbourhood of 0 in G' such that $W'_\epsilon W_\epsilon = W_\epsilon$. It is possible, since the topology in G and G' is given by the corresponding ultrametrics and there exists W_ϵ with $W_\epsilon W_\epsilon = W_\epsilon$, hence it is sufficient to take $W'_\epsilon \subset W_\epsilon$. For $g \in W_\epsilon$, $v = w_{\text{sep}}^{-1}(g)$, $\phi \in W'_\epsilon$ the following operator $S_\phi(v) := \Upsilon \circ L_\phi \circ \Upsilon^{-1}(v) - v$ is defined for each $(\phi, v) \in W'_\epsilon \times V'_0$, where $L_\phi(g) := \phi \circ g$. Then $S_\phi(v) \in V''_0 \subset V'_0$, where V''_0 is an open neighbourhood of the zero section either in the Banach subspace \tilde{H}' of $T_\epsilon G'$ for $\dim_K M < \infty$ or in the Banach subspace \tilde{H}' of $c_0(\{T_\epsilon G'_a : a \in N\})$ for $\dim_K M = \aleph_0$, where $G'_a = W_\xi^{(h)}(M_a, N)$. Moreover, $S_\phi(v)$ is the $C(\infty)$ -mapping by ϕ and v . The rest of the proof is quite analogous to that of Theorem I.3.6.

5.3. Note. O -groups can be defined in another topology with the help of $c_0(\{H_j : j \in N\})$, where $H_j := C_0(\xi; U_j \rightarrow Y)$. Then on such O -groups quasi-invariant and pseudo-differentiable measures can be constructed quite analogously.

6 Notation.

K is a local field; $N := \{1, 2, 3, \dots\}$; $N_0 := \{0, 1, 2, \dots\}$;

$B(X, x, r)$ and $B(X, x, r^-)$ are balls §I.2.2;

\tilde{Q}_m are polynomials §I.2.2;

$X = c_0(\alpha, K)$, $Y = c_0(\beta, K)$, $\{e_i : i \in \alpha\}$ and $\{q_i : i \in \beta\}$ are orthonormal bases in Banach spaces X and Y ; M and N are manifolds on X and Y

respectively §I.2.4;

$At(M) = \{(U_j, \phi_j) : j \in \Lambda_M\}$ and $AT(N) = \{(V_k, \psi_k) : k \in \Lambda_N\}$ are atlases §I.2.4;

$C(t, M \rightarrow Y)$ and $C_0(t, M \rightarrow Y)$ are spaces, $\|f\|_{C(t, M \rightarrow Y)} = \|f\|_t$ and $\|f\|_{C_0(t, M \rightarrow Y)}$ are norms §I.2.4;

$\rho^\xi(f, g)$ and $\rho_0^\xi(f, g)$ are ultrametrics in $C^\theta(\xi, M \rightarrow N)$ and $C_0^\theta(\xi, M \rightarrow N)$ respectively, $\xi = t$ or $\xi = (t, s)$, for $s > 0$ the manifold M is locally compact, for $s = 0$ the manifold M may be non-locally compact §I.2.4.3;

$Hom(M)$ is a homeomorphism group §I.2.4.4;

$G(\xi, M)$ and $Diff(\xi, M)$ are diffeomorphism groups §I.2.4.4;

$M = \bar{M} \setminus \{0\}$, $\bar{M} \hookrightarrow c_0(\omega_0, K)$, $At'(\bar{M}) = \{(\bar{U}_j, \bar{\phi}_j) : j \in \Lambda'_M\}$, $s_0 = 0$ and $y_0 = 0$ are marked points of \bar{M} and N respectively §I.2.5;

$\chi : M \vee M \rightarrow M$ is a mapping §I.2.6;

$G_0(\xi, M)$ is a subgroup and $C_0(\xi, (M, s_0) \rightarrow (N, y_0))$ is a subspace preserving marked points, K_ξ is an equivalence relation, $\langle f \rangle_{K_\xi}$ is a class of equivalent elements §I.2.6;

$\Omega_\xi(M, N)$ is a loop semigroup §I.2.6;

$P(l, s)$ is an antiderivation §I.2.11;

$Bf(X')$, $Af(X', \mu)$ and $Bco(X')$ are algebras of subsets of X' , N_μ is a function §I.3.1;

$\rho_\mu(h, g)$ is a quasi-invariance factor §I.3.3;

$S_\xi(M, N)$ is a path semigroup §II.4.1;

$L_\xi(M, N)$ is a loop group §II.2.1;

$P_\xi(M, N)$ is a path group §II.4.1;

$W_\xi(M, N)$ is an O -group §II.5.1.

References

- [1] W. Banaszczyk. "Additive subgroups of topological vector spaces" (Berlin: Springer, 1991).
- [2] N. Bourbaki. "Intégration". Livre VI. Fasc. XIII, XXI, XXIX, XXXV. Ch. 1-9 (Paris: Hermann; 1965, 1967, 1963, 1969).
- [3] N. Bourbaki. "Variétés différentielles et analytiques". Fasc. XXXIII (Paris: Hermann, 1967).

- [4] Yu.L. Dalecky, S.V. Fomin. "Measures and differential equations in infinite-dimensional space" (Dordrecht, The Netherlands: Kluwer, 1991).
- [5] N. Dunford, J.T. Schwartz. "Linear operators" (New York: Interscience Publ., V. 1, 1958; V. 2, 1963).
- [6] R. Engelking. "General topology". Second Edit., Sigma Ser. in Pure Math. V. 6 (Berlin: Heldermann Verlag, 1989).
- [7] H. Federer. "Geometric measure theory" (Berlin: Springer, 1968).
- [8] J.M.G. Fell, R.S. Doran. "Representations of $*$ -algebras, locally compact groups, and Banach $*$ -algebraic bundles". V. 1 and V. 2 (Boston.: Acad. Press, 1988).
- [9] I.M. Gel'fand, N.Ja. Vilenkin. "Generalized functions. V. 4. Applications of harmonic analysis" (New York: Academic Press, 1964).
- [10] E. Hewitt, K.A. Ross. "Abstract harmonic analysis" (Berlin: Springer, 1979).
- [11] W. Klingenberg. "Riemannian geometry" (Berlin: Walter de Gruyter, 1982).
- [12] S. Lang. "Algebra" (New York: Addison-Wesley Pub. Com, 1965).
- [13] S.V. Ludkovsky. "Quasi-invariant and pseudo-differentiable measures on a non-Archimedean Banach space". Int. Cent. Theor. Phys., Trieste, Italy, Preprint N^o IC/96/210, 1996.
- [14] S.V. Ludkovsky. "Embeddings of non-Archimedean Banach manifolds into non-Archimedean Banach spaces". Russ. Math. Surv. 53 (1998), 1097-1098.
- [15] L. Narici, E. Beckenstein. "Topological vector spaces" (New York: Marcel Dekker Inc., 1985).
- [16] A.C.M. van Rooij. "Non-Archimedean functional analysis" (New York: Marcel Dekker Inc., 1978).

- [17] R.C. Swan. "The Grothendieck ring of a finite group". *Topology*. **2** (1963), 85-110.
- [18] R.M. Switzer. "Algebraic topology - homotopy and homology" (Berlin: Springer, 1975).