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**SOME MORE STEINHAUS TYPE THEOREMS
OVER VALUED FIELDS**

by P.N. Natarajan

1. Preliminaries :

Throughout this paper, K denotes \mathbb{R} (the field of real numbers) or \mathbb{C} (the field of complex numbers) or a complete, non-trivially valued, non-archimedean field. In the relevant context, we mention explicitly which field is chosen. Entries of infinite matrices and sequences, which occur in the sequel, are in K . If X, Y are sequence spaces over K , by (X, Y) we mean the class of all infinite matrices $A = (a_{nk})$, $n, k = 0, 1, 2, \dots$ such that $Ax = \{(Ax)_n\} \in Y$ whenever $x = \{x_k\} \in X$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \quad n = 0, 1, 2, \dots,$$

it being assumed that the series on the right converges. Whenever there is some notion of limit or sum in X, Y , we denote by $(X, Y; P)$ that subclass of (X, Y) consisting of infinite matrices which preserve this limit or sum. Whatever be K , the sequence spaces $\ell, \gamma, c_0, c, \ell_{\infty}, \gamma_{\infty}$ are defined as :

$$\begin{aligned}
 \ell &= \left\{ \{x_k\} : \sum_{k=0}^{\infty} |x_k| \text{ converges} \right\}; \\
 \gamma &= \left\{ \{x_k\} : \sum_{k=0}^{\infty} x_k \text{ converges} \right\}; \\
 c_0 &= \left\{ \{x_k\} : \lim_{k \rightarrow \infty} x_k = 0 \right\}; \\
 c &= \left\{ \{x_k\} : \lim_{k \rightarrow \infty} x_k \text{ exists} \right\}; \\
 \ell_{\infty} &= \left\{ \{x_k\} : \sup_{k \geq 0} |x_k| < \infty \right\}; \\
 \gamma_{\infty} &= \left\{ \{x_k\} : \{s_k\} \in \ell_{\infty}, s_k = \sum_{i=0}^k x_i, k = 0, 1, 2, \dots \right\}.
 \end{aligned}$$

We note that $\ell \subset \gamma \subset c_0 \subset c \subset \ell_{\infty}$ and $\gamma_{\infty} \subset \ell_{\infty}$.

$(\ell, \gamma; P)$ denotes the class of all infinite matrices $A = (a_{nk})$ in (ℓ, γ) such that $\sum_{n=0}^{\infty} (Ax)_n =$

$$\sum_{k=0}^{\infty} x_k, x = \{x_k\} \in \ell.$$

2. The case $K = \mathbb{R}$ or \mathbb{C}

When $K = \mathbb{R}$ or \mathbb{C} , the following result is well-known (see [6], 48, p.7).

Theorem 2.1 *A matrix $A = (a_{nk})$ is in (ℓ, γ) if and only if*

$$\sup_{m,k} \left| \sum_{n=0}^m a_{nk} \right| < \infty ; \tag{1}$$

and

$$\sum_{n=0}^{\infty} a_{nk} \text{ converges}, k = 0, 1, 2, \dots \tag{2}$$

We now prove the following result when $K = \mathbb{R}$ or \mathbb{C} .

Theorem 2.2 *A matrix $A = (a_{nk})$ is in $(\ell, \gamma; P)$ if and only if it satisfies (1) and*

$$\sum_{n=0}^{\infty} a_{nk} = 1, k = 0, 1, 2, \dots \tag{3}$$

Proof. If A is in $(\ell, \gamma; P)$ then (1) holds. For $k = 0, 1, 2, \dots$, each $e_k = \{0, \dots, 0, 1, 0, \dots\}$, (1 occurring at the k th place), lies in ℓ and so $\sum_{n=0}^{\infty} (Ae_k)_n = 1$, i.e. , $\sum_{n=0}^{\infty} a_{nk} = 1$, $k = 0, 1, 2, \dots$. i.e., (3) holds.

Conversely, let (1) and (3) hold. It follows that A is in (ℓ, γ) in view of Theorem 2.1. Let $B = (b_{mk})$ where

$$b_{mk} = \sum_{n=0}^m a_{nk}, \quad m, k = 0, 1, 2, \dots .$$

Using (1) and (3), we have

$$\sup_{m,k} |b_{mk}| < \infty ; \tag{4}$$

and

$$\lim_{m \rightarrow \infty} b_{mk} = 1, \quad k = 0, 1, 2, \dots . \tag{5}$$

Thus B is in $(\ell, c; P')$ (see [5]). Let, now , $\{x_k\} \in \ell$. So

$$\lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} b_{mk} x_k \text{ exists and is equal to } \sum_{k=0}^{\infty} x_k,$$

$$\text{i.e., } \lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} \left(\sum_{n=0}^m a_{nk} \right) x_k = \sum_{k=0}^{\infty} x_k.$$

$$\text{i.e., } \lim_{m \rightarrow \infty} \sum_{n=0}^m \left(\sum_{k=0}^{\infty} a_{nk} x_k \right) = \sum_{k=0}^{\infty} x_k,$$

$$\text{i.e., } \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_{nk} x_k \right) = \sum_{k=0}^{\infty} x_k,$$

$$\text{i.e., } \sum_{n=0}^{\infty} (Ax)_n = \sum_{k=0}^{\infty} x_k.$$

In other words, A is in $(\ell, \gamma; P)$, which completes the proof of the theorem.

Maddox [3] proved that $(\gamma, \gamma; P) \cap (\gamma_{\infty}, \gamma) = \emptyset$. In this context, it is worthwhile to note that the identity matrix (i.e., $I = (i_{nk})$ where $i_{nk} = 1$, if $k = n$ and $i_{nk} = 0$, if $k \neq n$) is in $(\ell, \gamma; P) \cap (\gamma_{\infty}, \gamma)$ so that $(\ell, \gamma; P) \cap (\gamma_{\infty}, \gamma) \neq \emptyset$. Since $(\gamma, \gamma) \supset (\gamma_{\infty}, \gamma)$, it follows that $(\ell, \gamma; P) \cap (\gamma, \gamma) \neq \emptyset$. We note that $(\gamma, \gamma; P) \subset (\ell, \gamma; P)$ and $(c_0, \gamma) \subset (\gamma, \gamma)$. Having

“enlarged” the class $(\gamma, \gamma; P)$ to $(\ell, \gamma; P)$, we would like to “contract” the class (γ, γ) to (c_0, γ) and attempt a Steinhaus type theorem involving the classes $(\ell, \gamma; P)$ and (c_0, γ) .

Theorem 2.3 $(\ell, \gamma; P) \cap (c_0, \gamma) = \emptyset$.

Proof. Let $A = (a_{nk})$ be in $(\ell, \gamma; P) \cap (c_0, \gamma)$. Since A is in (c_0, γ) ,

$$\sup_m \sum_{k=0}^{\infty} \left| \sum_{n=0}^m a_{nk} \right| \leq M < \infty \quad (6)$$

(see [6], 43, p.6). Now, for $L = 0, 1, 2, \dots, m = 0, 1, 2, \dots$,

$$\sum_{k=0}^L \left| \sum_{n=0}^m a_{nk} \right| \leq \sum_{k=0}^{\infty} \left| \sum_{n=0}^m a_{nk} \right| \leq M.$$

Taking limit as $m \rightarrow \infty$, we have,

$$\sum_{k=0}^L \left| \sum_{n=0}^{\infty} a_{nk} \right| \leq M, \quad L = 0, 1, 2, \dots$$

Taking limit as $L \rightarrow \infty$, we get ,

$$\sum_{k=0}^{\infty} \left| \sum_{n=0}^{\infty} a_{nk} \right| \leq M.$$

which is contradiction, since $\sum_{n=0}^{\infty} a_{nk} = 1, k = 0, 1, 2, \dots$ in view of (3). This establishes our claim.

Corollary. Since $c_0 \subset c \subset \ell_{\infty}, (\ell_{\infty}, \gamma) \subset (c, \gamma) \subset (c_0, \gamma)$ so that $(\ell, \gamma; P) \cap (X, \gamma) = \emptyset$ for $X = c_0, c, \ell_{\infty}$.

3. The case when K is a complete, non trivially valued, non-archimedean field.

When K is a complete, non-trivially valued, non-archimedean field, we note that $\gamma = c_0$ and $\gamma_{\infty} = \ell_{\infty}$. In this case, it is easy to prove the following results.

Theorem 3.1 $(\ell, \gamma) = (\ell, c_0) = (c_0, c_0)$. A matrix $A = (a_{nk})$ is in (ℓ, c_0) if and only if it satisfies

$$\sup_{n,k} |a_{nk}| < \infty \tag{7}$$

and

$$\lim_{n \rightarrow \infty} a_{nk} = 0, \quad k = 0, 1, 2, \dots \tag{8}$$

Theorem 3.2 $(\ell, \gamma; P) = (\ell, c_0; P) = (c_0, c_0; P) = (\gamma, \gamma; P)$. A matrix $A = (a_{nk})$ is in $(\ell, c_0; P)$ if and only if it satisfies (3), (7) and (8).

Theorem 3.3 A matrix $A = (a_{nk})$ is in (c, c_0) if and only if it satisfies (7), (8) and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{nk} = 0. \tag{9}$$

Remark 3.4 Theorem 2.3 fails to hold when K is a complete, non-trivially valued, non-archimedean field since $(\ell, c_0) = (c_0, c_0)$. We also have

$$(\ell, c_0; P) \cap (c, c_0) \neq \emptyset,$$

as the following example illustrates. Consider the infinite matrix

$$A = (a_{nk}) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & -2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & -3 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & -4 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

i.e., $a_{nk} = n + 1$ if $k = n$;
 $= -(n + 1)$, if $k = n + 1$;
 $= 0$, otherwise .

Then (3) , (7) , (8) and (9) hold so that A is in $(\ell, c_0; P) \cap (c, c_0)$. These remarks point out significant departure from the case $K = \mathbb{R}$ or \mathbb{C} .

The following lemma is needed in the sequel.

Lemma 3.5 *The following statements are equivalent :*

- (a) A matrix $A = (a_{nk})$ is in (ℓ_{∞}, c_0) ;

$$(b) \quad (i) \quad \lim_{k \rightarrow \infty} a_{nk} = 0 ; \quad (10)$$

and

$$(ii) \quad \lim_{n \rightarrow \infty} \sup_{k \geq 0} |a_{nk}| = 0 ; \quad (11)$$

$$(c) \quad (i) \quad (8) \text{ holds}$$

and

$$(ii) \quad \lim_{k \rightarrow \infty} \sup_{n \geq 0} |a_{nk}| = 0 . \quad (12)$$

Proof. For the proof of “(a) is equivalent to (b)”, see ([4], 422). We now prove that (b) and (c) are equivalent. Let us suppose that (b) holds. For every fixed $k = 0, 1, 2, \dots$,

$$|a_{nk}| \leq \sup_{k' \geq 0} |a_{nk'}|.$$

Now (8) follows in view of (b) (ii). Again by (b) (ii), given $\varepsilon > 0$, we can choose a positive integer N such that

$$\sup_{k \geq 0} |a_{nk}| < \varepsilon, \quad n > N. \quad (13)$$

In view of (b) (i), for $n = 0, 1, 2, \dots, N$, we can find a positive integer L such that

$$|a_{nk}| < \varepsilon, \quad k > L. \quad (14)$$

(13) and (14) imply that

$$|a_{nk}| < \varepsilon, \quad n = 0, 1, 2, \dots, k > L.$$

i.e.,
$$\sup_{n \geq 0} |a_{nk}| < \varepsilon, \quad k > L$$

i.e.,
$$\lim_{k \rightarrow \infty} \sup_{n \geq 0} |a_{nk}| = 0,$$

so that (c) (ii) holds. Similarly we can prove that (c) implies (b). This establishes the lemma.

We now prove the following Steinhaus type result.

Theorem 3.6 *When K is a complete, non trivially valued, non-archimedean field, then*

$$(\ell, c_0; P) \cap (\ell_\infty, c_0) = \emptyset.$$

Proof. Let $A = (a_{nk})$ be in $(\ell, c_0; P) \cap (\ell_\infty, c_0)$. In view of (3), we have,

$$l = \left| \sum_{n=0}^{\infty} a_{nk} \right| \leq \sup_{n \geq 0} |a_{nk}|.$$

Taking limit as $k \rightarrow \infty$ and using (12), we get $1 \leq 0$, which is absurd. This proves the theorem.

In view of Theorem 3.2 and Theorem 3.6 we have the following.

Corollary. $(c_0, c_0; P) \cap (\ell_\infty, c_0) = \emptyset$.

We shall now take up an application of Theorem 3.6 to analytic functions. For the theory of analytic functions in non-archimedean fields, one can refer to [2]. Consider the space of bounded analytic functions inside the disk $d(0, 1^-)$ (usually denoted by $\mathcal{A}_b(d(0, 1^-))$), provided with the topology of uniform convergence in each disk $d(0, r)$, $r < 1$ and the space of analytic elements in the disk $d(0, 1)$ (usually denoted by $H(d(0, 1))$), provided with the topology of uniform convergence on $d(0, 1)$. Thanks to Lemma 3.5, one can check that (ℓ_∞, c_0) represents the space of continuous linear mappings from $\mathcal{A}_b(d(0, 1^-))$ into $H(d(0, 1))$. We now have the following result, which follows from Theorem 3.6.

Theorem 3.7 *There exists no continuous linear mapping ϕ from $\mathcal{A}_b(d(0, 1^-))$ into $H(d(0, 1))$ satisfying $\phi(f)(1) = f(1)$ for all $f \in \mathcal{A}_b(d(0, 1^-))$.*

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