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1 Introduction

The development of a non-Archimedean (especially, p-adic) mathematical physics [20]-[22], [1]-[4], [6], [8]-[13] induced some new mathematical structures over non-Archimedean fields. In particular, probability theory with p-adic valued probabilities was developed in [11], [8], [4][1].

The first theory with p-adic probabilities was the frequency theory in which probabilities were defined as limits of relative frequencies $\nu_N = n/N$ in the p-adic topology[2].This frequency probability theory was a natural extension of the frequency probability theory of R. von Mises [15], [16].

The next step was the creation of p-adic probability formalism on the basis of a theory of p-adic valued probability measures. It was natural to do this by following the fundamental work of A.N. Kolmogorov [14] in which he had proposed the measure-theoretical axiomatics of probability theory. Kolmogorov used properties of the frequency probability (non-negativity, normalization by 1 and additivity) as the basis of his axiomatics. Then he added the technical condition of σ-additivity for using Lebesgue's integration theory. In works [11],[8] we tried to follow A.N. Kolmogorov. p-adic frequency probability has also the properties of additivity. it is normalized by 1 and the set of possible values of this probability is the whole field of p-adic numbers $\mathbf{Q}_p$. Thus it was natural to define p-adic probability as a $\mathbf{Q}_p$-valued measure normalized by 1.

[1] p-adic probability theory appeared in connection with a model of quantum mechanics with p-adic valued wave functions [12]. The main task of this probability formalism was to present the probability interpretation for p-adic valued wave functions.

[2] The following trivial fact is the cornerstone of this theory: the relative frequencies belong to the field of rational numbers $\mathbf{Q}$; we can study their behaviour not only in the real topology on $\mathbf{Q}$, but also in the p-adic topologies on $\mathbf{Q}$.

[18], [19]. Therefore the creators of non-Archimedean integration theory (A. Monna and T. Springer [17]) did not try to develop abstract measure theory, but they proposed an integration formalism via Bourbaki based on integrals of continuous functions. This integration theory has been used for creating p-adic probability theory in the measure-theoretical framework [8]. The main disadvantage of this probability model is the strong connection with the topological structure of a sample space[3].

An abstract theory of non-Archimedean measures has been developed in [19]. The basic idea of this approach is to study measures defined on *rings* which in principle cannot be extended to measures on σ-rings. This gives the possibility for constructing non-discrete valued measures with values in non-Archimedean fields (and, in particular, in fields of p-adic numbers). On the other hand, the condition of continuity for measures in [19] implies the σ-additivity in all natural cases.

In this paper we develop a p-adic probability formalism based on measure theory of [19]. By probabilistic reasons we use the special case of this measure theory: (1) measures are defined on *algebras* (such measures have some special properties); (2) measures take values in fields of p-adic numbers (here values of probabilities can be approximated by rational relative frequencies).

However, probabilistic applications stimulate also the development of the general theory of non-Archimedean measures defined on rings. We prove the formula of the change of variables for these measures and use this formula for developing the formalism of conditional expectations for p-adic valued random variables.

2 Measures

Everywhere below K denotes a complete non-Archimedean field, $\mathbf{R}$ denotes the field of real numbers. The valuations on these fields are denoted by the same symbol $|\cdot|$. We set $U_R(a) = \{x \in K : |x-a| \leq R\}, a \in K, R \in \mathbf{R}, R > 0$. By definition these are balls in K.

Let X be an arbitrary set and let $\mathcal{R}$ be a ring of subsets of X. The pair $(X, \mathcal{R})$ is called a *measurable space.* The ring $\mathcal{R}$ is said to be *separating* if for every two distinct elements, x and y, of X there exists an $A \in \mathcal{R}$ such

[3]This is quite similar to the old probability formalisms of Frechet [6] and Cramer [5] in which the topological structure of the sample space played the important role.

that $x \in A, y \notin A$. We shall consider measurable spaces only over separating rings which cover the set X.

Every ring $\mathcal{R}$ can be used as a base for the zero-dimensional topology which we shall call the $\mathcal{R}$-topology. This topology is Hausdorff iff $\mathcal{R}$ is separating.

Throughout this section, $\mathcal{R}$ is a separating ring of a set X.

A subcollection $\mathcal{S}$ of $\mathcal{R}$ is said to be *shrinking* if the intersection of any two elements of $\mathcal{S}$ contains an element of $\mathcal{S}$. If $\mathcal{S}$ is shrinking, and if f is a map $\mathcal{R} \to K$ or $\mathcal{R} \to \mathbf{R}$, we say that $\lim_{A\in\mathcal{S}} f(A) = 0$ if for every $\epsilon > 0$, there exists an $A_0 \in \mathcal{S}$ such that $|f(A)| \leq \epsilon$ for all $A \in \mathcal{S}, A \subset A_0$.

A *measure* on $\mathcal{R}$ is a map $\mu : \mathcal{R} \to K$ with the properties: (i) μ is additive; (ii) for all $A \in \mathcal{R}$, $\|A\|_\mu = \sup\{|\mu(B)| : B \in \mathcal{R}, B \subset A\} < \infty$; (iii) if $\mathcal{S} \subset \mathcal{R}$ is shrinking and has empty intersection, then $\lim_{A\in\mathcal{S}} \mu(A) = 0$.

We call these conditions respectively *additivity, boundedness, continuity.* The latter condition is equivalent to the following: $\lim_{A\in\mathcal{S}} \|A\|_\mu = 0$ for every shrinking collection $\mathcal{S}$ with empty intersection. Further, we shall briefly discuss the main properties of measures, see [19] for the details.

For any set D, we denote its characteristic function (the indicator) by the symbol i_D. For $f : X \to K$ and $\phi : X \to [0, \infty)$, put $\|f\|_\phi = \sup_{x\in X} |f(x)|\phi(x)$. We set $N_\mu(x) = \inf_{U\in\mathcal{R}, x\in U} \|U\|_\mu$ for $x \in X$. Then $\|A\|_\mu = \|i_A\|_{N_\mu}$ for any $A \in \mathcal{R}$. We set $\|f\|_\mu = \|f\|_{N_\mu}$.

A *step function* (or $\mathcal{R}$-step function) is a function $f : X \to K$ of the form $f(x) = \sum_{k=1}^N c_k i_{A_k}(x)$ where $c_k \in K$ and $A_k \in \mathcal{R}, A_k \cap A_l = \emptyset, k \neq l$. We set for such a function $\int_X f(x)\mu(dx) = \sum_{k=1}^N c_k \mu(A_k)$. Denote the space of all step functions by the symbol $S(X)$. The integral $f \to \int_X f(x)\mu(dx)$ is the linear functional on $S(X)$ which satisfies the inequality

$$|\int_X f(x)\mu(dx)| \leq \|f\|_\mu. \qquad (1)$$

A function $f : X \to K$ is called *μ-integrable* if there exists a sequence of step functions $\{f_n\}$ such that $\lim_{n\to\infty} \|f - f_n\|_\mu = 0$. The μ-integrable functions form a vector space $L_1(X, \mu)$ (and $S(X) \subset L_1(X, \mu)$). The integral is extended from $S(X)$ on $L_1(X, \mu)$ by continuity. The inequality (1) holds for $f \in L_1(X, \mu)$.

Let $\mathcal{R}_\mu = \{A : A \subset X, i_A \in L_1(X, \mu)\}$. This is a ring. Elements of this ring are called μ-measurable sets. By setting $\mu(A) = \int_X i_A(x)\mu(dx)$ the measure μ is extended to a measure on $\mathcal{R}_\mu$. This is the *maximal extension* of μ, i.e., if we repeat the previous procedure starting with the ring $\mathcal{R}_\mu$, we will obtain this ring again.

Set $X_\epsilon = \{x \in X : N_\mu(x) \geq \epsilon\}, X_0 = \{x \in X : N_\mu(x) = 0\}, X_+ = X \setminus X_0$. Every $A \subset X_0$ belongs to $\mathcal{R}_\mu$. We call such sets *μ-negligible.*

Now we construct product measures. Let $\mu_j, j = 1, 2, ..., n$, be measures on (separating) rings $\mathcal{R}_j$ of subsets of sets X_j. The finite unions of the sets $A_1 \times \cdots \times A_n, A_j \in \mathcal{R}_j$, form a (separating) ring $\mathcal{R}_1 \times \cdots \times \mathcal{R}_n$ of $X_1 \times \cdots \times X_n$. Then there exists a unique measure $\mu_1 \times \cdots \times \mu_n$ on $\mathcal{R}_1 \times \cdots \times \mathcal{R}_n$ such that $\mu_1 \times \cdots \times \mu_n(A_1 \times \cdots \times A_n) = \mu_1(A_1) \times \cdots \times \mu_n(A_n)$. We have

$$N_{\mu_1 \times \cdots \times \mu_n}(x_1, ..., x_n) = N_{\mu_1}(x_1) \times \cdots \times N_{\mu_n}(x_n). \qquad (2)$$

Let X be a zero-dimensional topological space[4]. We denote the ring of *clopen* (i.e., at the same time open and closed) subsets of X by the symbol $B(X)$ (in fact, this is an algebra). We denote the space of continuous bounded functions $f : X \to K$ by the symbol $C_b(X)$. We use the norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$ on this space.

First we remark that if X is compact and $\mathcal{R} = B(X)$ then the condition (iii) in the definition of a measure is redundant. If X is not compact then there exist bounded additive set functions which are not continuous.

Let X be zero-dimensional N-compact topological space,i.e., there exists a set S such that X is homeomorphic to a closed subset of $\mathbf{N}^S$. We remark that every product of N-compact spaces is N-compact; every closed subspace of an N-compact space is N-compact. Then every bounded σ-additive function $\mu : B(X) \to K$ is a measure. On the other hand, if X is a zero-dimensional space such that every bounded σ-additive function $B(X) \to K$ is a measure, then X is N-compact.

In the theory of integration a crucial role is played by the $\mathcal{R}_\mu$-topology, i.e., the (zero-dimensional) topology that has $\mathcal{R}_\mu$ as a base. Of course, $\mathcal{R}_\mu$-topology is stronger that $\mathcal{R}$-topology. Every μ-negligible set is $\mathcal{R}_\mu$-clopen. The following two theorems [19] will be important for our considerations.

Theorem 2.1. (i) *If μ is a measure on $\mathcal{R}$, then N_μ is $\mathcal{R}$-upper semicontinuous, (hence, $\mathcal{R}_\mu$-upper semicontinuous) and for every $A \in \mathcal{R}_\mu$ and $\epsilon > 0$ the set $A_\epsilon = A \cap X_\epsilon$ is $\mathcal{R}_\mu$-compact.*

(ii) Conversely, let $\mu : \mathcal{R} \to K$ be additive. Assume that there exists an $\mathcal{R}$-upper semicontinuos $\phi : X \to [0, \infty)$ such that $|\mu(A)| \leq \sup_{x \in A} \phi(x), A \in \mathcal{R}$, and $\{x \in A : \phi(x) \geq \epsilon\}$ is $\mathcal{R}$-compact ($A \in \mathcal{R}, \epsilon > 0$). Then μ is a measure and $N_\mu \leq \phi$.

Theorem 2.2. *Let $\mu : \mathcal{R} \to K$ be a measure. A function $f : X \to K$ is μ-integrable iff it has the following two properties: (1) f is $\mathcal{R}_\mu$-continuous;*

[4]We consider only Hausdorff spaces.

(2) for every $\epsilon > 0$, *the set* $\{x : |f(x)|N_\mu(x) \geq \epsilon\}$ *is* $\mathcal{R}_\mu$*-compact.*

We shall also use the following fact.

Theorem 2.3. *Let* $f \in L_1(X, \mu)$ *and let*

$$\int_A f(x)\mu(dx) = 0 \ \textit{for every} \ A \in \mathcal{R}. \tag{3}$$

Then supp $f \subset X_0$.

Proof. Let us assume that f satisfies (3) and there exists $x_0 \in X_+$ (hence $N_\mu(x_0) = \alpha > 0$) such that $|f(x_0)| = c > 0$. Let $\{f_k\}$ be a sequence of $\mathcal{R}$-step functions which approximates f. For every $\epsilon > 0$ there exist N_ϵ such that $\|f - f_k\|_\mu < \alpha\epsilon$ for all $k \geq N_\epsilon$. In particular, this implies that $|f_k(x_0)| \geq c - \epsilon$, $k \geq N_\epsilon$. Then we have

$$\Delta_{B,k} = |\int_B f_k(x)\mu(dx)| = |\int_B f_k(x)\mu(dx) - \int_B f(x)\mu(dx)| < \alpha\epsilon, \ B \in \mathcal{R}.$$

Let

$$f_k(x) = \sum_j c_{kj} i_{B_{kj}}(x), c_{kj} \in K, B_{kj} \in \mathcal{R}, B_{kj} \cap B_{ki} = \emptyset, i \neq j,$$

and let $x_0 \in B_{kj_0}$. If $B \subset B_{kj_0}, B \in \mathcal{R}$, then $\Delta_{B,k} = |c_{kj}||\mu(B)| = |f_k(x_0)||\mu(B)| < \alpha\epsilon$. On the other hand, as $\|B_{kj_0}\|_\mu \geq \alpha$, then for every $\delta > 0$, there exists $B \subset B_{kj_0}, B \in \mathcal{R}$, such that $|\mu(B)| \geq (\alpha - \delta)$. Thus we obtain for this B: $\Delta_{B,k} \geq (\alpha - \delta)(c - \epsilon)$. By choosing $\epsilon > 0, \delta > 0$, such that $(\alpha - \delta)(c - \epsilon) > \alpha\epsilon$, we arrive to a contradiction.

Let $(X_j, \mathcal{R}_j), j = 1, 2$, be two measurable spaces. A function $f : X_1 \to X_2$ such that $f^{-1}(\mathcal{R}_2) \subset \mathcal{R}_1$ is said to be measurable ($(\mathcal{R}_1, \mathcal{R}_2)$-measurable). We shall use the following simple fact.

Lemma 2.1. *Let* $(X_j, \mathcal{R}_j), j = 1, 2$, *be measurable spaces and let* $f : X_1 \to X_2$ *be measurable. If* $\mathcal{S}$ *is shrinking in* $\mathcal{R}_2$ *then* $f^{-1}(\mathcal{S})$ *is shrinking in* $\mathcal{R}_1$. *If* $\mathcal{S}$ *has empty intersection, then* $f^{-1}(\mathcal{S})$ *has also empty intersection.*

Lemma 2.2. *Let* $(X_j, \mathcal{R}_j), j = 1, 2$, *be measurable spaces and let* $\eta : X_1 \to X_2$ *be a measurable function. Then, for every measure* $\mu : \mathcal{R}_1 \to K$, *the function* $\mu_\eta : \mathcal{R}_2 \to K$ *defined by the equality* $\mu_\eta(A) = \mu(\eta^{-1}(A))$ *is a measure on* $\mathcal{R}_2$ *and, for every* $\mathcal{R}_2$*-continuous function,* $h : X_2 \to K$ *the following inequality holds:*

$$\|h\|_{\mu_\eta} \leq \|h \circ \eta\|_\mu. \tag{4}$$

Proof. We have for every $A \in \mathcal{R}_2$,

$$\|A\|_{\mu_\eta} = \sup\{|\mu(\eta^{-1}(B)) : B \in \mathcal{R}_2, B \subset A\} \leq \|\eta^{-1}(A)\|_\mu < \infty. \tag{5}$$

Thus μ_η is bounded. We now prove that μ_η is continuous on $\mathcal{R}_2$. Let $\mathcal{S}$ be shrinking in $\mathcal{R}_2$ which has the empty intersection. By Lemma 2.1 $\eta^{-1}(\mathcal{S})$ is shrinking in $\mathcal{R}_1$ which has also the empty intersection. By (5) we obtain that $\lim_{A\in\mathcal{S}} \|A\|_{\mu_\eta} = 0$.

We prove inequality (4). Let $h : X_2 \to K$ be $\mathcal{R}_2$-continuous. We wish to prove that $|h(b)|N_{\mu_\eta}(b) \le \|h\circ\eta\|_\mu$ for all $b \in X_2$. So we choose $b \in X_2$ with $h(b) \ne 0$. Then the set $C_b = \{y \in X_2 : |h(y)| = |h(b)|\}$ is $\mathcal{R}_2$-open. Hence there is a $B \in \mathcal{R}_2$ with $b \in B \subset C_b$. Then

$$|h(b)|N_{\mu_\eta}(b) \le |h(b)|\|B\|_{\mu_\eta} \le |h(b)|\|\eta^{-1}(B)\|_\mu =$$

$$\sup_{x\in\eta^{-1}(B)} |h(b)|N_\mu(x) \le \sup_{x\in\eta^{-1}(B)} |(h\circ\eta)(x)|N_\mu(x) \le \|h\circ\eta\|_\mu.$$

Theorem 2.4. (Change of variables) *Let $(X_j, \mathcal{R}_j), j = 1, 2,$ be measurable spaces and let $\eta : X_1 \to X_2$ be a measurable function, and let $\mu : \mathcal{R}_1 \to K$ be a measure. If $f : X_2 \to K$ is an $\mathcal{R}_2$-continuous function such that the function $f\circ\eta$ belongs to $L_1(X_1, \mu)$, then $f \in L_1(X_2, \mu_\eta)$ and*

$$\int_{X_1} f(\eta(x))\mu(dx) = \int_{X_2} f(y)\mu_\eta(dy). \tag{6}$$

Proof. It suffices to prove that for every $\epsilon > 0$ there exists a $\mathcal{R}_2$-step function g such that $\|f-g\|_{\mu_\eta} \le \epsilon$ and $\|f\circ\eta - g\circ\eta\|_\mu \le \epsilon$. By (4) the first follows from the second. So we fix $\epsilon > 0$.

By Theorem 2.2 the set

$$A = \{x \in X_1 : |(f\circ\eta)(x)|N_\mu(x) \ge \epsilon\}$$

is $\mathcal{R}_1$-compact and therefore contained in an element of $\mathcal{R}_1$. But N_μ is bounded on every element of $\mathcal{R}_1$, so N_μ is bounded on A. We choose $\delta > 0$ so that

$$\delta N_\mu(x) \le \epsilon \text{ for all } x \in A.$$

As A is compact, $f(\eta(A))$ is also compact. We can cover $f(\eta(A))$ by disjoint closed balls of radius $\delta :$ $f(\eta(A)) \subset U_\delta(\alpha_0)\cup\ldots\cup U_\delta(\alpha_N)$, where α_0 is chosen to be 0 in order to obtain:

$$|\alpha_n| \le |t| \text{ for } t \in U_\delta(\alpha_n), n = 0, 1, \ldots, N. \tag{7}$$

For each n, $\mathcal{C}_n = \{C \in \mathcal{R}_2 : C \subset f^{-1}(U_\delta(\alpha_n))\}$ is a collection of open sets covering the compact set $\eta(A) \cap f^{-1}(U_\delta(\alpha_n))$. Thus, for each n there is a $C_n \in \mathcal{C}_n$ such that $\eta(A)\cap f^{-1}(U_\delta(\alpha_n)) \subset C_n$. We now have

$$C_0, \ldots, C_N \in \mathcal{R}_2, \tag{8}$$

$$C_n \subset f^{-1}(U_\delta(\alpha_n)), n = 0, 1, ..., N, \tag{9}$$

$$\eta(A) \subset C_0 \cup ... \cup C_N. \tag{10}$$

Put $g(x) = \sum_{n=0}^{N} \alpha_n i_{C_n}(x)$. Then g is a $\mathcal{R}_2$-step function. We wish to show that, for all $a \in X$,

$$\Delta(a) = |(f \circ \eta)(a) - (g \circ \eta)(a)| N_\mu(a) \leq \epsilon.$$

Thus, take $a \in X$:

(1) If $a \in A$, then there is a unique n with $\eta(a) \in C_n$. Then $\Delta(a) = |(f \circ \eta)(a) - \alpha_n| N_\mu(a) \leq \delta N_\mu(a) \leq \epsilon$.

(2) If $a \notin A$, but $\eta(a) \in C_n$ for some n, then by (7) we obtain that $\Delta(a) = |(f \circ \eta)(a) - \alpha_n| N_\mu(a) \leq |(f \circ \eta)(a)| N_\mu(a) \leq \epsilon$.

(3) If $a \notin C_0 \cup ... \cup C_N$, then $g(\eta(a)) = 0$. Thus $\Delta(a) = |(f \circ \eta)(a)| N_\mu(a) \leq \epsilon$ (as $a \notin A$).

Open problem. To find a condition for functions f which is weaker than continuity, but implies the formula of the change of variables.

Further we shall obtain some properties of measures which are specific for measures defined on algebras[5].

Throughout this paper, $\mathcal{A}$ is a separating algebra of a set X. First we remark that if we start with a measure μ defined on the algebra $\mathcal{A}$ then the system $\mathcal{A}_\mu$ of μ-integrable sets is again an algebra.

Proposition 2.1. *Let $\mu : \mathcal{A} \to K$ be a measure. Then for each $\epsilon > 0$, the set X_ϵ is $\mathcal{A}_\mu$-compact.*

This fact is a consequence of Theorem 2.1.

Proposition 2.2. *Let $\mu : \mathcal{A} \to K$ be a measure. Then the algebra $B(X)$ of $\mathcal{A}_\mu$-clopen sets coincides with the algebra $\mathcal{A}_\mu$.*

Proof. We use Theorem 2.2 and the previous proposition. Let $B \in B(X)$. Then i_B is $\mathcal{A}_\mu$-continuous and $\{x : |i_B(x)| N_\mu(x) \geq \epsilon\} = B \cap X_\epsilon$. As B is closed and X_ϵ is compact, $B \cap X_\epsilon$ is compact. Thus $B(X) \subset \mathcal{A}_\mu$.

As a consequence of Proposition 2.2, we obtain that $C_b(X) \subset L_1(X, \mu)$ (for the space X endowed with $\mathcal{A}_\mu$-topology) and the following inequality holds:

$$|\int_X f(x)\mu(dx)| \leq \|f\|_\infty \|X\|_\mu, \quad f \in C_b(X). \tag{11}$$

Let X be zero dimensional topological space. A measure μ defined on the algebra $B(X)$ of the clopen sets is called a *tight* measure. Thus by

[5]An algebra of X is a ring of subsets of X containing X.

Proposition 2.2 every measure $\mu : \mathcal{A} \to K$ is extended to a tight measure on the space X endowed with the $\mathcal{A}_\mu$-topology.

Proposition 2.3. *Let $\mu : \mathcal{A} \to K$ be a measure and let $f \in L_1(X, \mu)$. Then f is $(\mathcal{A}_\mu, B(K))$-measurable.*

Proof. By Theorem 2.2 f is $\mathcal{A}_\mu$-continuous. Thus $f^{-1}(B(K)) \subset B(X)$. But by Proposition 2.2 we have that $\mathcal{A}_\mu = B(X)$.

3 p-adic probability space

Let $\mu : \mathcal{A} \to \mathbf{Q}_p$ be a measure defined on a separating algebra $\mathcal{A}$ of subsets of the set Ω which satisfies the normalization condition $\mu(\Omega) = 1$. We set $\mathcal{F} = \mathcal{A}_\mu$ and denote the extension of μ on $\mathcal{F}$ by the symbol $\mathbf{P}$. A triple $(\Omega, \mathcal{F}, \mathbf{P})$ is said to be a p-adic *probability space* (Ω is a *sample space*, $\mathcal{F}$ is an algebra of *events*, $\mathbf{P}$ is a *probability*).

As in general measure theory we set $\Omega_\alpha = \{\omega \in \Omega : N_{\mathbf{P}}(\omega) \geq \alpha\}, \alpha > 0, \Omega_+ = \cup_{\alpha>0}\Omega_\alpha, \Omega_0 = \Omega \setminus \Omega_+$. Everywhere below, if a property Ξ is valid on the subset Ω_+ we say that Ξ is valid a.e. (mod $\mathbf{P}$).

Everywhere below (G, Γ) denotes a measurable space over the algebra Γ. Functions $\xi : \Omega \to G$ which are $(\mathcal{F}, \Gamma)$-measurable are said to be random variables.

Everywhere below Y is a zero dimensional topological space. We consider Y as the measurable space over the algebra $B(Y)$. Every random variable $\xi : \Omega \to Y$ is continuous in the $\mathcal{F}$-topology. In particular, $\mathbf{Q}_p$-valued random variables are $(\mathcal{F}, B(\mathbf{Q}_p))$-measurable functions. If $\xi \in L_1(\Omega, \mathbf{P})$, we introduce an *expectation* of this random variable by setting $\mathbf{E}\xi = \int_\Omega \xi(\omega)\mathbf{P}(d\omega)$. We note that every bounded random variable $\xi : \Omega \to \mathbf{Q}_p$ belongs to $L_1(\Omega, \mathbf{P})$.

Let $\eta : \Omega \to G$ be a random variable. The measure $\mathbf{P}_\eta$ is said to be a *distribution* of the random variable. By Theorem 2.4 we have that

$$\mathbf{E}f(\eta) = \int_{\mathbf{Q}_p} f(y)\mathbf{P}_\eta(dy) \tag{12}$$

for every Γ-continuous function $f : G \to \mathbf{Q}_p$ such that $f \circ \eta \in L_1(\Omega, \mathbf{P})$. In particular, we have the following result.

Proposition 3.1. *Let $\eta : \Omega \to Y$ be a random variable and let $f \in C_b(Y)$. Then the formula (12) holds.*

We shall also use the following technical result.

Proposition 3.2. *Let $\eta : \Omega \to Y$ be a random variable and let $\zeta \in L_1(\Omega, \mathbf{P})$, and let $f \in C_b(Y)$. Then $\xi(\omega) = \zeta(\omega) f(\eta(\omega))$ belongs $L_1(\Omega, \mathbf{P})$ and*

$$\mathbf{E}\xi = \int_{\mathbf{Q}_p \times Y} x f(y) \mathbf{P}_z(dx dy), \quad z(\omega) = (\zeta(\omega), \eta(\omega)).$$

Proof. We have only to show that $\xi \in L_1(\Omega, \mathbf{P})$. This fact is a consequence of Theorem 2.2.

The random variables $\xi, \eta : \Omega \to G$ are called independent if

$$\mathbf{P}(\xi \in A, \eta \in B) = \mathbf{P}(\xi \in A)\mathbf{P}(\eta \in B) \text{ for all } A, B \in \Gamma. \tag{13}$$

Proposition 3.3. *Let $\xi, \eta : \Omega \to Y$ be independent random variables and functions $f, g \in C_b(Y)$. Then we have:*

$$\mathbf{E} f(\xi) g(\eta) = \mathbf{E} f(\xi) \mathbf{E} g(\eta). \tag{14}$$

Proof. If f and g are locally constant functions then (14) is a consequence of (13). Arbitrary functions $f, g \in C_b(Y)$ can be approximated by locally constant functions (with the convergence of corresponding integrals) by using the technique developed in the proof of Theorem 2.4.

Remark 3.1. In fact, the formula (14) is valid for the continuous f, g such that the random variables $f(\xi), g(\eta)$ and $f(\xi)g(\eta)$ belong to $L_1(\Omega, \mathbf{P})$.

Proposition 3.4. *Let ξ and η be independent random variables. Then the random vector $z = (\xi, \eta)$ has the probability distribution $\mathbf{P}_z = \mathbf{P}_\eta \times \mathbf{P}_\xi$.*

This fact is a direct consequence of (13).

Let ξ and η be respectively $\mathbf{Q}_p$ and G valued random variables and $\xi \in L_1(\Omega, \mathbf{P})$. A *conditional expectation* $\mathbf{E}[\xi | \eta = y]$ is defined as a function $m \in L_1(G, \mathbf{P}_\eta)$ such that

$$\int_{\{\omega \in \Omega : \eta(\omega) \in B\}} \xi(\omega) \mathbf{P}(d\omega) = \int_B m(y) \mathbf{P}_\eta(dy) \text{ for every } B \in \Gamma.$$

Proposition 3.5. *The conditional expectation if it exists, is defined uniquely a.e.* mod $\mathbf{P}_\eta$.

Proof. We assume that there exist two conditional expectations $m_j \in L_1(G, \mathbf{P}_\eta)$ and $m_1(x_0) \neq m_2(x_0)$ at some point x_0 and $N_{\mathbf{P}_\eta}(x_0) > 0$. Set $m(x) = m_1(x) - m_2(x)$. We have : $\int_B m(x) \mathbf{P}_\eta(dx) = 0$ for every $B \in \Gamma$. To obtain the contradiction, it is sufficient to use Theorem 2.3.

As there is no analogue of the Radon-Nikodym theorem in the non-Archimedean case [17], [18], [19], it may happens that a conditional expectation does not exist. Everywhere below we assume that $m(y) = \mathbf{E}[\xi | \eta = y]$ is well defined and moreover, that it belongs to the class $C_b(Y)$.

Proposition 3.6. *Let $\xi : \Omega \to \mathbf{Q}_p, \eta : \Omega \to Y$ be random variables, and $\xi \in L_1(\Omega, \mathbf{P})$. The equality*

$$\mathbf{E}f(\eta)\xi = \mathbf{E}f(\eta(\omega))\mathbf{E}[\xi(\omega)|\eta = \eta(\omega)] \tag{15}$$

holds for every function $f \in C_b(Y)$.

Proof. By Proposition 3.2 we obtain $\mathbf{E}\xi f(\eta) = \int_{\mathbf{Q}_p \times Y} x f(y)\mathbf{P}_z(dxdy)$, where $z(\omega) = (\xi(\omega), \eta(\omega))$. Set for $A \in B(Y)$,

$$\lambda(A) = \int_{\mathbf{Q}_p \times Y} x i_A(y)\mathbf{P}_z(dxdy).$$

As $\lambda(A) = \int_{\eta^{-1}(A)} \xi(\omega)\mathbf{P}(d\omega) = \int_Y m(y)\mathbf{P}_\eta(dy)$, it is a tight measure on Y. Then

$$\int_{\mathbf{Q}_p \times Y} x f(y)\mathbf{P}_z(dxdy) = \int_Y f(y)\lambda(dy) = \int_Y f(y)m(y)\mathbf{P}_\eta(dy) = Ef(\eta)m(\eta).$$

The authors plan to apply the measure-theoretical framework developed in this paper for studying of the limits theorems, random walks for p-adic probabilities (compare with the paper [3] in that p-adic random walk was studied on the basis of conventional probability theory).

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