ANNALES MATHÉMATIQUES BLAISE PASCAL

STANY DE SMEDT

Local invertibility of non-archimedean vector-valued functions

Annales mathématiques Blaise Pascal, tome 5, nº 1 (1998), p. 13-23

<http://www.numdam.org/item?id=AMBP_1998__5_1_13_0>

Ann. Math. Blaise Pascal, Vol. 5, N° 1, 1998, pp.13-23

LOCAL INVERTIBILITY OF NON-ARCHIMEDEAN VECTOR-VALUED FUNCTIONS

Stany De Smedt

Abstract We are well acquainted with the local invertibility of C^n-functions of one variable (see [4]). Surprisingly this is not the case for vector-valued functions. In this paper we will fill this gap in non-archimedean functional analysis.

1991 Mathematics Subject Classification : 46S10

1 Introduction

Let K be a complete non-archimedean non-trivially valued field, and $C(X \to K)$ the space of all continuous functions from X to K, where X is a nonempty subset of K without isolated points.

Let $f : X \to K$ and $\nabla^2 X = X \times X \setminus \{ (x,x) \mid x \in X \}$. The (first) difference quotient $\phi_1 f : \nabla^2 X \to K$ is defined by

$$\phi_1 f(x,y) = \frac{f(y) - f(x)}{y - x}$$

f is called continuously differentiable (or strictly differentiable, or uniformly differentiable) at $a \in X$ if $\lim_{(x,y)\to(a,a)} \phi_1 f(x,y)$ exists. We will also say that f is C^1 at a.
In a similar way, we may define C^n-functions as follows :
For $n \in$, we define $\nabla^{n+1} X = \{ (x_1, ..., x_{n+1}) \in X^{n+1} \mid x_i \neq x_j \text{ if } i \neq j \}$ and the n-th difference quotient $\phi_n f : \nabla^{n+1} X \to K$ by $\phi_0 f = f$ and

$$\phi_n f(x_1, x_2, ..., x_{n+1}) = \frac{\phi_{n-1} f(x_2, x_3, ..., x_{n+1}) - \phi_{n-1} f(x_1, x_3, ..., x_{n+1})}{x_2 - x_1}$$

A function f is called a C^n-function if $\phi_n f$ can be extended to a continuous function $\overline{\phi_n f}$ on X^{n+1}.
The set of all C^n-functions from X to K will be denoted by $C^n(X \to K)$.
For $f : X \times X \to K$, the first difference quotients $\phi_1^{(1)} f$ and $\phi_1^{(2)} f$ are defined as

$$\phi_1^{(1)} f(x, x', y) = \frac{f(x,y) - f(x',y)}{x - x'}$$

and

$$\phi_1^{(2)} f(x, y, y') = \frac{f(x,y) - f(x,y')}{y - y'}$$

(for $x \neq x'$ and $y \neq y'$).
If $\phi_1^{(1)} f$ and $\phi_1^{(2)} f$ can be extended to continuous functions $\overline{\phi_1^{(1)} f}$, $\overline{\phi_1^{(2)} f}$ respectively, defined on X^3 then f is called a C^1-function. The space of all these C^1-functions, will be denoted $C^1(X \times X \to K)$.
We have obviously

$$\frac{\partial f}{\partial x}(x, y) = \overline{\phi_1^{(1)} f}(x, x, y)$$
$$\frac{\partial f}{\partial y}(x, y) = \overline{\phi_1^{(2)} f}(x, y, y)$$

for all $x, y \in X$.
For the difference quotients of second order, we get

$$\phi_2^{(11)} f(x, x', x'', y) = \frac{\phi_1^{(1)} f(x, x', y) - \phi_1^{(1)} f(x, x'', y)}{x' - x''}$$

$$\phi_2^{(21)} f(x, x', y, y') = \frac{\phi_1^{(1)} f(x, x', y) - \phi_1^{(1)} f(x, x', y')}{y - y'}$$

$$\phi_2^{(12)} f(x, x', y, y') = \frac{\phi_1^{(2)} f(x, y, y') - \phi_1^{(2)} f(x', y, y')}{x - x'}$$

$$\phi_2^{(22)} f(x, y, y', y'') = \frac{\phi_1^{(2)} f(x, y, y') - \phi_1^{(2)} f(x, y, y'')}{y' - y''}$$

and f is a C^2-function if those four functions can be extended to continuous functions on X^4. Remark that $\phi_2^{(21)} f(x, x', y, y') = \phi_2^{(12)} f(x, x', y, y')$. Following the notations above, we denote $C^2(X \times X \to K)$ for the space of all C^2-functions. Continuing in the same way, we define the difference quotients of n-th order and the C^n-functions.

More information about C^n-functions can be found in [1], [2], [3] and [4].

2 Vector-valued C^1-functions

Now let us take a look to vector-valued functions.
Let E be an n-dimensional non-archimedean vectorspace, F an m-dimensional non-archimedean vectorspace, and let $f : E \to F$. Let x be a point of E. For all vectors h such that $|h|$ is small (and $h \neq 0$), the point $x + h$ also lies in E. However we cannot form a quotient $\frac{f(x+h) - f(x)}{h}$ because it is meaningless to divide by a vector. In order to define what we mean for a function f to be continuously differentiable, we must find a way which does not involve dividing by h. We reconsider the case of functions f of one variable :

$$f \text{ is continuously differentiable at a point } a \text{ if } \lim_{(x,y)\to(a,a)} \frac{f(y) - f(x)}{y - x} \text{ exists}$$

In other words f is C^1 at a if f is differentiable at a and if for each $\epsilon > 0$, there exists $\delta > 0$ such that if $|x-a| < \delta, |y-a| < \delta, x \neq y$ then $\left|\frac{f(y) - f(x)}{y - x} - f'(a)\right| < \epsilon$

This can also be written as :
f is C^1 at a if there exists a function g such that for all $x \neq y$:

$$f(y) - f(x) = f'(a).(y - x) + g(y - x) \text{ whereby } \lim_{(x,y)\to(a,a)} \frac{g(y - x)}{y - x} = 0$$

We now consider a function of n variables.

Definition
Let X be a nonempty subset of E without isolated points. The function $f : X \to F$ is *continuously differentiable at* a if there is a linear function L (depending on a) such that for all $x \neq y \in X$:

$$f(x) - f(y) = L(x - y) + g(x - y) \text{ with } \lim_{(x,y)\to(a,a)} \frac{|g(y - x)|}{|y - x|} = 0$$

The linear function L is then denoted by df_a and is called the *differential of f at a*. Its matrix $f'(a)$ is called *the derivative of f at a*. Thus $f'(a)$ is the unique $m \times n$-matrix such that $df_a(x) = f'(a).x$ for all $x \in X$. The set of all C^1-functions $f : X \to F$ is denoted $C^1(X \to F)$.

Lemma
The function $f : X \to F$ is continuously differentiable at a if and only if each of its component functions $f_1, \ldots, f_m$ is.

This lemma follows immediately from a componentwise reading of the vector equation in the definition above.

Proposition
The composition of two C^1-functions is a C^1-function.

Proof
We prove the proposition for n = m = 2. The proof is completely similar for the general case.

$$
\begin{aligned}
&\phi_1^{(1)}(g\circ f)_1(x_1,x_1',x_2)\\
&= \frac{(g\circ f)_1(x_1,x_2)-(g\circ f)_1(x_1',x_2))}{x_1-x_1'}\\
&= \frac{g_1(f_1(x_1,x_2),f_2(x_1,x_2))-g_1(f_1(x_1',x_2),f_2(x_1',x_2))}{x_1-x_1'}\\
&= \frac{g_1(f_1(x_1,x_2),f_2(x_1,x_2))-g_1(f_1(x_1',x_2),f_2(x_1,x_2))}{x_1-x_1'}\\
&\quad + \frac{g_1(f_1(x_1',x_2),f_2(x_1,x_2))-g_1(f_1(x_1',x_2),f_2(x_1',x_2))}{x_1-x_1'}\\
&= \frac{g_1(f_1(x_1,x_2),f_2(x_1,x_2))-g_1(f_1(x_1',x_2),f_2(x_1,x_2))}{f_1(x_1,x_2)-f_1(x_1',x_2)}.\frac{f_1(x_1,x_2)-f_1(x_1',x_2)}{x_1-x_1'}\\
&\quad + \frac{g_1(f_1(x_1',x_2),f_2(x_1,x_2))-g_1(f_1(x_1',x_2),f_2(x_1',x_2))}{f_2(x_1,x_2)-f_2(x_1',x_2)}.\frac{f_2(x_1,x_2)-f_2(x_1',x_2)}{x_1-x_1'}\\
&= \phi_1^{(1)}g_1(f_1(x_1,x_2),f_1(x_1',x_2),f_2(x_1,x_2)).\phi_1^{(1)}f_1(x_1,x_1',x_2)\\
&\quad + \phi_1^{(2)}g_1(f_1(x_1',x_2),f_2(x_1,x_2),f_2(x_1',x_2)).\phi_1^{(1)}f_2(x_1,x_1',x_2)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\phi_1^{(1)}(g\circ f)_2(x_1,x_1',x_2)\\
&= \phi_1^{(1)}g_2(f_1(x_1,x_2),f_1(x_1',x_2),f_2(x_1,x_2)).\phi_1^{(1)}f_1(x_1,x_1',x_2)\\
&\quad + \phi_1^{(2)}g_2(f_1(x_1',x_2),f_2(x_1,x_2),f_2(x_1',x_2)).\phi_1^{(1)}f_2(x_1,x_1',x_2)
\end{aligned}
$$

$$
\begin{aligned}
&\phi_1^{(2)}(g\circ f)_1(x_1,x_2,x_2')\\
&= \phi_1^{(1)}g_1(f_1(x_1,x_2),f_1(x_1,x_2'),f_2(x_1,x_2)).\phi_1^{(2)}f_1(x_1,x_2,x_2')\\
&\quad + \phi_1^{(2)}g_1(f_1(x_1,x_2'),f_2(x_1,x_2),f_2(x_1,x_2')).\phi_1^{(2)}f_2(x_1,x_2,x_2')
\end{aligned}
$$

$$
\begin{aligned}
&\phi_1^{(2)}(g\circ f)_2(x_1,x_2,x_2')\\
&= \phi_1^{(1)}g_2(f_1(x_1,x_2),f_1(x_1,x_2'),f_2(x_1,x_2)).\phi_1^{(2)}f_1(x_1,x_2,x_2')\\
&\quad + \phi_1^{(2)}g_2(f_1(x_1,x_2'),f_2(x_1,x_2),f_2(x_1,x_2')).\phi_1^{(2)}f_2(x_1,x_2,x_2')
\end{aligned}
$$

And these are continuous extendable since f and g are continuously differentiable functions.

Theorem
Let $f : E \to E$ be C^1 at a and suppose $det f'(a) \neq 0$, then f is locally invertible. The local inverse g of f is C^1 at $f(a)$ and $g'(f(a)) = (f'(a))^{-1}$.

Proof
We may assume that $a = 0 = f(a)$. If this is not the case, we first make translations, replacing $f(x)$ by $F(x) = f(x+a) - f(a)$.
Let $L = df_0$ and $\Phi = L^{-1} \circ f$. Note that the inverse L^{-1} exists since $det f'(0) \neq 0$.
Moreover $\Phi(0) = 0$ and $d\Phi_0 = L^{-1} \circ df_0 = L^{-1} \circ L = I$.
L^{-1} is a linear function, so L^{-1} is C^1 and thus also Φ is C^1. From the definition of C^1-functions we then get that given ϵ there exists a neighbourhood U of 0 of radius δ such that $|\Phi(s) - \Phi(t) - (s-t)| \leq \epsilon |s-t|$ for every $s, t \in U$.
Choose $\epsilon < 1$, then f is injective on U.
Indeed, suppose $f(s) = f(t)$ with $s, t \in U$, then $\Phi(s) = L^{-1}(f(s)) = L^{-1}(f(t)) = \Phi(t)$. And thus $|s-t| = |\Phi(s) - \Phi(t) - (s-t)| \leq \epsilon|s-t|$, so $s = t$.
Given $y \in U$, define $\varphi_y : E \to E$ by $\varphi_y(x) = x - \Phi(x) + y$.
We want to show that φ_y is a contraction on U, its unique fixed point will then be the desired point $x \in U$, such that $\Phi(x) = y$.
For $x \in U$:

$$|\varphi_y(x)| = |x - \Phi(x) + y| \leq \max(|x - \Phi(x)|, |y|) < \max(|x|, |y|) < \delta$$

So φ_y maps U into itself.

$$|\varphi_y(s) - \varphi_y(t)| = |s - \Phi(s) - t + \Phi(t)| \leq \epsilon|s-t|$$

Thus $\varphi_y : U \to U$ is indeed a contraction, and therefore has a unique fixed point $x \in U$ such that $\Phi(x) = y$.
The fixed point $x = \Psi(y)$ is the limit of the sequence (x_m) defined by

$$x_0 = 0,\ x_{m+1} = x_m - \Phi(x_m) + y$$

We now apply that $|\Phi(s) - \Phi(t) - (s-t)| \leq \epsilon|s-t|$ with $s = \Psi(x), t = \Psi(y)$ for any $\epsilon < 1$, then

$$|\Phi(\Psi(x)) - \Phi(\Psi(y)) - (\Psi(x) - \Psi(y))| \leq \epsilon|\Psi(x) - \Psi(y)| < |\Psi(x) - \Psi(y)|$$

or equivalently

$$|x - y - (\Psi(x) - \Psi(y))| < |\Psi(x) - \Psi(y)|$$

Because of the strict inequality, this means that $|x-y| = |\Psi(x) - \Psi(y)|$, so

$$|\Psi(x) - \Psi(y) - (x-y)| \leq \epsilon|x-y|$$

This proves that Ψ is continuously differentiable with $d\Psi_0 = I$.
Let $\eta = \dfrac{\delta}{\|L^{-1}\|}$ and let V be the η-neighbourhood of 0.
Define g by $g(x) = \Psi(L^{-1}(x))$ for all $x \in V$. Thus g is C^1 as composition of two C^1-functions and $g(V) \subset U$.
Moreover,

$$f(g(x)) = f(\Psi(L^{-1}(x))) = L \circ \Phi \circ \Psi \circ L^{-1}(x) = x \quad \text{for all } \; x \in V$$

And

$$g(f(t)) = \Psi \circ L^{-1} \circ f(t) = \Psi \circ \Phi(t) = t \quad \text{for all } \; t \in U$$

The chain rule gives $g'(f(a)) = (f'(a))^{-1}$, which ends our proof.

3 Local invertibility of C^k-functions

As in classical analysis, C^2-functions can be defined as follows.

Definition
The function $f : E \to F$ is C^2 *at* a if there is a linear function L_1 and a bilinear function L_2 (both depending on a) such that for all $x \neq y \in E$:

$$f(x)-f(y) = L_1(x-y)+L_2(x-y)(x-y)+g(x-y) \text{ with } \lim_{(x,y)\to(a,a)} \frac{|g(y-x)|}{|y-x|^2} = 0$$

In order to give a similar definition for C^k-functions, we have to formulate an extra assumption on the domain of our function (see [4], section 83). Whereas the proof of the local inverbility theorem for C^1-functions is based on such a definition, we will prefer the following definition to prove the local inverbility theorem for C^k-functions.

Definiton
The function $f : E \to F$ is C^k at a if and only if each of its components $f_1, \ldots, f_m$ is.

We then get

Theorem
Let $f : E \to E$ be C^k at a and suppose $\det f'(a) \neq 0$, then f is locally invertible. The local inverse g of f is C^k at $f(a)$.

Proof

We will restrict our proof to the case $k = 2, n = 2$, since otherwise the notations become too complicated.

Since f is a C^2-function by assumption, f is also a C^1-function. So the previous theorem tells us that f is locally invertible and the local inverse g of f is C^1 at $f(a)$. To prove that g is also C^2 at $f(a)$ we will give an expression for the second order difference quotients of the components g_1 and g_2 of g.

We start form the formulas in the propositiȯn of section 2, for example

$$\begin{aligned}
&\phi_1^{(1)}(f\circ g)_1(x_1,x_1',x_2)\\
&\quad= \phi_1^{(1)}f_1(g_1(x_1,x_2),g_1(x_1',x_2),g_2(x_1,x_2)).\phi_1^{(1)}g_1(x_1,x_1',x_2)\\
&\qquad+\phi_1^{(2)}f_1(g_1(x_1',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)}g_2(x_1,x_1',x_2)
\end{aligned}$$

Since f and g are inverse functions $f\circ g = I$, the identity function. So all second order difference quotients of $f\circ g$ are identically 0. We thus get

$$\begin{aligned}
0 &= \phi_2^{(11)}(f\circ g)_1(x_1,x_1',x_1'',x_2)\\
&= \frac{\phi_1^{(1)}(f\circ g)_1(x_1,x_1',x_2)-\phi_1^{(1)}(f\circ g)_1(x_1,x_1'',x_2)}{x_1'-x_1''}\\
&= \Big(\phi_1^{(1)}f_1(g_1(x_1,x_2),g_1(x_1',x_2),g_2(x_1,x_2)).\phi_1^{(1)}g_1(x_1,x_1',x_2)\\
&\quad+\phi_1^{(2)}f_1(g_1(x_1',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)}g_2(x_1,x_1',x_2)\\
&\quad-\phi_1^{(1)}f_1(g_1(x_1,x_2),g_1(x_1'',x_2),g_2(x_1,x_2)).\phi_1^{(1)}g_1(x_1,x_1'',x_2)\\
&\quad-\phi_1^{(2)}f_1(g_1(x_1'',x_2),g_2(x_1,x_2),g_2(x_1'',x_2)).\phi_1^{(1)}g_2(x_1,x_1'',x_2)\Big)\cdot\frac{1}{x_1'-x_1''}\\
&= \Big(\phi_1^{(1)}f_1(g_1(x_1,x_2),g_1(x_1',x_2),g_2(x_1,x_2)).\phi_1^{(1)}g_1(x_1,x_1',x_2)\\
&\quad-\phi_1^{(1)}f_1(g_1(x_1,x_2),g_1(x_1',x_2),g_2(x_1,x_2)).\phi_1^{(1)}g_1(x_1,x_1'',x_2)\\
&\quad+\phi_1^{(1)}f_1(g_1(x_1,x_2),g_1(x_1',x_2),g_2(x_1,x_2)).\phi_1^{(1)}g_1(x_1,x_1'',x_2)\\
&\quad+\phi_1^{(2)}f_1(g_1(x_1',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)}g_2(x_1,x_1',x_2)\\
&\quad-\phi_1^{(2)}f_1(g_1(x_1',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)}g_2(x_1,x_1'',x_2)\\
&\quad+\phi_1^{(2)}f_1(g_1(x_1',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)}g_2(x_1,x_1'',x_2)\\
&\quad-\phi_1^{(1)}f_1(g_1(x_1,x_2),g_1(x_1'',x_2),g_2(x_1,x_2)).\phi_1^{(1)}g_1(x_1,x_1'',x_2)\\
&\quad-\phi_1^{(2)}f_1(g_1(x_1'',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)}g_2(x_1,x_1'',x_2)\\
&\quad+\phi_1^{(2)}f_1(g_1(x_1'',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)}g_2(x_1,x_1'',x_2)\\
&\quad-\phi_1^{(2)}f_1(g_1(x_1'',x_2),g_2(x_1,x_2),g_2(x_1'',x_2)).\phi_1^{(1)}g_2(x_1,x_1'',x_2)\Big)\cdot\frac{1}{x_1'-x_1''}\\
&= \phi_1^{(1)}f_1(g_1(x_1,x_2),g_1(x_1',x_2),g_2(x_1,x_2)).\phi_2^{(11)}g_1(x_1,x_1',x_1'',x_2)
\end{aligned}$$

$$
\begin{aligned}
&+ \phi_1^{(2)} f_1(g_1(x_1',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_2^{(11)} g_2(x_1,x_1',x_1'',x_2)\\
&+ \phi_2^{(11)} f_1(g_1(x_1,x_2),g_1(x_1',x_2),g_1(x_1'',x_2),g_2(x_1,x_2)).\phi_1^{(1)} g_1(x_1,x_1'',x_2).\phi_1^{(1)} g_1(x_1',x_1'',x_2)\\
&+ \phi_2^{(12)} f_1(g_1(x_1',x_2),g_1(x_1'',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)} g_2(x_1,x_1'',x_2).\phi_1^{(1)} g_1(x_1',x_1'',x_2)\\
&+ \phi_2^{(22)} f_1(g_1(x_1'',x_2),g_2(x_1,x_2),g_2(x_1',x_2),g_2(x_1'',x_2)).\phi_1^{(1)} g_2(x_1,x_1'',x_2).\phi_1^{(1)} g_2(x_1',x_1'',x_2)
\end{aligned}
$$

In the same way we can prove that

$$
\begin{aligned}
0 &= \phi_2^{(11)}(f\circ g)_2(x_1,x_1',x_1'',x_2)\\
&= \phi_1^{(1)} f_2(g_1(x_1,x_2),g_1(x_1',x_2),g_2(x_1,x_2)).\phi_2^{(11)} g_1(x_1,x_1',x_1'',x_2)\\
&\quad + \phi_1^{(2)} f_2(g_1(x_1',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_2^{(11)} g_2(x_1,x_1',x_1'',x_2)\\
&\quad + \phi_2^{(11)} f_2(g_1(x_1,x_2),g_1(x_1',x_2),g_1(x_1'',x_2),g_2(x_1,x_2)).\phi_1^{(1)} g_1(x_1,x_1'',x_2).\phi_1^{(1)} g_1(x_1',x_1'',x_2)\\
&\quad + \phi_2^{(12)} f_2(g_1(x_1',x_2),g_1(x_1'',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)} g_2(x_1,x_1'',x_2).\phi_1^{(1)} g_1(x_1',x_1'',x_2)\\
&\quad + \phi_2^{(22)} f_2(g_1(x_1'',x_2),g_2(x_1,x_2),g_2(x_1',x_2),g_2(x_1'',x_2)).\phi_1^{(1)} g_2(x_1,x_1'',x_2).\phi_1^{(1)} g_2(x_1',x_1'',x_2)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \phi_2^{(21)}(f\circ g)_1(x_1,x_1',x_2,x_2')\\
&= \phi_1^{(1)} f_1(g_1(x_1,x_2),g_1(x_1',x_2),g_2(x_1,x_2)).\phi_2^{(21)} g_1(x_1,x_1',x_2,x_2')\\
&\quad + \phi_1^{(2)} f_1(g_1(x_1',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_2^{(21)} g_2(x_1,x_1',x_2,x_2')\\
&\quad + \phi_2^{(11)} f_1(g_1(x_1,x_2),g_1(x_1,x_2'),g_1(x_1',x_2),g_2(x_1,x_2)).\phi_1^{(1)} g_1(x_1,x_1',x_2').\phi_1^{(2)} g_1(x_1,x_2,x_2')\\
&\quad + \phi_2^{(11)} f_1(g_1(x_1,x_2'),g_1(x_1',x_2),g_1(x_1',x_2'),g_2(x_1,x_2)).\phi_1^{(1)} g_1(x_1,x_1',x_2').\phi_1^{(2)} g_1(x_1',x_2,x_2')\\
&\quad + \phi_2^{(21)} f_1(g_1(x_1,x_2'),g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1,x_2')).\phi_1^{(1)} g_1(x_1,x_1',x_2').\phi_1^{(2)} g_2(x_1,x_2,x_2')\\
&\quad + \phi_2^{(12)} f_1(g_1(x_1',x_2),g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)} g_2(x_1,x_1',x_2').\phi_1^{(2)} g_1(x_1',x_2,x_2')\\
&\quad + \phi_2^{(22)} f_1(g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1,x_2'),g_2(x_1',x_2)).\phi_1^{(1)} g_2(x_1,x_1',x_2').\phi_1^{(2)} g_2(x_1,x_2,x_2')\\
&\quad + \phi_2^{(22)} f_1(g_1(x_1',x_2'),g_2(x_1,x_2'),g_2(x_1',x_2),g_2(x_1',x_2')).\phi_1^{(1)} g_2(x_1,x_1',x_2').\phi_1^{(2)} g_2(x_1',x_2,x_2')
\end{aligned}
$$

$$
\begin{aligned}
0 &= \phi_2^{(21)}(f\circ g)_2(x_1,x_1',x_2,x_2')\\
&= \phi_1^{(1)} f_2(g_1(x_1,x_2),g_1(x_1',x_2),g_2(x_1,x_2)).\phi_2^{(21)} g_1(x_1,x_1',x_2,x_2')\\
&\quad + \phi_1^{(2)} f_2(g_1(x_1',x_2),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_2^{(21)} g_2(x_1,x_1',x_2,x_2')\\
&\quad + \phi_2^{(11)} f_2(g_1(x_1,x_2),g_1(x_1,x_2'),g_1(x_1',x_2),g_2(x_1,x_2)).\phi_1^{(1)} g_1(x_1,x_1',x_2').\phi_1^{(2)} g_1(x_1,x_2,x_2')\\
&\quad + \phi_2^{(11)} f_2(g_1(x_1,x_2'),g_1(x_1',x_2),g_1(x_1',x_2'),g_2(x_1,x_2)).\phi_1^{(1)} g_1(x_1,x_1',x_2').\phi_1^{(2)} g_1(x_1',x_2,x_2')\\
&\quad + \phi_2^{(21)} f_2(g_1(x_1,x_2'),g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1,x_2')).\phi_1^{(1)} g_1(x_1,x_1',x_2').\phi_1^{(2)} g_2(x_1,x_2,x_2')\\
&\quad + \phi_2^{(12)} f_2(g_1(x_1',x_2),g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)} g_2(x_1,x_1',x_2').\phi_1^{(2)} g_1(x_1',x_2,x_2')\\
&\quad + \phi_2^{(22)} f_2(g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1,x_2'),g_2(x_1',x_2)).\phi_1^{(1)} g_2(x_1,x_1',x_2').\phi_1^{(2)} g_2(x_1,x_2,x_2')\\
&\quad + \phi_2^{(22)} f_2(g_1(x_1',x_2'),g_2(x_1,x_2'),g_2(x_1',x_2),g_2(x_1',x_2')).\phi_1^{(1)} g_2(x_1,x_1',x_2').\phi_1^{(2)} g_2(x_1',x_2,x_2')
\end{aligned}
$$

$$
\begin{aligned}
0 &= \phi_2^{(12)}(f\circ g)_1(x_1,x_1',x_2,x_2') \\
&= \phi_1^{(1)}f_1(g_1(x_1,x_2),g_1(x_1,x_2'),g_2(x_1,x_2)).\phi_2^{(12)}g_1(x_1,x_1',x_2,x_2') \\
&\quad + \phi_1^{(2)}f_1(g_1(x_1,x_2'),g_2(x_1,x_2),g_2(x_1,x_2')).\phi_2^{(12)}g_2(x_1,x_1',x_2,x_2') \\
&\quad + \phi_2^{(11)}f_1(g_1(x_1,x_2),g_1(x_1',x_2),g_1(x_1,x_2'),g_2(x_1,x_2)).\phi_1^{(1)}g_1(x_1,x_1',x_2).\phi_1^{(2)}g_1(x_1',x_2,x_2') \\
&\quad + \phi_2^{(11)}f_1(g_1(x_1',x_2),g_1(x_1,x_2'),g_1(x_1',x_2'),g_2(x_1,x_2)).\phi_1^{(1)}g_1(x_1,x_1',x_2').\phi_1^{(2)}g_1(x_1',x_2,x_2') \\
&\quad + \phi_2^{(21)}f_1(g_1(x_1',x_2),g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)}g_2(x_1,x_1',x_2).\phi_1^{(2)}g_1(x_1',x_2,x_2') \\
&\quad + \phi_2^{(12)}f_1(g_1(x_1,x_2'),g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1,x_2')).\phi_1^{(1)}g_1(x_1,x_1',x_2').\phi_1^{(2)}g_2(x_1',x_2,x_2') \\
&\quad + \phi_2^{(22)}f_1(g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1',x_2),g_2(x_1,x_2')).\phi_1^{(1)}g_2(x_1,x_1',x_2).\phi_1^{(2)}g_2(x_1',x_2,x_2') \\
&\quad + \phi_2^{(22)}f_1(g_1(x_1',x_2'),g_2(x_1',x_2),g_2(x_1,x_2'),g_2(x_1',x_2')).\phi_1^{(1)}g_2(x_1,x_1',x_2').\phi_1^{(2)}g_2(x_1',x_2,x_2')
\end{aligned}
$$

$$
\begin{aligned}
0 &= \phi_2^{(12)}(f\circ g)_2(x_1,x_1',x_2,x_2') \\
&= \phi_1^{(1)}f_2(g_1(x_1,x_2),g_1(x_1,x_2'),g_2(x_1,x_2)).\phi_2^{(12)}g_1(x_1,x_1',x_2,x_2') \\
&\quad + \phi_1^{(2)}f_2(g_1(x_1,x_2'),g_2(x_1,x_2),g_2(x_1,x_2')).\phi_2^{(12)}g_2(x_1,x_1',x_2,x_2') \\
&\quad + \phi_2^{(11)}f_2(g_1(x_1,x_2),g_1(x_1',x_2),g_1(x_1,x_2'),g_2(x_1,x_2)).\phi_1^{(1)}g_1(x_1,x_1',x_2).\phi_1^{(2)}g_1(x_1',x_2,x_2') \\
&\quad + \phi_2^{(11)}f_2(g_1(x_1',x_2),g_1(x_1,x_2'),g_1(x_1',x_2'),g_2(x_1,x_2)).\phi_1^{(1)}g_1(x_1,x_1',x_2').\phi_1^{(2)}g_1(x_1',x_2,x_2') \\
&\quad + \phi_2^{(21)}f_2(g_1(x_1',x_2),g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1',x_2)).\phi_1^{(1)}g_2(x_1,x_1',x_2).\phi_1^{(2)}g_1(x_1',x_2,x_2') \\
&\quad + \phi_2^{(12)}f_2(g_1(x_1,x_2'),g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1,x_2')).\phi_1^{(1)}g_1(x_1,x_1',x_2').\phi_1^{(2)}g_2(x_1',x_2,x_2') \\
&\quad + \phi_2^{(22)}f_2(g_1(x_1',x_2'),g_2(x_1,x_2),g_2(x_1',x_2),g_2(x_1,x_2')).\phi_1^{(1)}g_2(x_1,x_1',x_2).\phi_1^{(2)}g_2(x_1',x_2,x_2') \\
&\quad + \phi_2^{(22)}f_2(g_1(x_1',x_2'),g_2(x_1',x_2),g_2(x_1,x_2'),g_2(x_1',x_2')).\phi_1^{(1)}g_2(x_1,x_1',x_2').\phi_1^{(2)}g_2(x_1',x_2,x_2')
\end{aligned}
$$

$$
\begin{aligned}
0 &= \phi_2^{(22)}(f\circ g)_1(x_1,x_2,x_2',x_2'') \\
&= \phi_1^{(1)}f_1(g_1(x_1,x_2),g_1(x_1,x_2'),g_2(x_1,x_2)).\phi_2^{(22)}g_1(x_1,x_2,x_2',x_2'') \\
&\quad + \phi_1^{(2)}f_1(g_1(x_1,x_2'),g_2(x_1,x_2),g_2(x_1,x_2')).\phi_2^{(22)}g_2(x_1,x_2,x_2',x_2'') \\
&\quad + \phi_2^{(11)}f_1(g_1(x_1,x_2),g_1(x_1,x_2'),g_1(x_1,x_2''),g_2(x_1,x_2)).\phi_1^{(2)}g_1(x_1,x_2,x_2'').\phi_1^{(2)}g_1(x_1,x_2',x_2'') \\
&\quad + \phi_2^{(12)}f_1(g_1(x_1,x_2'),g_1(x_1,x_2''),g_2(x_1,x_2),g_2(x_1,x_2')).\phi_1^{(2)}g_2(x_1,x_2,x_2'').\phi_1^{(2)}g_1(x_1,x_2',x_2'') \\
&\quad + \phi_2^{(22)}f_1(g_1(x_1,x_2''),g_2(x_1,x_2),g_2(x_1,x_2'),g_2(x_1,x_2'')).\phi_1^{(2)}g_2(x_1,x_2,x_2'').\phi_1^{(2)}g_2(x_1,x_2',x_2'')
\end{aligned}
$$

$$
\begin{aligned}
0 &= \phi_2^{(22)}(f\circ g)_2(x_1,x_2,x_2',x_2'') \\
&= \phi_1^{(1)}f_2(g_1(x_1,x_2),g_1(x_1,x_2'),g_2(x_1,x_2)).\phi_2^{(22)}g_1(x_1,x_2,x_2',x_2'') \\
&\quad + \phi_1^{(2)}f_2(g_1(x_1,x_2'),g_2(x_1,x_2),g_2(x_1,x_2')).\phi_2^{(22)}g_2(x_1,x_2,x_2',x_2'') \\
&\quad + \phi_2^{(11)}f_2(g_1(x_1,x_2),g_1(x_1,x_2'),g_1(x_1,x_2''),g_2(x_1,x_2)).\phi_1^{(2)}g_1(x_1,x_2,x_2'').\phi_1^{(2)}g_1(x_1,x_2',x_2'')
\end{aligned}
$$

$$+ \phi_2^{(12)} f_2(g_1(x_1,x_2'),g_1(x_1,x_2''),g_2(x_1,x_2),g_2(x_1,x_2')).\phi_1^{(2)} g_2(x_1,x_2,x_2'').\phi_1^{(2)} g_1(x_1,x_2',x_2'')$$
$$+ \phi_2^{(22)} f_2(g_1(x_1,x_2''),g_2(x_1,x_2),g_2(x_1,x_2'),g_2(x_1,x_2'')).\phi_1^{(2)} g_2(x_1,x_2,x_2'').\phi_1^{(2)} g_2(x_1,x_2',x_2'')$$

We now get a system of 8 ($= n^{k+1}$) equations in the 8 unknowns
$\phi_2^{(11)} g_1(x_1,x_1',x_1'',x_2), \phi_2^{(11)} g_2(x_1,x_1',x_1'',x_2), \phi_2^{(21)} g_1(x_1,x_1',x_2,x_2'), \phi_2^{(21)} g_2(x_1,x_1',x_2,x_2'),$
$\phi_2^{(12)} g_1(x_1,x_1',x_2,x_2'), \phi_2^{(12)} g_2(x_1,x_1',x_2,x_2'), \phi_2^{(22)} g_1(x_1,x_2,x_2',x_2''), \phi_2^{(22)} g_2(x_1,x_2,x_2',x_2'')$
with determinant

$$D = \begin{vmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{vmatrix}$$

where

$$A = \begin{pmatrix} \phi_1^{(1)} f_1(g_1(x_1,x_2),g_1(x_1',x_2),g_2(x_1,x_2)) & \phi_1^{(2)} f_1(g_1(x_1',x_2),g_2(x_1,x_2),g_2(x_1',x_2)) \\ \phi_1^{(1)} f_2(g_1(x_1,x_2),g_1(x_1',x_2),g_2(x_1,x_2)) & \phi_1^{(2)} f_2(g_1(x_1',x_2),g_2(x_1,x_2),g_2(x_1',x_2)) \end{pmatrix}$$

and

$$B = \begin{pmatrix} \phi_1^{(1)} f_1(g_1(x_1,x_2),g_1(x_1,x_2'),g_2(x_1,x_2)) & \phi_1^{(2)} f_1(g_1(x_1,x_2'),g_2(x_1,x_2),g_2(x_1,x_2')) \\ \phi_1^{(1)} f_2(g_1(x_1,x_2),g_1(x_1,x_2'),g_2(x_1,x_2)) & \phi_1^{(2)} f_2(g_1(x_1,x_2'),g_2(x_1,x_2),g_2(x_1,x_2')) \end{pmatrix}$$

In the limit that $(x_1,x_2),(x_1',x_2),(x_1,x_2')$ tend to $f(a)$, D becomes equal to $(\det f'(a))^4$ which is different from 0. (Herein $4 = n^k$). So $D \neq 0$ in a neighbourhood of $f(a)$.
Since f is a C^2-function and g is a C^1-function, and using Cramers rule to solve a system of linear equations, we see that all second order difference quotients of g are continuously extendable. Thus g is a C^2-function.

Remark
In the proof above we could reduce our system to a system of 6 equations in 6 unknowns, since $\phi_2^{(21)} g_1(x_1,x_1',x_2,x_2') = \phi_2^{(12)} g_1(x_1,x_1',x_2,x_2')$ and $\phi_2^{(21)} g_2(x_1,x_1',x_2,x_2') = \phi_2^{(12)} g_2(x_1,x_1',x_2,x_2')$, but then it is more difficult to see the generalization to arbitrary n and k.

ACKNOWLEDGEMENT : The author wishes to thank professor W. Schikhof for pointing this subject to his attention.

4 References

[1] S. DE SMEDT, The van der Put base for C^n-functions, Bulletin of the Belgian Mathematical Society - Simon Stevin 1(1), 1994, p.85-98

[2] S. DE SMEDT, p-adic continuously differentiable functions of several variables, Collectanea Mathematica 45(2), 1994, p.137-152

[3] W.H. SCHIKHOF, Non-Archimedean calculus (Lecture notes), Report 7812, Katholieke Universiteit Nijmegen, 1978

[4] W.H. SCHIKHOF, Ultrametric Calculus : an introduction to p-adic analysis, Cambridge University Press, 1984

Stany De Smedt
Free University of Brussels
Faculty of Applied Sciences
Pleinlaan 2
B-1050 Brussels
Belgium

Manuscrit reçu en Décembre 1997