

P.N. NATARAJAN

V. SRINIVASAN

## **Convolution of Nörlund methods in non-archimedean fields**

*Annales mathématiques Blaise Pascal*, tome 4, n° 2 (1997), p. 41-47

<[http://www.numdam.org/item?id=AMBP\\_1997\\_\\_4\\_2\\_41\\_0](http://www.numdam.org/item?id=AMBP_1997__4_2_41_0)>

© Annales mathématiques Blaise Pascal, 1997, tous droits réservés.

L'accès aux archives de la revue « Annales mathématiques Blaise Pascal » (<http://math.univ-bpclermont.fr/ambp/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

## Convolution of Nörlund methods in non-archimedean fields

P.N. Natarajan

Department of Mathematics

Ramakrishna Mission Vivekananda College

Chennai - 600 004

India

V. Srinivasan

Department of Mathematics

V. Ramakrishna Polytechnic

Chennai - 600 019

India

### Abstract

In the present paper we obtain a few inclusion theorems for the convolution of Nörlund methods in the form  $(N, r_n) \subseteq (N, p_n) * (N, q_n)$  in complete, non-trivially valued, non-archimedean fields.

Throughout the present paper  $K$  denotes a complete, non-trivially valued, non-archimedean field. Infinite matrices and sequences, which are considered in the sequel, have entries in  $K$ . If  $A = (a_{nk})$ ,  $a_{nk} \in K$ ,  $n, k = 0, 1, 2, \dots$  is an infinite matrix, the  $A$ -transform  $Ax = \{(Ax)_n\}$  of the sequence  $x = \{x_k\}$ ,  $x_k \in K$ ,  $k = 0, 1, 2, \dots$  is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \quad n = 0, 1, 2, \dots,$$

where it is assumed that the series on the right converge. If  $\lim_{n \rightarrow \infty} (Ax)_n = s$ , we say that  $\{x_k\}$  is  $A$ -summable to  $s$ , written as  $x_k \rightarrow s(A)$  or  $A\text{-lim } x_k = s$ . If  $\lim_{n \rightarrow \infty} (Ax)_n = s$  whenever  $\lim_{k \rightarrow \infty} x_k = s$ , we say that  $A$  is regular. The following result is well-known (see [2], [4]).

**Theorem 1**  $A = (a_{nk})$  is regular if and only if

$$\sup_{n,k} |a_{nk}| < \infty; \quad (1)$$

$$\lim_{n \rightarrow \infty} a_{nk} = 0, \quad \text{for every fixed } k; \quad (2)$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{nk} = 1. \quad (3)$$

Any matrix  $A$  for which (1) holds is called a  $K_r$ -matrix. If  $A$  and  $B$  are two infinite matrices such that  $x_k \rightarrow s(A)$  implies  $x_k \rightarrow s(B)$ , we say that  $A$  is included in  $B$ , written as  $A \subseteq B$ .  $A$  is said to be row-finite if for  $n = 0, 1, 2, \dots$ , there exists a positive integer  $k_n$  such that  $a_{nk} = 0, k > k_n$ .

Given two infinite matrices  $A$  and  $B$ , their convolution is defined as the matrix  $C = (c_{nk})$ , where

$$c_{nk} = \sum_{i=0}^k a_{ni} b_{n,k-i}, \quad n, k = 0, 1, 2, \dots \quad (4)$$

In such a case we write  $C = A * B$ .

The following properties of the convolution can be easily proved.

1. If  $A$  and  $B$  are both row-finite or both  $K_r$ , then their convolution  $C$  is row-finite or  $K_r$  respectively and their row sums satisfy

$$\sum_{k=0}^{\infty} c_{nk} = \left( \sum_{k=0}^{\infty} a_{nk} \right) \left( \sum_{k=0}^{\infty} b_{nk} \right), \quad n = 0, 1, 2, \dots \quad (5)$$

2. If  $A, B$  are both regular, then  $C$  is regular too.

The Nörlund method of summability i.e.,  $(N, p_n)$  method in  $K$  is defined as follows (see [5]):  $(N, p_n)$  is defined by the infinite matrix  $(a_{nk})$  where

$$\begin{aligned} a_{nk} &= \frac{p_{n-k}}{P_n}, \quad k \leq n; \\ &= 0, \quad k > n, \end{aligned}$$

where  $p_0 \neq 0, |p_0| > |p_j|, j = 1, 2, \dots$  and  $P_n = \sum_{k=0}^n p_k, n = 0, 1, 2, \dots$ . It is to be noted that  $|P_n| = |p_0| \neq 0$  so that  $P_n \neq 0, n = 0, 1, 2, \dots$ .

The following result is very useful in the sequel.

**Theorem 2** (See [5], Theorem 1.)  $(N, p_n)$  is regular if and only if

$$p_n \rightarrow 0, \quad n \rightarrow \infty. \quad (6)$$

The purpose of the present paper is to prove a few inclusion theorems for the convolution of Nörlund methods in the form  $(N, r_n) \subseteq (N, p_n) * (N, q_n)$ .

We need to define  $\{\bar{p}_n\}$  by

$$p_0\bar{p}_0 = 1, \quad p_0\bar{p}_n + p_1\bar{p}_{n-1} + \cdots + p_n\bar{p}_0 = 0, \quad n \geq 1 \quad (7)$$

i.e.,  $\bar{p}(x) = \sum_{n=0}^{\infty} \bar{p}_n x^n = \frac{1}{\sum_{n=0}^{\infty} p_n x^n} = \frac{1}{p(x)}$ , assuming that these series converge.

The following result is an easy consequence of Kojima-Schur theorem (see [2], [4]).

**Lemma 1** *Let  $A = (a_{nk})$  be a row-finite matrix and  $(N, p_n)$  be a regular Nörlund method. Then  $A$ -lim  $x_k$  exists whenever  $(N, p_n)$ -lim  $x_k$  exists if and only if*

$$\sup_{0 \leq \gamma \leq k_n} |P_\gamma \sum_{k=\gamma}^{k_n} a_{nk} \bar{p}_{k-\gamma}| = O(1), \quad n \rightarrow \infty; \quad (8)$$

$$\lim_{n \rightarrow \infty} \sum_{k=\gamma}^{k_n} a_{nk} \bar{p}_{k-\gamma} = \delta_\gamma, \quad \text{for every fixed } \gamma; \quad (9)$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{k_n} a_{nk} = \delta. \quad (10)$$

**Corollary 1**  $(N, p_n) \subseteq A$  if and only if (8), (9) and (10) hold with  $\delta_\gamma = 0$ ,  $\gamma = 0, 1, 2, \dots$  and  $\delta = 1$ .

**Corollary 2** If  $(N, p_n)$  and  $(N, q_n)$  are regular Nörlund methods, then  $(N, p_n) \subseteq (N, q_n)$  if and only if  $h_n \rightarrow 0$ ,  $n \rightarrow \infty$  where

$$h(x) = \sum_{n=0}^{\infty} h_n x^n = \frac{\sum_{n=0}^{\infty} q_n x^n}{\sum_{n=0}^{\infty} p_n x^n} = \frac{q(x)}{p(x)}$$

(see [5]).

Let  $(N, p_n)$ ,  $(N, q_n)$ ,  $(N, r_n)$  be regular Nörlund methods. Let  $p_n(x) = \sum_{i=n}^{\infty} p_i x^i$ ,  $p_0(x) = p(x)$  with similar expressions for  $q_n(x)$ ,  $r_n(x)$ . Let

$$\begin{aligned} \frac{p(x)q(x)}{r(x)} &= \sum_{\gamma=0}^{\infty} \theta_\gamma x^\gamma; \\ \frac{p_{n+1}(x)q(x)}{r(x)} &= \sum_{\gamma=0}^{\infty} \alpha_{n\gamma} x^\gamma; \\ \frac{p(x)q_{n+1}(x)}{r(x)} &= \sum_{\gamma=0}^{\infty} \beta_{n\gamma} x^\gamma, \end{aligned} \quad (11)$$

and

$$\frac{1}{r(x)} \{p(x)q(x) - p_{n+1}(x)q(x) - p(x)q_{n+1}(x)\} = \sum_{\gamma=0}^{\infty} \varphi_{n\gamma} x^\gamma. \quad (12)$$

It now follows that

$$\varphi_{n\gamma} = \theta_\gamma - \alpha_{n\gamma} - \beta_{n\gamma} \quad (13)$$

and

$$\alpha_{n\gamma} = \beta_{n\gamma} = 0, \quad 0 \leq \gamma \leq n. \quad (14)$$

Taking  $C = (N, p_n) * (N, q_n)$ ,  $C$  is a row-finite matrix with

$$c_{nk} = \frac{1}{P_n Q_n} \sum_{i=\max(0, k-n)}^{\min(k, n)} p_{n-i} q_{n-k+i} \quad \text{with } k_n = 2n. \quad (15)$$

We write

$$f_{n\gamma} = \sum_{k=\max(0, 2n-\gamma)}^{2n} c_{nk} \bar{r}_{k+\gamma-2n}, \quad n, \gamma \geq 0. \quad (16)$$

**Lemma 2**

$$P_n Q_n f_{n\gamma} = \varphi_{n\gamma}, \quad 0 \leq \gamma \leq 2n + 1. \quad (17)$$

**Proof.** The result follows as in [6].

**Theorem 3**  $(N, p_n) * (N, q_n)$ -lim  $x_k$  exists whenever  $(N, r_n)$ -lim  $x_k$  exists if and only if

$$\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} \varphi_{n\gamma}| = O(1), \quad n \rightarrow \infty; \quad (18)$$

and

$$\lim_{n \rightarrow \infty} \frac{\varphi_{n, 2n-\gamma}}{P_n Q_n} = \delta_\gamma, \quad \text{for every fixed } \gamma. \quad (19)$$

**Proof.** Let  $(N, p_n) * (N, q_n)$ -lim  $x_k$  exist whenever  $(N, r_n)$ -lim  $x_k$  exists. Applying Lemma 1 with  $(N, p_n) = (N, r_n)$  and  $A = (N, p_n) * (N, q_n) = (c_{nk})$ , we have,

$$\sup_{0 \leq \gamma \leq 2n} |R_\gamma \sum_{k=\gamma}^{2n} c_{nk} \bar{r}_{k-\gamma}| = O(1), \quad n \rightarrow \infty \quad (20)$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=\gamma}^{2n} c_{nk} \bar{r}_{k-\gamma} = \delta_\gamma, \quad \text{for every fixed } \gamma. \quad (21)$$

Using (16) and (20), we get

$$\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} f_{n\gamma}| = O(1), \quad n \rightarrow \infty. \quad (22)$$

Using (17), we note that  $|f_{n\gamma}| = \frac{|\varphi_{n\gamma}|}{|p_0| |q_0|}$  since  $|P_n| = |p_0|$  and  $|Q_n| = |q_0|$ . Consequently, in view of (22), we get

$$\sup_{0 \leq \gamma \leq 2n} |R_{2n-\gamma} \varphi_{n\gamma}| = O(1), \quad n \rightarrow \infty.$$

In view of (16) and (21), we have

$$\lim_{n \rightarrow \infty} f_{n, 2n-\gamma} = \delta_\gamma, \quad \text{for every fixed } \gamma.$$

Now, using (17), we get

$$\lim_{n \rightarrow \infty} \frac{\varphi_{n, 2n-\gamma}}{P_n Q_n} = \delta_\gamma, \quad \text{for every fixed } \gamma.$$

Thus (18) and (19) hold. Conversely (18) and (19) imply (20) and (21) respectively. Using (5), we have,  $\lim_{n \rightarrow \infty} \sum_{k=0}^{2n} c_{nk} = 1$ . Using Lemma 1, the result follows, completing the proof of the theorem.

**Corollary 3**  $(N, r_n) \subseteq (N, p_n) * (N, q_n)$  if and only if (18) and (19) hold with  $\delta_\gamma = 0$ .

**Corollary 4** If  $\lim_{n \rightarrow \infty} \bar{r}_n = 0$ , then  $(N, r_n) \subseteq (N, p_n) * (N, q_n)$  if and only if (18) holds.

**Proof.** The result follows using (9) and the fact that  $(N, p_n) * (N, q_n)$  is regular.

**Theorem 4** If

$$\varphi_{n, 2n-\gamma} = o(1), \quad n \rightarrow \infty, \quad \text{for every fixed } \gamma, \quad (23)$$

and either

$$\varphi_{n\gamma} = O(1), \quad n, \gamma \rightarrow \infty, \quad (24)$$

or

$$\theta_\gamma, \alpha_{n\gamma}, \beta_{n\gamma} = O(1), \quad n, \gamma \rightarrow \infty, \quad (25)$$

then

$$(N, r_n) \subseteq (N, p_n) * (N, q_n).$$

**Proof.** Using (23), (19) follows with  $\delta_\gamma = 0$  since  $|P_n| = |p_0|$  and  $|Q_n| = |q_0|$ . Because of (13) and (25), (24) holds. So if (24) or (25) holds, (18) holds since  $R_n = O(1)$ ,  $n \rightarrow \infty$ ,  $(N, r_n)$  being a regular method. The result now follows from Corollary 3.

We shall now take up an application of Theorem 4.

**Theorem 5** Let  $\bar{p}_n, \bar{q}_n \rightarrow 0$ ,  $n \rightarrow \infty$  and  $t_n = p_0q_n + p_1q_{n-1} + \dots + p_nq_0$ ,  $n = 0, 1, 2, \dots$ . Then

$$(N, t_n) \subseteq (N, p_n) * (N, q_n)$$

and

$$(N, p_n) \subseteq (N, t_n), \quad (N, q_n) \subseteq (N, t_n).$$

**Proof.** We apply Theorem 4 with  $r_n = t_n$ . With the usual notation we have  $t(x) = p(x)q(x)$  and  $\bar{t}(x) = \bar{p}(x)\bar{q}(x)$ . Since  $\bar{p}_n, \bar{q}_n \rightarrow 0$ ,  $n \rightarrow \infty$ ,  $\bar{t}_n \rightarrow 0$ ,  $n \rightarrow \infty$  (see [3], Theorem 1). Consequently (23) follows using (9). In view of (11), we have,

$$\sum_{\gamma=0}^{\infty} \theta_{\gamma} x^{\gamma} = \frac{p(x)q(x)}{t(x)} = 1,$$

so that

$$\theta_0 = 1 \text{ and } \theta_{\gamma} = 0, \quad \gamma \geq 1;$$

$$\sum_{\gamma=0}^{\infty} \alpha_{n\gamma} x^{\gamma} = \frac{p_{n+1}(x)q(x)}{t(x)} = p_{n+1}(x)\bar{p}(x),$$

so that

$$\begin{aligned} \alpha_{n\gamma} &= \sum_{\lambda=0}^{\gamma-(n+1)} \bar{p}_{\lambda} p_{\gamma-\lambda}, \quad \gamma \geq n+1; \\ &= 0, \quad 0 \leq \gamma \leq n. \end{aligned}$$

Consequently  $\alpha_{n\gamma} = O(1)$ ,  $n, \gamma \rightarrow \infty$ . Similarly  $\beta_{n\gamma} = O(1)$ ,  $n, \gamma \rightarrow \infty$ . In view of Theorem 4,  $(N, t_n) \subseteq (N, p_n) * (N, q_n)$ . Now  $\frac{t(x)}{p(x)} = q(x)$  and  $q_n \rightarrow 0$ ,  $n \rightarrow \infty$ ,  $(N, q_n)$  being regular, by (6). So by Corollary 2,  $(N, p_n) \subseteq (N, t_n)$ . Similarly  $(N, q_n) \subseteq (N, t_n)$ . The proof of the theorem is now complete.

## References

- [1] G. Bachman, *Introduction to p-adic numbers and valuation theory*, Academic Press, 1964.
- [2] A.F. Monna, Sur le théorème de Banach-Steinhaus, *Indag. Math.* 25 (1963), 121-131.
- [3] P.N. Natarajan, Multiplication of series with terms in a non-archimedean field, *Simon Stevin* 52 (1978), 157-160.
- [4] P.N. Natarajan, Criterion for regular matrices in non-archimedean fields, *J. Ramanujan Math. Soc.* 6 (1991), 185-195.

- [5] P.N. Natarajan, On Nörlund method of summability in non-archimedean fields, *J. Analysis* 2 (1994), 97-102.
- [6] D.C. Russell, Convolution of Nörlund summability methods, *Proc. London Math. Soc.* 9 (1959), 1-20.

Manuscrit reçu en Septembre 1997