ANNALES MATHÉMATIQUES BLAISE PASCAL

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Annales mathématiques Blaise Pascal, tome 4, nº 2 (1997), p. 11-25
<http://www.numdam.org/item?id=AMBP_1997__4_2_11_0>

Ann. Math. Blaise Pascal, Vol. 4, N° 2, 1997, pp.11-25

Wick product and stochastic partial differential equations with Poisson measure

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Abstract. We establish the existence and uniqueness of the solution for one conservation equation perturbed by a Poisson noise and when the initial value is affine. We give also the existence and uniqueness of the solution for a multidimensional linear Skorohod stochastic differential equation driven by a Poisson measure.

Résumé. En utilisant la théorie de noyaux et symboles sur l'espace de Fock, nous obtenons un résultat d'existence et d'unicité de la solution à une équation de conservation perturbée par un bruit Poissonien et lorsque la condition initiale est affine. Nous donnons aussi un résultat d'existence et d'unicité de la solution à une équation différentielle stochastique linéaire multidimensionnelle au sens de Skorohod et perturbée par un bruit Poissonien.

1 Introduction

In stochastic analysis on the Wiener space the Wick product is natural because it is implicit in the Itô integral (and, more generally, in the Skorohod integral if the integrand is anticipating). The Skorohod integral coincides with the Itô integral if the integrand is non-anticipating see e.g. Nualart, Pardoux (1988). Today the Wick product and the Skorohod integral are important in the study of stochastic ordinary and partial differential equations see e.g. Nualart, Zakai (1989), Holden, Lindstrom, Oksendal, Uboe, Zhang (1994). In the Poisson case some works were also done on this subject Dermoune, Krée, Wu (1988), Carlen, Pardoux (1988), Nualart, Vives (1990), Dermoune (1995). Here we study one stochastic partial differential equation (SPDE) and one stochastic differential equation (SDE) in order to show that the Wick product also arrives naturally in the Poisson case. The first SPDE is the following conservation equation perturbed by a Poisson measure (see e.g. Whitham (1974)) :

$$\partial_t(u(x,t) - q((0,t])) + \partial_x(a\frac{u(x,t)^2}{2} + bu(x,t)) = 0 \quad x \in \mathbb{R}, t > 0, \tag{1}$$

where $q(t) = \sum_i \delta_{t_i}(t) - dt$ is the centered Poisson measure on $\mathbb{R}_+$, and a, b are two random variables on Poisson space. ∂_t, ∂_x denote the partial derivatives with respect to t and x. Since $t \to q(t)$ is a distribution and not a smooth function, we interpret the equation (1) in the distribution sense, and the products $(au(x,t)+b)\partial_x u(x,t)$ are interpreted as the Wick product of generalized random variables. The second SDE concerns the following problem : Let E be a locally compact set, $\mathcal{B}$ be the Borel σ-field over $[0,1]\times E$, and ν be a positive Radon measure on $([0,1]\times E,\mathcal{B})$ such that $\nu(\{(t,x)\})=0$, $\forall (t,x)\in[0,1]\times E$. We denote by q the centered Poisson random measure on $[0,1]\times E$ with intensity ν, and $\mathcal{F}_t, 0\le t\le 1$, is the natural filtration of q. Consider the SDE of the form

$$X_t = X_0 + \int_0^t \int_E A(s,x)X_{s-}dq(s,x) + \int_0^t B_s X_s ds, \quad 0\le t\le 1, \qquad (2)$$

where X_0 is a d-dimensional random vector, $A(s,x)$ and B_s are $d\times d$ deterministic matrices. The latter SDE was studied by Buckdahn, Nualart (1994), when the Wiener noise takes the place of the Poisson measure q. If X_0 is not a deterministic vector then the process (X_{s-}) is anticipating, and the integral $\int_0^t\int_E A(s,x)X_{s-}dq(s,x)$ can not be defined in Itô's sense. Nevertheless if $A\in L^1([0,1]\times E,\nu)$, and $B\in L^1([0,1],dt)$ then this integral can be interpreted as a pathwise integral with respect to q, and the equation (2) has a unique pathwise solution

$$X_t = \left[I_d + \sum_{n=1}^{\infty}\int_{0<t_1<\dots<t_n<t}\left(\int_E A(t_1,x_1)dq(t_1,x_1) + B(t_1)dt_1\right)\right.$$
$$\left.\dots\left(\int_E A(t_n,x_n)dq(t_n,x_n) + B(t_n)dt_n\right)\right] X_0,$$

where I_d is the identity of $d\times d$ matrices. But if $A\in L^2([0,1]\times E,\nu)$ then the equation (2) has neither pathwise interpretation nor Itô's interpretation. In this case we propose to interpret the stochastic integral $\int_0^t\int_E A(s,x)X_{s-}dq(s,x)$ in the Skorohod sense given by Dermoune, Krée, Wu (1988), Nualart, Vives (1990), and to study the equation (2) in this sense. Note that if X_{s-} is non-anticipating then the Skorohod integral of $(A(s,x)X_{s-})$ coincides with its Itô's integral. But if (X_{s-}) is anticipating and $A \in L^2([0,1] \times E, \nu) \bigcap L^1([0,1] \times E, \nu)$, then the pathwise and the Skorohod interpretations do not coincide.

The plan and the main results of this work are the following : in the section 2 we recall some definitions and results Krée (1986), Dermoune, Krée, Wu (1988), Nualart, Vives (1990), which will be used in the section 3. The combined characteristics and symbol method is used in the first part of the section 3 to establish the existence and the uniqueness of the solution for the SPDE (1) when the initial value is affine. In the second part of section 3 we show that nearly all the results of Buckdahn, Nualart (1994) rest valid for the SDE (2) and we emphasize some differences.

2 Differential calculus relative to the Poisson measure

Let X be a locally compact set, $\mathcal{B}$ be the Borel σ-field over X, and μ be a positive Radon measure on $(X, \mathcal{B})$ such that $\mu(\{x\}) = 0$, $\forall x \in X$. Let U be a subspace of $L^1(X, \mathcal{B}, \mu) \bigcap L^2(X, \mathcal{B}, \mu)$, which is dense in $L^2(X, \mathcal{B}, \mu)$. The set $\Omega := M_p(X)$ is the space of Radon point measures on X. A generic element of U is denoted by z, and a generic element of Ω is denoted by ω. The duality between Ω and U is denoted by $< \omega, z >$. We have the triplet $U \hookrightarrow H = L^2(X, \mathcal{B}, \mu) \hookrightarrow \Omega$. The scalar product over H is denoted by $< \cdot, \cdot >$, and the norm by $|| \cdot ||_2$. The triplet $(\Omega, \mathcal{F}, P)$ is the probability space of the Poisson measure on $(X, \mathcal{B}, \mu)$.

We denote by $L^2(\Omega)$ the space of the square integrable random variables with respect to P, and $I\!E$ denotes the expectation. The centered Poisson measure q is defined by $q(z) =< \omega, z > - \int_X z(x)d\mu(x)$. It follows from the characteristic functions of P that $I\!E\left[|q(z)|^2\right] = ||z||_2^2$. From this we see that if $z \in H$, and we choose $z_n \in U$ such that $z_n \to z$ in H then $q(z) := \lim_{n\to\infty} q(z_n)$ in H and the limit is independent of the choice of $\{z_n\}$.

Wiener-Itô isomorphism. The symmetric Fock space over H is defined by $Fock(H) = \oplus_{k=0}^{\infty} H^{\odot k}$, $H^{\odot 0} = I\!R$ and for $k \in I\!N*$, the space $H^{\odot k}$ is the set of the class of square integrable functions with respect to $\mu^{\otimes k}$, which are symmetric with respect to the k parameters $x_1, ..., x_k$. We will denote the norm over $H^{\odot k}$ also by $||\cdot||_2$. The scalar product $< \cdot, \cdot >$ over $Fock(H)$ is defined by $< (f_k), (g_k) >= \sum_{k=0}^{\infty} k! < f_k, g_k >$. For $h \in H$, we denote by e^h the exponential vector, element of $Fock(H)$, defined by $e^h = \oplus_{k=0}^{\infty} \frac{h^{\otimes k}}{k!}$; $h^{\otimes 0} := 1$. For $k \in I\!N*$ and for $f_k \in H^{\odot k}$, the random variable $I_k(f_k)$ is the symmetric multiple integral with respect to q defined in Surgailis (1984) and denoted formally by $I_k(f_k) = \int_{X^k} f_k(x_1, ..., x_k)\, dq(x_1)...dq(x_k)$. The random variables $F_k = I_k(f_k)$; $k \in I\!N$ are such that

$$I\!E[\,|F_k|^2\,] = k!||f_k||_2^2, \quad I\!E[\,F_k F_j\,] = 0, \quad \text{for } j \neq k. \tag{3}$$

The Wiener-Itô expansion for the centered Poisson measure q means the isomorphism I from $Fock(H)$ into $L^2(\Omega)$ defined by $(f_k) \in Fock(H) \to F = \sum_{k=0}^{\infty} I_k(f_k)$. For $z \in H \bigcap L^1(X, \mu)$, the image $\mathcal{E}(z)$ of the exponential vector

e^z by the Wiener-Itô isomorphism I is given by

$$\mathcal{E}(z) := I(e^z)(\omega) = e^{-\int_X z(x)d\mu(x)} \prod_j^{n(\omega)} (1 + z(x_j)) \tag{4}$$

where $\omega = \sum_{j=1}^{n(\omega)} \delta_{x_j}$. We can see from (3) that for all $f_k \in H^{\odot k}, z \in H$,

$$\mathbb{E}\left[I_k(f_k)\mathcal{E}(z)\right] = < f_k, z^{\otimes k} > . \tag{5}$$

Distributions on $(\Omega, \mathcal{F}, \mathbf{P})$. Let $S(U)$ be the space of all tensors $z^{\otimes n}$ over U. It is well known that the set $S(U)^*$ of linear forms on $S(U)$ is the space of formal series on U, and we have the triplet

$$S(U) \hookrightarrow Fock(H) \hookrightarrow S(U)^* \tag{6}$$

We define the duality between $S(U)$ and $S(U)^*$ as an extension of the scalar product over $Fock(H)$, i.e. for all $z \in U$, $F = \sum_n^\infty F_n \in S(U)^*$, $< F, z^{\otimes n} >= n!F_n(z)$. From (6), an element of $Fock(H)$ is interpreted as a formal series on U defined by $z \in H \rightarrow < F, e^z > := F(z)$.

Generalized random variables. For simplicity we put $z^{\otimes n} = z^n$. The image of $S(U)$ by the isometry I is the space $P(\Omega)$ spanned by $I_k(z^k)$, $k \in \mathbb{N}$, and $z \in U$. Moreover, if $z \in \bigcap_{1 \le p < \infty} L^p(X)$ then $I_k(z^k) \in \bigcap_{1 \le p < \infty} L^p(\Omega)$, see proposition 3.1 in Surgailis (1984). The set $P(\Omega)^*$ of linear forms on $P(\Omega)$ is called the set of generalized random variables. From that we obtain the triplet

$$P(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow P(\Omega)^*. \tag{7}$$

The isomorphism I from $Fock(H)$ into $L^2(\Omega)$ is such that $I(S(U)) = P(\Omega)$. Thus the transpose I^*, of the restriction of I from $S(U)$ into $P(\Omega)$, defines an isomorphism from $P(\Omega)^*$ into $S(U)^*$. The Wiener-Itô expansion I is extended to an isomorphism between the triplets (6) and (7), and we denote it also by I. For $\lambda > 0$ the space H_λ, defined by $F = \sum_{n=0}^{\infty} I_n(f_n) \in H_\lambda$ if $||F||_\lambda = \sum_{n=0}^{\infty} n!\lambda^{2n}||f_n||_2^2 < \infty$, is a subspace of $P(\Omega)^*$.

Gradient operator and divergence operator. In the triplet (6) the gradient operator D is defined from $S(U)$ to $S(U)\otimes U$ by $Dz^n = nz^{n-1}\otimes z, \quad \forall z \in U, n \in \mathbb{N}*$. The transpose D^T of D defines the divergence operator from $(S(U)\otimes U)^*$ to $S(U)^*$, and we have for all $F \in (S(U)\otimes U)^*$ and $z \in U$ the following equality in the formal series sense $\sum_{n=0}^{\infty} \frac{1}{n!} < D^T F, z^n >= \sum_{n=0}^{\infty} \frac{1}{n!} < F, Dz^n >$.

Skorohod integral with respect to the centered Poisson measure. Using the isomorphism I between the triplets (6) and (7) we define the gradient operator ∇ on $P(\Omega)$ and the divergence operator δ on $(P(\Omega)\otimes U)^*$. An element of $(P(\Omega)\otimes U)^*$ is called a generalized stochastic process. For $F \in (P(\Omega)\otimes U)^*$ we have the generalized integration by parts formula $< \delta(F), I_n(z^n) >=$ $< F, \nabla I_n(z^n) >$, where $< \cdot, \cdot >$ denotes the duality between $P(\Omega)^*$ and $P(\Omega)$. If $X = [0,1]\times E$ then the Itô integral with respect to the centered Poisson measure q is extended by the divergence operator δ, Dermoune, Krée, Wu (1988).

Wick product of two generalized random variables. A generalized random variable $F \in P(\Omega)^*$ is characterized by its symbol $z \in U \to F(z) =$ $\sum_{n=0}^{\infty} \frac{1}{n!} < F, I_n(z^n) >$. If $F, G \in P(\Omega)^*$ then the Wick product $F\diamond G$ of F and G is the element of $P(\Omega)^*$, defined by $F\diamond G(z) = F(z)G(z), \forall z \in U$.

Example. i) Let $x \in X$, the generalized random variable $q(x)$ is defined by the symbol $z \to q(x)(z) = z(x)$. If F is a generalized random variable then the Wick product $F \diamond q(x)$ has the symbol $z \to F(z)z(x)$.

ii) For $n \in \mathbb{N}*$, and $F \in P(\Omega)^*$, we denote by $F^{\diamond n}$ the Wick product $F \diamond \ldots \diamond F$ n-times. If $f(t) = \sum_n a_n t^n$ is an analytic function, then $f(F)$ means the sum $\sum_n a_n F^{\diamond n}$.

3 The main results

Here $U = C_c^1(\mathbb{R}_+)$ is the set of continuously differentiable functions from $\mathbb{R}_+$ to $\mathbb{R}$, with compact support, and $C^1(\mathbb{R} \times \mathbb{R}_+, P(\Omega)^*)$ is the set of functions $(x,t) \to u(x,t) \in P(\Omega)^*$ which have the symbol $u(x,t,z)$ continuously differentiable with respect to $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. We suppose that $a, b \in P(\Omega)^*$, and we seek solution of the SPDE (1) in the space $C^1(\mathbb{R} \times \mathbb{R}_+, P(\Omega)^*)$, with initial value $x \to u(x,0)$ in $C^1(\mathbb{R}, P(\Omega)^*)$. If u is such a solution then its symbol $u(x,t,z)$ is solution of the deterministic following quasi-linear PDE :

$$\partial_t u(x,t,z) + (a(z)u(x,t,z) + b(z))\partial_x u(x,t,z) = z(t), \tag{8}$$

the initial value $x \to u(x,0,z)$ belongs to $C^1(\mathbb{R})$. The system of characteristic equations, for the PDE (8), see e.g. Whitham (1974), are

$$\frac{dX}{dt} = a(z)V + b(z), \quad \frac{dV}{dt} = z(t). \tag{9}$$

Under the initial conditions

$$X(t=0) = y, \quad V(t=0) = u(y,0,z)$$

the solution of (9) is given by

$$X(y,t,z) = a(z)\int_0^t (t-s)z(s)ds + (a(z)u(y,0,z) + b(z))t + y, \tag{10}$$

and

$$V(y,t,z) = u(y,0,z) + \int_0^t z(s)ds. \tag{11}$$

Thus, the solution $u(x,t,z)$ of (8) may be described in two different ways: on the one hand we may consider it at a fixed point of space, at time t. This is the so-called Eulerian description. On the other hand, we may follow the wave evolution along the characteristic $X(y,t,z)$, defined by the initial coordinate y. This description is a Lagrangian one, with y the Lagrangian coordinate. To pass from the Lagrangian approach to the Eulerian one it is necessary to find an initial Lagrangian coordinate $y(x,t)$ such that $x = X(y(x,t),t,z)$. From (10), (11) we have

$$u(x,t,z) = u(y,0,z) + \int_0^t z(s)ds, \tag{12}$$

and

$$y = x - a(z)\int_0^t (t-s)z(s)ds - ta(z)V(y,t,z) + ta(z)\int_0^t z(s)ds - tb(z).$$

If the initial value is affine then we have the following result.

Proposition 3.1. *Let the initial value $u(y,0) = \alpha y + \beta$, where $\alpha, \beta \in P(\Omega)^*$. Suppose that $\alpha(0)a(0) \geq 0$, then $z \to (a(z)\alpha(z)t + 1)^{-1}$ is the symbol of a generalized random variable denoted by $A(\alpha,a,t)$, and the SPDE (1) has a* *unique solution in* $C^1(\mathbb{R} \times \mathbb{R}_+, P(\Omega)^*)$ *given by*

$$u(x,t) = A(\alpha, a, t) \diamond \left(x\alpha - t\alpha \diamond b + \alpha \diamond a \diamond \int_0^t s dq(s) + \beta + q((0,t]) \right).$$

In particular if a, α *are deterministic and such that* $\alpha a \geq 0$ *then*

$$u(x,t) = (a\alpha t + 1)^{-1}(x\alpha - t\alpha b + a\alpha \int_0^t s dq(s) + \beta + q((0,t]).$$

But if $\alpha a < 0$ *then this solution is only defined for* $0 < t < -\alpha^{-1} a^{-1}$.

Proof. First, it is well known that a series $z \to S(z)$, such that $S(0) \neq 0$, is invertible in $S(U)^*$. From that and from the condition $a(0)\alpha(0) \geq 0$ we conclude that for all $t \in \mathbb{R}_+$, $z \to (a(z)\alpha(z)t + 1)^{-1}$ belongs to $S(U)^*$. It follows that $z \to (a(z)\alpha(z)t + 1)^{-1}$ is the symbol of a generalized random variable $A(\alpha, a, t)$. Under the hypothesis on the initial value and from (12) we have

$$u(x,t,z) = (t\alpha(z)a(z) + 1)^{-1} \left(x\alpha - tb(z)\alpha(z) + a(z)\alpha(z) \int_0^t s z(s) ds + \beta(z) + \int_0^t z(s) ds \right).$$

It is easy to see from (5) that $\int_0^t s z(s) ds, \int_0^t z(s) ds$ are respectively the symbol of the random variables $\int_0^t s dq(s)$ and $q((0,t])$. From that and from the definition of the Wick product we have the result.

Example. Suppose that α is deterministic, $a = q((0,T])$ and $T > 0$. The symbol $A(\alpha, a, t, z)$ of the generalized random variable $A(\alpha, a, t)$ is given by

$$\left(1 + \int_0^T z(s) ds) t\alpha \right)^{-1} = \sum_{n=0}^{\infty} (-\alpha t)^n (\int_0^T z(s) ds)^n.$$

It is easy to prove that $(\int_0^T z(s) ds)^n$ is the symbol of $I_n(1_{(0,T]}^{\otimes n})$. From that we have $A(\alpha, a, t) = \sum_{n=0}^{\infty} (-t\alpha)^n I_n(1_{(0,T]}^{\otimes n})$, and does not belong to $\bigcup_{\lambda > 0} H_\lambda$.

Now we study the SDE (2). We denote by M^d the space of $d \times d$ matrices, $H = L^2([0,1] \times E, \mathcal{B}, \nu)$ and U is a dense subset of H. The set H_f is the space spanned by $I_n(f_n), n \in \mathbb{N}, f_n \in H^{\odot n}$, and $H_f(\mathbb{R}^d), H_\lambda(\mathbb{R}^d)$ are the corresponding spaces of $\mathbb{R}^d$-valued random variables (H_λ is defined in paragraph generalized random variable section 2). Let $A \in H \otimes M^d$, $B \in L^1([0,1]; dt) \otimes M^d$ and X_0 be a generalized random vector such that the formal series $X_0(z)$ converges for all $z \in U$. The linear stochastic differential equation

$$X_t = X_0 + \int_0^t \int_E A(s,x) X_{s-} dq(s,x) + \int_0^t B_s X_s ds$$

has a unique solution in $(P(\Omega) \otimes U)^*$ given by $X_t = \xi_t \diamond X_0$, where ξ is the solution of the linear stochastic differential equation

$$\xi_t = I_d + \int_0^t \int_E A(s,x) \xi_{s-} dq(s,x) + \int_0^t B_s \xi_s ds.$$

Let us suppose that for all $s, t \in [0,1], x \in E$ the matrices $A(s,x), B_t, B_s$ commute, under this hypothesis the solution X_t of the SDE (2) is given by $\exp(\int_0^t B_s ds)\,(\mathcal{E}_t(A) \diamond X_0)$, where $\mathcal{E}_t(A) = \mathcal{E}(A 1_{(0,t]})$. We can express $\mathcal{E}_t(A) \diamond X_0$ as a product $\mathcal{E}_t(A) T_t^{-A}(X_0)$, where T_t^{-A} is a linear transformation on $H_f(\mathbb{R}^d)$. The proof of these results and nearly all the results in the Wiener case go exactly as in Buckdahn, Nualart (1994), but there is one difference, due to the fact that the product formula $\mathcal{E}(z)\mathcal{E}(z')$ is not the same. In the Poisson space we have

$$\mathcal{E}(z)\mathcal{E}(z') = \exp(\int z(x) z'(x) d\mu(x)) \mathcal{E}(z + z' + zz'). \qquad (13)$$

The following proposition shows that in the Poisson case we obtain an additional term for the operator T_t^{-A}.

Proposition 3.2. *If $(I_d + A)^{-1} \in L^\infty([0,1] \times E, \nu) \otimes M^d$ then for all $F \in H_f(\mathbb{R}^d)$, $\mathcal{E}_t(A) \diamond F = \mathcal{E}_t(A) T_t^{-A}(F)$. The linear operator T_t^{-A} is defined for all $m \in \mathbb{N}$, $f_m \in H^{\odot m}(\mathbb{R}^d)$ by*

$$T_t^{-A}(I_m(f_m)) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k I_{m-k} \left(< (A1_{[0,t[})^{\otimes k}, ((I_d + A1_{[0,t[})^{-1})^{\otimes m} f_m >\right).$$

Proof. Let $z \in U$, from (13), (5) we have

$$\mathbb{E}\left[(\mathcal{E}_t(A) \diamond I_m(f_m))\mathcal{E}(z)\right] = \exp\left(\int_0^t \int_E z(s,x)A(s,x)d\nu(s,x)\right) < f_m, z^m >$$

and,

$$\mathbb{E}\left[\mathcal{E}_t(A) T_t^{-A}(I_m(f_m))\,\mathcal{E}(z)\right] =$$

$$\exp\left(\int_0^t \int_E z(s,x)A(s,x)d\nu(s,x)\right)\mathbb{E}\left[\mathcal{E}(A1_{[0,t[} + z(I_d + A1_{[0,t[}))\, T_t^{-A}(I_m(f_m))\right].$$

We want to get for all $z \in U$

$$\mathbb{E}\left[\mathcal{E}(A1_{[0,t[} + z(I_d + A1_{[0,t[}))\, T_t^{-A}(I_m(f_m))\right] = < f_m, z^m > .$$

By putting $Z = A1_{[0,t[} + z(I_d + A1_{[0,t[})$ we obtain

$$\mathbb{E}[\mathcal{E}(Z) T_t^{-A}(I_m(f_m))] = \left\langle ((Z - A1_{[0,t[})(I_d + A1_{[0,t[})^{-1})^{\otimes m}, f_m \right\rangle$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^k \left\langle (A1_{[0,t[})^{\otimes k} \otimes Z^{\otimes m-k}, ((I_d + A1_{[0,t[})^{-1})^{\otimes m} f_m \right\rangle$$

By identification we obtain the expression of $T_t^{-A}(I_m(f_m))$.

Example. If $f, A \in H$ such that $(1 + A)^{-1} \in L^\infty([0,1] \times E, \nu)$, then

$$T_t^{-A}(I_1(f)) = I_1\left((1 + A1_{[0,t]})^{-1} f\right) - \int_0^t \int_E \frac{A(s,x)}{1 + A(s,x)} f(s,x)d\nu(s,x).$$

If $\omega = \sum_j \delta_{(t_j,x_j)} \in \Omega$ then $T_t^{-A}(I_1(f))(\omega) = I_1(f)\left(T_t^{-A}(\omega)\right)$, where $T_t^{-A}(\omega) = \sum_j (1 + A(t_j,x_j)1_{[0,t]}(t_j))^{-1}\delta_{(t_j,x_j)}$. It follows that contrary to the Wiener case the transformation $T_t^{-A}(\omega) \notin \Omega$, and T_t^{-A} is only a linear operator on $H_f(I\!R^d)$. The operator T_t^{-A} can be extend to $H_\lambda(I\!R^d)$, but the estimate of its norm is slightly different from the Wiener case.

Proposition 3.3. *Let $F \in H_f(I\!R^d)$, $\lambda > 0$, and $A \in H \otimes M^d$ such that $(I_d + A)^{-1} \in L^\infty([0,1] \times E, \nu) \otimes M^d$, then*

$$\left|\left|T_t^{-A}(F)\right|\right|_\lambda^2 \leq 2\exp\left(\frac{||A1_{[0,t[})||_2^2}{\lambda^2}\right)||F||^2_{2\lambda||(I_d+A1_{[0,t[})^{-1}||_\infty}.$$

Proof. Let $F = \sum_{n=0}^N I_n(f_n) \in H_f(I\!R^d)$, from the definition of T_t^{-A} we have

$$||T_t^{-A}(\sum_{n=0}^N I_n(f_n))||_\lambda^2 =$$

$$\sum_{j=0}^N j!\lambda^{2j}\left|\left|\sum_{n=j}^N \binom{n}{j}(-1)^{n-j} < (A1_{[0,t]})^{\otimes n-j}, ((I_d + A1_{[0,t[})^{-1})^{\otimes n} f_n >\right|\right|_2^2$$

$$\leq 2\sum_{j=0}^N j!\lambda^{2j}\sum_{n=j}^N \binom{n}{j}^2 ||A1_{[0,t[}||_2^{2(n-j)}\,||((I_d + A1_{[0,t[})^{-1})^{\otimes n} f_n||_2^2\, 2^n$$

$$= 2\sum_{n=0}^N\sum_{k=0}^n \binom{n}{k}\frac{n!}{k!}\lambda^{2n}\left(\frac{||A1_{[0,t[}||_2^2}{\lambda^2}\right)^k 2^n\,||((I_d + A1_{[0,t[})^{-1})^{\otimes n} f_n||_2^2.$$

Using the inequality

$$\frac{1}{k!}\left(\frac{||A1_{[0,t[}||_2^2}{\lambda^2}\right)^k \leq \exp\left(\frac{||A1_{[0,t[})||_2^2}{\lambda^2}\right)$$

we obtain

$$\left|\left|T_t^A(\sum_{n=0}^N I_n(f_n))\right|\right|_\lambda^2 \leq$$

$$2\exp\left(\frac{\|A1_{[0,t[}\|_2^2}{\lambda^2}\right)\sum_{n=0}^{N}\left\{\sum_{k=0}^{n}\binom{n}{k}\right\}n!2^n\lambda^{2n}\left\|(I_d+A1_{[0,t[})^{-1}\right\|_\infty^{2n}\|f_n\|_2^2$$
$$\leq 2\exp\left(\frac{\|A1_{[0,t[}\|_2^2}{\lambda^2}\right)\left\|\sum_{n=0}^{N}I_n(f_n)\right\|^2_{2\lambda\|(I_d+A1_{[0,t[})^{-1}\|_\infty}$$

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Manuscrit reçu en Mars 1997