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THE FEYNMAN INTEGRAL AND FEYNMAN'S OPERATIONAL CALCULUS: A HEURISTIC AND MATHEMATICAL INTRODUCTION

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A la mémoire d'Albert Badrikian,
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Résumé. Nous donnons une courte introduction heuristique au calcul opérationnel de Feynman pour des opérateurs qui ne commutent pas. Nous discutons également (tout aussi brièvement) une approche mathématique de ce calcul opérationnel développée en collaboration avec Gerald W. Johnson. Ce faisant, nous évoquons quelques-uns des liens entre ce sujet et les intégrales de Feynman de la physique quantique. En particulier, la notion d'intégrale de Feynman analytique et des opérations non-commutatives convenables sur certaines algèbres de fonctionnelles de Wiener (appelées "*algèbres de démêlement*") jouent ici un rôle essentiel. Le lecteur intéressé pourra trouver une discussion beaucoup plus approfondie de ce sujet dans les chapitres 14 à 19 du livre "*The Feynman Integral and Feynman's Operational Calculus*" [JoLa5] par G. W. Johnson et l'auteur, à paraître chez *Oxford University Press.*

Abstract. We provide a short heuristic introduction to Feynman's operational calculus for noncommuting operators, as well as discuss briefly a mathematical approach to this subject developed by Gerald W. Johnson and the author. We also evoke some of the connections between this topic and the "Feynman path integrals" from quantum physics. In particular, analytic (operator-valued) Feynman path integrals, along with (suitable) noncommutative operations on certain algebras of Wiener functionals (called "*disentangling algebras*"), play a prominent role in this context. The interested reader can find a much more thorough discussion of these and related developments in Chapters 14 through 19 of the book by G. W. Johnson and the author entitled "*The Feynman Integral and Feynman's Operational Calculus*" [JoLa5], to be published by *Oxford University Press.*

Our goal in this paper is two-fold: first, in Section 1, to provide a short heuristic introduction to Feynman's operational calculus for noncommuting operators [Fe2]. Then, in Section 2, to give a very brief discussion of a mathematical approach to this calculus, as developed originally by G. W. Johnson and the author by means

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of analytic (operator-valued) Feynman path integrals and suitable noncommutative operations on "disentangling algebras" of Wiener functionals ([JoLa1-4] and [JoLa5, esp. Chaps. 14-19]). (See also [La1-6] for related works and, for more recent developments, [dFJoLa1-2], [JeJo].)

1. Feynman's Operational Calculus: Heuristic Approach

In his 1951 paper, entitled "*An operator calculus having applications in quantum electrodynamics*" [Fe2], Feynman suggested to construct a functional calculus for noncommuting operators which may in some sense be viewed as a generalized kind of (or else a substitute for) path integration. More precisely, to this aim, he proposed to use the following *heuristic* rules:

(1) (Feynman's Time-Ordering Convention.) Attach "time indices" to the operators involved to specify the order of operations in products.

Hence, if A and B are given (possibly noncommuting) operators, then "*Feynman's time-ordering convention*" can be stated as follows:

$$A(s_1)B(s_2) := \begin{cases} BA, & \text{if } s_1 < s_2, \\ AB, & \text{if } s_2 < s_1, \\ \text{undefined}, & \text{if } s_1 = s_2, \end{cases} \tag{1.1}$$

even though A and B themselves may be time independent.

(2) With these time indices attached, form functions of these operators by treating them as if they were commuting.

[So that, in (1.1), for example, $A(s_1)B(s_2) = B(s_2)A(s_1) = BA$, provided that $s_1 < s_2$.]

(3) (Disentangling Process.) Finally, return to the real world where operators

do not commute in general: "*disentangle*" the resulting expressions; that is, restore the conventional ordering of the operators.

Feynman [Fe2, p. 110] says of the disentangling process involved in Step 3: "*The process is not always easy to perform, and in fact is the central problem of this operator calculus.*"

We now illustrate these "rules" by a very simple example. We write successively:

$$
\begin{aligned}
\text{“}A\cdot B\text{”} &= \left(\int_0^1 A(s_1)ds_1\right)\left(\int_0^1 B(s_2)ds_2\right) \\
&= \int\int_{[0,1]\times[0,1]} A(s_1)B(s_2)ds_1ds_2 \\
&= \int\int_{0<s_1<s_2<1} A(s_1)B(s_2)ds_1ds_2 \\
&+ \int\int_{0<s_2<s_1<1} A(s_1)B(s_2)ds_1ds_2 \\
&= \int\int_{s_1<s_2} BAds_1ds_2 + \int\int_{s_2<s_1} ABds_1ds_2 \\
&= \frac{1}{2}BA + \frac{1}{2}AB \qquad (1.2) \\
&= \frac{1}{2}(AB+BA),
\end{aligned}
$$

where, of course, we have used Feynman's time-ordering convention (1.1) in the second to last equality.

In summary,

$$\text{“}A\cdot B\text{”} = \left(\int_0^1 A(s_1)ds_1\right)\left(\int_0^1 B(s_2)ds_2\right) = \frac{1}{2}(AB+BA), \tag{1.3}$$

the anti-commutator of A and B.

More sophisticated examples lead to various kinds of "*time-ordered perturbation series*". For instance, we give explicitly (but without further explanation) the first two terms of such a series resulting from the "disentangling" of the formal expression "$\exp\{-tA + \int_0^t B(s)ds\}$", for some $t > 0$ (for more details, see [JoLa5, Chap. 14] or the introduction to [dFJoLa2]):

$$
\begin{aligned}
&\exp\{-tA + \int_0^t B(s)ds\} \\
&= \int_0^t \exp(-(t-s)A)B(s)\exp(-sA)ds \\
&\quad + \int\int_{0<s_1<s_2<t} \exp(-(t-s_2)A)B(s_2)\exp(-(s_2-s_1)A)B(s_1)ds_1 ds_2 \\
&\quad + \cdots .
\end{aligned}
\tag{1.4}
$$

Let us now specialize to $A = iH_0$ (where $H_0 = -\frac{1}{2}\Delta$ denotes the free Hamiltonian) and $B = -iV$ (where $V = V(s,\cdot)$ is the multiplication operator by a bounded, possibly time-dependent potential function $V : [0,t] \times \mathbb{R}^d \to \mathbb{R}$), with $i = \sqrt{-1}$. Then the full time-ordered exponential series corresponding to (1.4) is nothing but the "*classical Dyson series*" [Dy] encountered in the perturbative approach to quantum mechanics and quantum electrodynamics. (See, *e.g.*, [GlJa], [Si], [JoLa1] and the references therein.)

A number of "*generalized Dyson series*" (GDS)—possessing often a much more complex combinatorial structure—are obtained in the rigorous approach to Feynman's operational calculus developed in ([JoLa1-5], [La1-5], [dFJoLa1-2]), and to be briefly discussed in the next section. (See especially Chapters 15, 17 and 19 in [JoLa5].)

Many "*paradoxical formulas*" (as we like to call them) appear in the context of

Feynman's operational calculus. For example, the most striking such formula is given by

$$\exp(A+B) = \exp(A)\cdot\exp(B) \tag{1.5a}$$

or rather

$$\exp\left\{\int_0^1 A(s)ds + \int_0^1 B(s)ds\right\} = \exp\left\{\int_0^1 A(s)ds\right\}\left\{\exp(\int_0^1 B(s)ds\right\}, \tag{1.5b}$$

which every student of linear differential equations quickly learns to be wrong. Naturally, such formulas must be taken with a grain of salt. However, in our joint work [JoLa3,4], we have proposed "deforming" certain commutative operations (on the space of Wiener functionals) into noncommutative ones in order to reintepret rigorously (1.5) and other "paradoxical formulas". (See also [JoLa5, Chap. 18].)

In closing this section, we should stress that Feynman's original paper on this subject [Fe2] is not easy to read; probably less so (to most readers, at least) than his celebrated paper [Fel] on the "Feynman path integral". It is very rich in ideas but is also in need of further clarifications and mathematical developments. Indeed, Feynman himself wrote [Fe2, p. 108] about his operational calculus: "*The mathematics is not completely satisfactory. No attempt has been made to maintain mathematical rigor. The excuse is not that it is expected that rigorous demonstrations can be easily supplied. Quite the contrary, it is believed that to put the present methods on a rigorous basis may be quite a difficult task, beyond the abilities of the author*".

The above quote may be surprising to the reader familiar with some of Feynman's other statements regarding mathematics, as reported in the press or printed in some of his own books or articles. However, as the author could verify during a number of private conversations—including the initial one during which Feynman urged him to further develop mathematically his operational calculus—he was completely sincere in this quote. It was a pleasure for me to begin to carry out this program a few years later, jointly with my friend and (now) long-term collaborator, Gerald W. Johnson. Of course, many intricate problems remain to be tackled and eventually to be solved in this area.

2. Feynman's Operational Calculus, the Feynman Integral, and Disentangling Algebras

We will briefly discuss in this section some of out joint work with Gerald W. Johnson on Feynman's operational calculus for noncommuting operators [JoLa1-5], as well as related work of the author [La1-6] and further extensions (also joint with B. DeFacio) to more abstract settings [dFJoLa1-2]. We define a (commutative) Banach algebra $\mathcal{A}_t$ (indexed by time t) of Wiener functionals F such that the associated analytic operator-valued Feynman integral $K_\lambda^t(F)$ (*e.g.*, [CaSt], [JoSk], [JoLa1]) can be "*disentangled*" *via* time-ordered perturbation expansions, called *generalized Dyson series* (GDS, in short). These series (which can be visualized by means of certain generalized Feynman diagrams) have a rich combinatorial structure due to the presence of Lebesgue-Stieltjes measures with nonzero discrete part, in the definition of the functional F. (See [JoLa1] or [JoLa5, Chaps. 15 and 16].) The use of such measures enables us to blend continuous and discrete structures as well as to unify known phenomena and discover new ones.

More specifically, for $\lambda > 0$, the operator $K_\lambda^t(F)$ is given by a *bona fide* Wiener integral. It is then defined by analytic continuation to the open right-half plane $Re\,\lambda > 0$, followed by strong continuity for $Re\,\lambda \geq 0, \lambda > 0$ (*i.e.*, by passage to the limit along the imaginary axis). Of course, in view of the well-known *no go* theorem of Cameron [Ca] establishing the nonexistence of "Feynman's measure", $K_\lambda^t(F)$ is no longer given by a Wiener-type functional integral for nonreal values of the parameter λ. Nevertheless, for all $\lambda \in \mathbb{C}$ with $Re\,\lambda \geq 0, \lambda \neq 0$, the (bounded linear) operator $K_\lambda^t(F)$ is still given (or "disentangled") by the aforementioned GDS. (We note that probabilistically, $\lambda^{-1/2}$ can be thought of as a "diffusion constant", for $\lambda > 0$.)

In the terminology of ([JoLa1-5], [La1-5]), the *quantum-mechanical* (or Feynman) *case* corresponds to $\lambda = -i$ (or λ purely imaginary), whereas the *probabilistic* (or diffusion) *case* corresponds to $\lambda = 1$ (or $\lambda > 0$).

[For measure-theoretic reasons, the commutative Banach algebras $\mathcal{A}_t$ consist of (suitable) equivalence classes of Wiener functionals, where the equivalence relation [JoSk] is associated to the scale-change corresponding to different positive values of the parameter λ. Further, the natural operations within each algebra $\mathcal{A}_t$ correspond to the usual addition and multiplication of Borel measurable functions on Wiener space.]

We also introduce (in [JoLa3, 4]) noncommutative operations on the space of Wiener functionals (namely, a noncommutative multiplication $*$ and addition $\dot{+}$) and on the aforementioned "*disentangling algebras*" $\mathcal{A}_t$ $(t > 0)$ such that if $F \in \mathcal{A}_{t_1}$ and $G \in \mathcal{A}_{t_2}$, then $F * G \in \mathcal{A}_{t_1+t_2}$ (as well as $F \dot{+} G \in \mathcal{A}_{t_1+t_2}$) and (for $Re\,\lambda \geq$

0, $\lambda \neq 0$)

$$K_\lambda^{t_1+t_2}(F * G) = K_\lambda^{t_1}(F) K_\lambda^{t_2}(G). \tag{2.1}$$

Further, under the same assumptions, we have for example,

$$\exp(F \dot{+} G) = \exp(F) * \exp(G) \tag{2.2a}$$

and hence, by (2.1),

$$K_\lambda^{t_1+t_2}(\exp(F \dot{+} G)) = K_\lambda^{t_1}(\exp(F)) K_\lambda^{t_2}(\exp(G)). \tag{2.2b}$$

As was alluded to in Section 1, we may use formula (2.2) to give a rigorous interpretation of Feynman's "paradoxical formula" (1.5). (See also [JoLa5, Chap. 18].) Note that in (2.2a) [which is formally identical to (1.5a)] we are now working at the level of the functionals (rather than of the operators) and have replaced the usual (commutative) operations by new noncommutative ones. Intuitively, one may think of the relationship between the operator $K_\lambda^t(F)$ and the functional F as analogous to that between a pseudodifferential operator and its symbol. (Naturally, one can use similarly Equation (2.1) along with the properties of the noncommutative multiplication and addition to justify (and reinterpret) other "paradoxical formulas" in [Fe2], as well as new ones.

We also discuss closely related work of the author [La2-5] on the "*Feynman-Kac formula with a Lebesgue-Stieltjes measure*" in which we determine both in the diffusion and in the quantum-mechanical case, the integral equation, the distributional differential equation and the corresponding product integral representation, associated with the (generalized) Feynman-Kac functional

$$F(x) := \exp\left\{-\int_0^t V(s, x(s))\eta(ds)\right\},$$

where η is an arbitrary Lebesgue-Stieltjes measure on the time interval $[0, t)$.

If we write $\eta = \mu + \nu$, where μ is *continuous* (*i.e.*, "diffuse" or "nonatomic") and ν is *discrete* (possibly an infinite linear combination of Dirac measures δ_{τ_p}), then we can analyze precisely the effect of μ and ν on the solution of the differential (or integral) equation. In particular, the (unique, bounded) solution is shown to have (multiplicative) time-discontinuities at each instant τ_p in the support of the discrete part ν of η. Physically, these can be interpreted in the quantum-mechanical case as "instantaneous interactions", "shocks" or "scatterings" occurring at precisely those times (see [La3]). (The integral equation is derived in [La2,3] or [La4], when ν is finitely supported or in the general case, respectively, while the associated distributional equation and a product integral representation of the solution are obtained in [La5]. We stress that the time-ordered perturbation series (GDS) obtained in [JoLa1] were crucial in the derivation of these results, especially in the quantum-mechanical case where a standard functional integral representation is no longer available.) We also refer to [JoLa5, Chap. 17] for further discussion of this topic, as well as of its relationship with various aspects of Feynman's operational calculus.

An attempt to capture the essence of the algebraic and analytical structures underlying the construction (carried out in [JoLa1,4]) of $(\{\mathcal{A}_t\}_{t>0}, \dot{+}, *)$, the family of "disentangling algebras" equipped with its noncommutative operations $\dot{+}$ and $*$, is made in [La6], where a possible set of axioms for (parts of) Feynman's operational calculus is proposed. The counterpart of the mapping $F \mapsto K_\lambda^t(F)$ is then viewed as a "*quantization map*" defined *via* a kind of "generalized (Feynman) path integral". The difficulty is, of course, to construct such a map in each concrete situation (as was done in [JoLa1-4] in a setting corresponding to ordinary quantum mechanics).

We also briefly mention more recent work [dFJoLa1,2] (joint with G. W. Johnson and B. DeFacio) in which we determine, in particular, the evolution equation (in integrated form) associated with the exponential of sums of noncommuting operators. The setting of [dFJoLa2] is more general (in some respects) than, for example, in ([JoLa1-4], [La1-5]) because we now deal with (suitable) abstract operators in Hilbert spaces and thus no underlying path integral is then assumed (or even available). (An abstract measure-theoretic result [BaJoYo]—due to Albert Badrikian, G. W. Johnson and Yoo, and extending that of Johnson in [Jo]—is used in the setting of [dFJoLa] in order to simplify our hypotheses.) In addition, the work in [dFJoLa2] seems to provide a suitable theoretical framework to understand the efficient use of perturbation series associated with "nonlocal potentials" in phenomelogical nuclear physics (as studied, *e.g.*, in [ChSa], [McC], [Ta]).

Aspects of the approach in [dFJoLa] are extended (in different directions) in the works in preparation ([Re], [JeJo]). It is hoped that future research (probably joint with B. Jefferies and G. W. Johnson) will enable us to combine the basic features of ([JoLa], [dFJoLa], [JeJo]) in order to construct the above noncommutative operations $\dotplus$ and $*$ in this more general framework, and thereby provide a fairly general class of examples for which Feynman's operational calculus can be carried out in this manner.

Other works dealing with various aspects of Feynman's operational calculus—but using rather different approaches from those described here—include ([Ma], [Ne], [Ar], [Gi], [GiZa]).

The content of this paper is the subject of Chapters 14 through 19 of a book in preparation by G. W. Johnson and M. L. Lapidus, entitled "*The Feynman Integral and Feynman's Operational Calculus*", to be published by Oxford University Press in the Oxford Mathematical Monographs Series [JoLa5]. The interested reader can also find in other parts of [JoLa5] a more thorough discussion of several approaches to the Feynman path integral, using either probabilistic or operator-theoretic techniques.

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