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ABSOLUTE VALUES ON ALGEBRAS $H(D)$

by Kamal Boussaf and Alain Escassut

Abstract. Let K be an algebraically closed complete ultrametric field, and let D be an infraconnected set in K such that the set $H(D)$ of the analytic elements on D is a ring. Among the continuous multiplicative semi-norms on $H(D)$, we look for the ones that are absolute values. They are characterized by the location of the T-filters on D. Besides, we characterize the sets D such that $H(D)$ admits at least one continuous absolute value $|\ .\ |$.

Notations: Let K be an algebraically closed field complete for an ultrametric absolute value.
Given $a \in K$ and $r > 0$, $d(a,r)$ (resp. $d(a,r^-)$, resp. $C(a,r)$) denotes the disk $\{x \in K|\ |x-a| \leq r\}$ (resp. $\{x \in K|\ |x-a| < r\}$, resp. the circle $\{x \in K|\ |x-a| = r\}$).
Given $a \in K$, $r' > 0$ and $r'' > r'$, $\Gamma(a,r',r'')$ denotes the annulus $\{x \in K|\ r' < |x-a| < r''\}$.
Given a set A in K and a point $a \in K$, we denote by $\delta(a,A)$ the distance from a to A.
Let E be an infinite set in K, and let $a \in E$. If E is bounded of diameter r, we denote by $\widetilde{E}$ the disk $d(a,r)$, and if E is not bounded, we put $\widetilde{E} = K$. Then, $\widetilde{E} \setminus \overline{E}$ is known to admit a partition of the form $(d(a_i,r_i^-)_{i\in J})$, with $r_i = \delta(a_i,D)$ for each $i \in J$. The disks $d(a_i,r_i^-)_{i\in J}$, are named *the holes of* E.

$R(E)$ denotes the set of rational functions $h \in K(x)$ with no poles in E. This is a K-subalgebra of the algebra K^D of all functions from E into K. Then $R(E)$ is provided with the topology $\mathcal{U}_E$ of uniform convergence on E, and is a topological group for this topology. $H(E)$ denotes the completion of $R(E)$ for this topolgy and its elements are named *the analytic elements on* E [1], [2], [3], [9].

By [3], we remember that $H(E)$ is a K-subalgera of the algebra K^D if and only if E satisfies the following conditions:

A) $\widetilde{E} \setminus \overline{E}$ *is bounded,*

B) $\overline{E} \setminus E \subset \overset{\circ}{\widetilde{E}}$.

Henceforth, D will denote an infraconnected set satisfying Conditions A) *and* B).

In [7], [4], the continuous multiplicative semi-norms of an algebra $H(D)$ were characterized by means of the circular filters on D. So we have to recall the definitions of monotonous and circular filters.

Definitions and notations: The set of those multiplicative semi-norms that are continuous with respect to the topology of uniform convergence on D is denoted by $Mult(H(D),\mathcal{U}_D)$. Given a continuous multiplicative semi-norm $\psi \in Mult(H(D),\mathcal{U}_D)$ we denote by $Ker\psi$ the closed prime ideal of the $f \in H(D)$ such that $\psi(f) = 0$.

ψ will be said to be *punctual* if $Ker\psi$ is a maximal ideal of codimension 1 of $H(D)$. We know that there exists a bijection M from D onto the set of maximal ideals of codimension 1 of $H(D)$, defined as $M(a) = \{f \in H(D) | f(a) = 0\}$ (indeed, this was shown in [3], Proposition II.6, when D is closed and bounded, and it is easily extended to all sets D satisfying Conditions A) and B)). As a consequence, there exists a bijection S from D onto the set of punctual continuous multiplicative semi-norms of $H(D)$ defined as $S(a)(f) = |f(a)|$, whenever $f \in H(D)$.

In order to recall the characterization of the continuous multiplicative semi-norms of $H(D)$, we first have to recall the definition of monotonous and circular filters.

Given a filter $\mathcal{F}$ on D, we will denote by $\mathcal{I}(\mathcal{F})$ the ideal of the $f \in H(D)$ such that $\lim_{\mathcal{F}} f(x) = 0$.

Let $a \in \widetilde{D}$ and $S \in \mathbb{R}_+^*$ be such that $\Gamma(a,r,S) \cap D \neq \emptyset$ whenever $r \in]0,S[$ (resp. $\Gamma(a,S,r) \cap D \neq \emptyset$ whenever $r > S$). We call *an increasing* (resp. *a decreasing*) *filter of center a and diameter S, on D* the filter $\mathcal{F}$ on D that admits for base the family of sets $\Gamma(a,r,S) \cap D$ (resp. $\Gamma(a,S,r) \cap D$) . For every sequence $(r_n)_{n\in\mathbb{N}}$ such that $r_n < r_{n+1}$ (resp. $r_n > r_{n+1}$) and $\lim_{n\to\infty} r_n = S$, it is seen that the sequence $\Gamma(a,r_n,S) \cap D$ (resp. $\Gamma(a,S,r_n) \cap D$) is a base of $\mathcal{F}$ and such a base is called *a canonical base* .

We call *a decreasing filter with no center of canonical base $(D_n)_{n\in\mathbb{N}}$ and diameter $S > 0$, on D* a filter $\mathcal{F}$ on D that admits for base a sequence $(D_n)_{n\in\mathbb{N}}$ of the form $D_n = d(a_n,r_n) \cap D$ with $D_{n+1} \subset D_n$, $r_{n+1} < r_n$, $\lim_{n\to\infty} r_n = S$, and $\bigcap_{n\in\mathbb{N}} d(a_n,r_n) = \emptyset$.

Given an increasing (resp. a decreasing) filter $\mathcal{F}$ on D of center a and diameter r, we will denote by $\mathcal{P}(\mathcal{F})$ the set $\{x \in D | \ |x-a| \geq r\}$ (resp. the set $\{x \in D | \ |x-a| \leq r\}$ and by $\mathcal{C}(\mathcal{F})$ the set $\{x \in D | \ |x-a| < r\}$ (resp. the set $\{x \in D | \ |x-a| > r\}$. Besides $\mathcal{C}(\mathcal{F})$ will be named *the body of $\mathcal{F}$* and $\mathcal{P}(\mathcal{F})$ will be named *the beach of $\mathcal{F}$*.
We call *a monotonous filter on D* a filter which is either an increasing filter or a decreasing filter (with or without a center).
Given a monotonous filter $\mathcal{F}$ we will denote by $diam(\mathcal{F})$ its diameter.

The field K is said to be *spherically complete* if every decreasing filter on K has a center in K. (The field $\mathbb{C}_p$ for example is not spherically complete). However, every algebraically closed complete ultrametric field admits a spherically complete algebraically closed extension [10], [11].

Two monotonous filters $\mathcal{F}$ and $\mathcal{G}$ are said to be *complementary* if $\mathcal{P}(\mathcal{F}) \cup \mathcal{P}(\mathcal{G}) = D$.
Let $\mathcal{F}$ be an increasing (resp. a decreasing) filter of center a and diameter S on D. $\mathcal{F}$ is said *to be pierced* if for every $r \in]0, S[$, (resp. $r < S$), $\Gamma(a, r, S)$ (resp. $\Gamma(a, S, r)$) contains some hole T_m of D. A decreasing filter with no center $\mathcal{F}$, and canonical base $(D_n)_{n\in\mathbb{N}}$, on D is said *to be pierced* if for every $m \in \mathbb{N}$, $\widetilde{D}_m \setminus \widetilde{D}_{m+1}$ contains some hole T_m of D.

Let $a \in \widetilde{D}$, let $\rho = \delta(a, D)$ be such that $\rho \leq S \leq diam(D)$. We call *circular filter of center a and diameter S on D* the filter $\mathcal{F}$ which admits as a generating system the family of sets $\Gamma(\alpha, r', r'') \cap D$ with $\alpha \in d(a, S), r' < S < r''$, i.e. $\mathcal{F}$ is the filter which admits for base the family of sets of the form $D \cap (\bigcap_{i=1}^{q} \Gamma(\alpha_i, r'_i, r''_i))$ with $\alpha_i \in d(a, S), r'_i < S < r''_i$ $(1 \leq i \leq q\ ,\ q \in \mathbb{N})$.

A decreasing filter with no center, of canonical base $(D_n)_{n\in\mathbb{N}}$ is also called *circular filter on D with no center, of canonical base* $(D_n)_{n\in\mathbb{N}}$.
Finally the filter of the neighbourhoods of a point $a \in D$ will be called *circular filter of the neighbourhoods of a on D*. It will be also named *circular filter of center a and diameter* 0.
A circular filter on D will be said to be *large* if it has diameter different from 0.
Given a circular filter $\mathcal{F}$, its diameter will be denoted by $diam(\mathcal{F})$.

The set of the circular filters on D will be denoted by $\Phi(D)$.

Now let $\mathcal{F}$ be a circular filter on D. By [7], [4], we have the following characterization of continuous multiplicative semi-norms of $H(D)$.

Theorem 0: *Let $\mathcal{F}$ be a circular filter on D. For every $f \in H(D)$, $|f(x)|$ admits a limit along $\mathcal{F}$, and this limit, denoted by $\varphi_{\mathcal{F}}(f)$, defines a continuous multiplicative semi-norm $\varphi_{\mathcal{F}}$ on $H(D)$. Further, the mapping Θ from $\Phi(D)$ into $Mult(H(D), \mathcal{U}_D)$ defined as $\Theta(\mathcal{F}) = \varphi_{\mathcal{F}}$ is a bijection.*

Notations: For convenience, when $\mathcal{F}$ is the circular filter of center a and diameter r, we also denote by $\varphi_{a,r}$ the multiplicative semi-norm $\varphi_{\mathcal{F}}$.

Here, assuming $H(D)$ to be a K-algebra, we study what continuous multiplicative semi-norms of $H(D)$ are norms, i.e. are absolute values on $H(D)$. Of course, this requires $H(D)$ to have no divisors of zero. But then, as a trenscendental extension of the field K, the field of quotients L of $H(D)$ does admit absolute values extending the one of K. Hence so does $H(D)$. The problem, here, is whether such absolute values are continuous with respect to the topology of $H(D)$, i.e. are defined by circular filters on D. So, we will give the condition a circular filter has to satisfy in order that its continuous multiplicative semi-norm be an absolute value, and next, we will characterize the sets D such that at least one of the continuous multiplicative semi-norms is an absolute value.

All this study involves T-filters, and now we have to introduce them.

Definition: Let $\mathcal{F}$ be an increasing (resp. a decreasing) filter on D, of center a and diameter s. An element $f \in \mathcal{I}(\mathcal{F})$ is said to be *strictly vanishing along* $\mathcal{F}$ if there exists $t < s$ (resp. $t > s$) such that $\varphi_{a,r}(f) > 0$ for all $r \in [t, s[$, (resp. $]s, t]$).
Let $\mathcal{F}$ be a decreasing filter on D, with no center, of diameter r, of canonical base $(D_n)_{n \in \mathbb{N}}$, with $D_n = d(a_n, r_n) \cap D$. Then an element $f \in \mathcal{I}(\mathcal{F})$ is said to be *strictly vanishing along* $\mathcal{F}$ if there exists $t > s$ such that $\varphi_{a_n,r}(f) > 0$ for all $r \in [r_n, t]$, for every $n \in \mathbb{N}$.

Let $\mathcal{F}$ be a filter on D, and let $A \subset D$. $\mathcal{F}$ will be said to be *secant with* A if for every $F \in \mathcal{F}$, $A \cap F$ is not empty.

T-filters are certain pierced monotonous filters satisfying particular properties linked to the holes of D, and were defined in [1], [2], [5]. Here we will only use the following characterization:

A monotonous filter $\mathcal{F}$ is a T-filter if and only if there exists $f \in H(D)$ strictly vanishing along $\mathcal{F}$.

From [2], [5], [6]. we can easily deduce the following thechnical propositions that will be indispensable.

Proposition P: *Let $b \in D$, $l > 0$ and let $f \in H(D)$ satisfy $f(b) \neq 0$ and $\varphi_{b,l} = 0$. There exists an increasing T-filter $\mathcal{F}$ of center b and diameter $t \in]0, l[$ such that f is strictly vanishing along $\mathcal{F}$ and satisfies $\varphi_{b,s}(f) > 0$ for every $s \in]0, t[$.*

Let $a \in \widetilde{D}$ and let $r, s \in \mathbb{R}$ satisfy $\delta(a, D) \leq r \leq diam(D)$. If f satisfies $f(x) = 0$ whenever $x \in d(a, r) \cap D$, then f is strictly vanishing along a T-filter $\mathcal{F}$ such that $d(a, r) \cap D \subset \mathcal{P}(\mathcal{F})$ and $b \in \mathcal{C}(\mathcal{F})$.

Now, we can characterize absolute values among continuous multiplicative semi-norms.

Theorem 1: *Let $\mathcal{F}$ be a large circular filter on D. Then $\varphi_{\mathcal{F}}$ is not an absolute value if and only if it satisfies one of the following conditions:*

a) There exists a T-filter $\mathcal{G}$ on D such that $\mathcal{F}$ is secant with $\mathcal{P}(\mathcal{G})$,

b) $\mathcal{F}$ is a T-filter.

Proof: First, suppose that there exists a T-filter $\mathcal{G}$ satisfying a). By Lemma I.6 A of [5], there exists $f \in H(D)$, strictly vanishing along $\mathcal{G}$, equal to 0 in all of $\mathcal{P}(\mathcal{G})$. Hence we have $\varphi_{\mathcal{F}}(f) = 0$ and then $\varphi_{\mathcal{F}}$ is not a norm. Now if $\mathcal{F}$ is a T-filter, then there exists

$f \in H(D)$ strictly vanishing along $\mathcal{F}$ and therefore we have $\lim_{\mathcal{F}} f(x) = 0$, hence $\varphi_{\mathcal{F}}$ is not a norm.

Now we suppose that there exists no T-filter $\mathcal{G}$ satisfying $a)$ and that $\mathcal{F}$ is not a T-filter, and we suppose that $\varphi_{\mathcal{F}}$ is not a norm. Let $f \in H(D) \setminus \{0\}$ satisfying $\varphi_{\mathcal{F}}(f) = 0$. Let $S = diam(\mathcal{F})$. Let $b \in D$ be such that $f(b) \neq 0$.

We first assume that $\mathcal{F}$ has a center a.
On the first hand, we suppose that $b \in d(a, S)$. Since $\varphi_{a,r}(f) \neq 0$, when r approaches 0 there does exist $s \in]0, S[$ such that $\varphi_{b,s}(f) = 0$ and $\varphi_{b,r}(f) \neq 0$ whenever $r \in]0, s[$. Hence f is strictly vanishing along the increasing filter $\mathcal{G}$ of center b and diameter s, and therefore $\mathcal{F}$ is secant with $\mathcal{P}(\mathcal{G})$.
On the second hand, we suppose that $|a - b| > S$. Let $t = |a - b|$. If $\varphi_{b,t}(f) = 0$, there exists $s \in]0, t]$ such that $\varphi_{b,s}(f) = 0$ and $\varphi_{b,r}(f) \neq 0$ whenever $r \in]0, s[$, hence f is strictly vanishing along an increasing T-filter $\mathcal{G}$ of center b and diameter s and therefore $\mathcal{F}$ is secant with $\mathcal{P}(\mathcal{G})$. Now we may assume $\varphi_{b,t}(f) \neq 0$. But $\varphi_{b,t}(f) = \varphi_{a,t}(f)$ and therefore there exists $s \in [S, t[$ such that $\varphi_{a,s}(f) = 0$ and $\varphi_{a,r}(f) \neq 0$ whenever $r \in]s, t]$. Hence f is strictly vanishing along a decreasing T-filter $\mathcal{G}$ of center a and diameter s, and $\mathcal{F}$ is secant with $\mathcal{P}(\mathcal{G})$.

Now, assume that $\mathcal{F}$ is a decreasing filter with no center. Let $(D_n)_{n \in \mathbb{N}}$ be a canonical base of $\mathcal{F}$, and for each $n \in \mathbb{N}$, let $D_n = d(a_n, r_n) \cap D$ and let $u_n = \varphi_{a_n, r_n}(f)$. If there exists $q \in \mathbb{N}$ such that $u_q = 0$, by Proposition P, D admits a T-filter $\mathcal{G}$ such that D_q is included in $\mathcal{P}(\mathcal{G})$, and therefore, $\mathcal{F}$ is obviously secant with $\mathcal{P}(\mathcal{G})$. Hence we can assume that $u_n > 0$ for every $n \in \mathbb{N}$. Now, suppose that there exist $q \in \mathbb{N}$ and $r \in [r_{q+1}, r_q]$ such that $\varphi_{a_{q+1}, r}(f) = 0$. As we just saw, there exists a T-filter $\mathcal{G}$ on D such that the circular filter $\mathcal{F}_q$ is secant with $\mathcal{P}(\mathcal{G})$, and therefore so is $\mathcal{F}$.
Thus, without loss of generality, we can assume that $\varphi_{a_{q+1}, r}(f) > 0$ for every $r \in [r_{q+1}, r_q]$, for every $q \in \mathbb{N}$. Hence f is just strictly vanishing along the decreasing filter $\mathcal{F}$ and therefore $\mathcal{F}$ is a T-filter. This ends the proof of Theorem 1.

Corollary a: *All the not punctual continuous multiplicative semi-norms of $H(D)$ are absolute values if and only if D has no T-filter.*

Definitions and notations: Let $incT(D)$ (resp. $decT(D)$) be the set of increasisng (resp. decreasing) T-filters on D. We will denote by $\preceq$ the relation defined on $incT(D)$ (resp. $decT(D)$) by $\mathcal{F}_1 \preceq \mathcal{F}_2$ if $\mathcal{C}(\mathcal{F}_2) \subset \mathcal{C}(\mathcal{F}_1)$. This relation is obviously seen to be an order relation on $incT(D)$ (resp. $decT(D)$) .
An increasing (resp. a decreasing) T-filter $\mathcal{F}$ will be said to be *maximal* if it is maximal in $incT(D)$ (resp. in $decT(D)$) with respect to this relation.
We will denote by $\prec$ the strict order associated to $\preceq$ by $\mathcal{F}_1 \prec \mathcal{F}_2$ if $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\mathcal{F}_1 \neq \mathcal{F}_2$.
We will call *an ascending chain of increasing (resp. decreasing) T-filters* a sequence of increasing (resp. decreasing) T-filters $(\mathcal{F}_n)_{n \in \mathbb{N}}$ such that $\mathcal{F}_n \prec \mathcal{F}_{n+1}$ whenever $n \in \mathbb{N}$.

Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be an ascending chain of increasing T-filters. For each $n \in \mathbb{N}$ let $r_n = diam(\mathcal{F}_n)$. Since the sequence $(r_n)_{n\in\mathbb{N}}$ is decreasing, we put $r = \lim\limits_{n\to\infty} r_n$, and then r will be named *the diameter of the chain.*
We put $A := \bigcap\limits_{n\in\mathbb{N}} \mathcal{C}(\mathcal{F}_n)$ and for each $n \in \mathbb{N}, D_n := \mathcal{C}(\mathcal{F}_n) \setminus A$. The sequence $(D_n)_{n\in\mathbb{N}}$ is then a base of a filter $\mathcal{F}$ on D of diameter r.
If $r = 0$, since D Condition Condition B), A is a point a of D, hence $\mathcal{F}$ is the filter of the neighbourhoods of a in D.
If $r > 0$ $\mathcal{F}$ is a decreasing filter on D of diameter r.
In both cases $\mathcal{F}$ will be called *the returning filter of the ascending chain* $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Now let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be an ascending chain of decreasing T-filters and let $a \in \mathcal{P}(\mathcal{F}_n)$ for some $n \in \mathbb{N}$. The sequence $(r_n)_{n\in\mathbb{N}}$ is an increasing sequence of limit $r \in]0, +\infty]$, and r will be named *the diameter of the chain.* Since D belongs to $\mathcal{A}$, by Condition A) we notice that $r < +\infty$, and then we will call *the returning filter of the ascending chain* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ the increasing filter $\mathcal{F}$ of center a and diameter r (it is seen that $\mathcal{F}$ does not depend on the point $a \in \mathcal{P}(\mathcal{F}_n)$, whenever $n \in \mathbb{N}$).

Lemma 1: *Let $H(D)$ have no divisors of zero. Then $incT(D)$ is totally ordered with respect to the order $\preceq$.*

Proof: Suppose that $\mathcal{F}, \mathcal{G}$ are increasing T-filters on D that are not comparable. We put $A = \mathcal{C}(\mathcal{F})$ and $B = \mathcal{C}(\mathcal{G})$. Then A, B are two disks of D which satisfy neither $A \subset B$, nor $B \subset A$. Hence we have $A \cap B = \emptyset$. As a consequence, $\mathcal{P}(\mathcal{F}) \cup \mathcal{P}(\mathcal{G})$ is equal to D and then by [6] $H(D)$ has divisors of zero. Hence this contradicts the hypothesis and ends the proof.

Corollary b: *Let $H(D)$ have no divisors of zero, and let $\mathcal{F}$ (resp. $\mathcal{G}$) be an increasing T-filter on D, of diameter r (resp. s). If $r > s$ then $\mathcal{F} \prec \mathcal{G}$. If $r = s$, then $\mathcal{F} = \mathcal{G}$.*

We are now able to characterize the sets D such that $H(D)$ admits continuous absolute values. Let us recall the following theorem of [6]:

The algebra $H(D)$ has no divisors of zero if and only if D does not admit two complementary T-filters.

We will use comparison between filters. Here, a filter $\mathcal{F}$ will be said *thinner than* a filter $\mathcal{G}$ every element of $\mathcal{G}$ belongs to $\mathcal{F}$.
Theorem 2: *Let $H(D)$ have no divisors of zero. Then $Mult(H(D), \mathcal{U}_D)$ contains no norm if and only if D admits an ascending chain of T-filters $(\mathcal{F}_n)_{n\in\mathbb{N}}$ whose returning filter is either a T-filter or a Cauchy filter.*

Proof : On the first hand, we suppose that D admits an ascending chain of T-filters $(\mathcal{F}_n)_{n\in\mathbb{N}}$ whose returning filter $\mathcal{F}$ is either a T-filter or a Cauchy filter and we will prove that $Mult(H(D), \mathcal{U}_D)$ contains no norm. We denote by r the diameter of this ascending chain $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Let $\mathcal{G}$ be a circular filter on D, of diameter $s > 0$. By Theorem 1, we only have to show that either $\mathcal{G}$ is a T-filter, or $\mathcal{G}$ is secant with the beach of a T-filter.

First we suppose $\mathcal{F}$ is a Cauchy filter. Then the $\mathcal{F}_n$ are increasing T-filters. Let $q \in \mathbb{N}$ be such that $r_q < s$. Then $\mathcal{G}$ is clearly secant with $\mathcal{P}(\mathcal{F}_q)$.

Now we suppose that $\mathcal{F}$ is a T-filter. Then we just have to consider the case when $\mathcal{G}$ is secant with $\mathcal{C}(\mathcal{F})$ and is not equal to $\mathcal{F}$.
First we suppose $\mathcal{F}$ increasing, of center b and diameter r. Hence we have $\mathcal{C}(\mathcal{F}) = d(b, r^-)\cap D$. Then $\mathcal{G}$ has a diameter $s \in]0, r[$, and then it admits elements E of diameter $t \in]s, r[$, included in $d(b, r^-) \cap D$. Since $E \cap \mathcal{C}(\mathcal{F}) \neq \emptyset$, given a point $a \in E \cap \mathcal{C}(\mathcal{F})$ we have $E \subset d(a, t)$. Let $q \in \mathbb{N}$ be such that $r_q > \max(t, |a - b|)$. Then E is included in $d(b, r_q)$ and therefore is included in $\mathcal{P}(\mathcal{F}_q)$. Hence $\mathcal{G}$ is secant with $\mathcal{P}(\mathcal{F}_q)$.
Finally we suppose $\mathcal{F}$ decreasing, of diameter r. Since $\mathcal{G}$ is secant with $\mathcal{C}(\mathcal{F})$, and is not less thin than $\mathcal{F}$, we have $r < s$, and there exists $t \subset]r, s[$ and $a \in D$ such that $\mathcal{G}$ is secant with $(K \setminus d(a, t)) \cap D$ while $\mathcal{F}$ is secant with $d(a, t^-) \cap D$. Let $q \in \mathbb{N}$ be such that $r_q < t$ and let a_q be a center of $\mathcal{F}_q$. Then a_q belongs to $d(a, t)$ and we have $\mathcal{P}(\mathcal{F}_q) = (K \setminus d(a, r_q^-)) \cap D$. Hence $\mathcal{G}$ is secant with $\mathcal{P}(\mathcal{F}_q)$ and this finishes showing that $Mult(H(D), \mathcal{U}_D)$ contains no norm.

On the second hand, reciprocally, we suppose that $Mult(H(D), \mathcal{U}_D)$ contains no norm and we will show that D admits an ascending chain of T-filters $(\mathcal{F}_n)_{n\in\mathbb{N}}$ whose returning filter is either a T-filter or a Cauchy filter.

We denote by $\mathcal{R}'$ the set of the diameters of the $\mathcal{F} \in incT(D)$, by $\mathcal{R}''$ the set of the diameters of the $\mathcal{F} \in decT(D)$, and we put $\mathcal{R} = \mathcal{R}' \cup \mathcal{R}''$. Since $H(D)$ has no norm, by Theorem 1 $\mathcal{R}$ is not empty. Since D belongs to $\mathcal{A}$, by Condition A) $\mathcal{R}$ is obviously bounded. We put $t = \sup(\mathcal{R})$. Let $a \in D$. We will show that $incT(D) \neq \emptyset$. Indeed, suppose $incT(D) = \emptyset$. First let D be bounded, of diameter S. Any decreasing filter on D has a diameter $r < S$, and therefore the circular filter $\mathcal{G}$ of center a and diameter S is secant with D, but (of course) is not a T-filter on D, and is not secant with the beach of any decreasing T-filter on D. As a consequence, by Theorem 1 $\varphi_\mathcal{G}$ is a norm. Thus we see that D is not bounded. Then, any circular filter $\mathcal{G}$ of center a and diameter $r > t$ is not a T-filter and is not secant with the beach of any T-filter. Finally this shows that $\varphi_\mathcal{G}$ is a norm again. Thus we see that $incT(D)$ is not empty, and neither is $\mathcal{R}'$.

Now, we put $s = \inf(\mathcal{R}')$. First we suppose $s \in \mathcal{R}'$. Let $\mathcal{T} \in incT(D)$ satisfy $diam(\mathcal{T}) = s$, and let $b \in \mathcal{C}(\mathcal{T})$. Then for every $r \in]0, s[$, the circular filter of center b and diameter r is not a decreasing T-filter, and therefore is secant with the beach of a decreasing T-filter. Hence there exists a decreasing T-filter $\mathcal{F}$ of center b and diameter $\ell \geq s$. But since $H(D)$ has no divisors of zero, $\mathcal{F}$ is not complementary with $\mathcal{T}$, hence we have $s \leq \ell$, i.e. $s \leq \ell < r$. So, we clearly deduce the existence of a sequence of decreasing T-filters $(\mathcal{F}_n)_{n\in\mathbb{N}}$, such that each one admits b as a center and has a diameter r_n satisfying $r_n < r_{n+1} < s$, $\lim_{n\to\infty} r_n = s$. Therefore the sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$

is an ascending chain of decreasing T-filters such that $\bigcap_{n\in\mathbb{N}} \mathcal{P}(\mathcal{F}_n) = \mathcal{C}(\mathcal{T})$, hence the returning filter of the ascending chain $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a T-filter.

Now, we suppose $s \notin \mathcal{R}'$. Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence in $incT(D)$ such that $\lim_{n\to\infty} diam(\mathcal{T}_n) = s$, with $diam(\mathcal{T}_{n+1}) < diam(\mathcal{T}_n)$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we put $A_n = \widetilde{\mathcal{C}(\mathcal{T}_n)}$, and $r_n = diam(A_n)$. By Lemma 1 the sequence $\mathcal{C}(\mathcal{T}_n)_{n\in\mathbb{N}}$ is strictly decreasing, and so is the sequence $(A_n)_{n\in\mathbb{N}}$. Besides, the sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is an ascending chain of increasing T-filters. Obviously each A_n contains a hole T_n of D. If $s = 0$, then we have $\delta(b, T_n) \le r_n$, and therefore b does not belong to $\overset{\circ}{\overline{D}}$, but then, by Condition A) b must belong to $\overline{D}$. Thus, the sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is an ascending chain of increasing T-filters that converges to b, and therefore the returning filter of the ascending chain $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is a Cauchy filter.

Finally, it only remains to consider the case when $s > 0$, with $s \notin \mathcal{R}'$. Let $\mathcal{F}$ be the returning filter of the sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$. If $\mathcal{F}$ were a T-filter, or were secant with the beach of an increasing T-filter, this increasing T-filter would have a diameter inferior or equal to s. Hence $\mathcal{F}$ either is a decreasing T-filter or is secant with the beach of a decreasing T-filter. Of course, if $\mathcal{F}$ is a T-filter, it is just the returning filter of the chain $(\mathcal{T}_n)_{n\in\mathbb{N}}$. Finally if $\mathcal{F}$ is secant with the beach of a decreasing T-filter $\mathcal{G}$, then we have $diam(\mathcal{G}) \le s$ because if $diam(\mathcal{G})$ were strictly superior to s, then $\mathcal{G}$ would be complementary to $\mathcal{T}_n$ when n is big enough, and therefore $H(D)$ would have divisors of zero. Hence we have $diam(\mathcal{G}) = s$, and therefore $\mathcal{G}$ is just the returning filter of the sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$. This finishes proving that D admits an ascending chain of T-filters $(\mathcal{F}_n)_{n\in\mathbb{N}}$ whose returning filter is either a T-filter or a Cauchy filter, and this ends the proof of Theorem 2.

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