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## MINIMAL-HAUSDORFF $p$ -ADIC LOCALLY

### CONVEX SPACES

W.H. Schikhof

**ABSTRACT.** In this note we characterize, in various ways, those Hausdorff locally convex spaces over a non-archimedean valued field  $K$  that do not admit a strictly weaker Hausdorff locally convex topology (Theorems 7 and 9.2). Our results extend the ones obtained by N. De Grande-De Kimpe in [1], Proposition 8-11, for spherically complete  $K$ . For an analogous theory for compactoids instead of locally convex spaces we refer to [6].

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**PRELIMINARIES.** Throughout  $K$  is a complete non-archimedean valued field whose valuation  $|\cdot|$  is non-trivial. All spaces are over  $K$ . We will use the notations and terminology of [2] for Banach spaces and of [7] and [3] for locally convex spaces. In particular, for a subset  $X$  of a locally convex space  $E$  we write  $[X]$  for the linear span of  $X$  and  $\overline{X}$  for the closure of  $X$ . The algebraic dual of  $E$  is denoted by  $E^*$ , its topological dual by  $E'$ . For locally convex spaces  $E$  and  $F$  the expression  $E \sim F$  indicates that  $E$  and  $F$  are linearly homeomorphic ('isomorphic'). The completion of a Hausdorff locally convex space  $E$  is denoted  $\widehat{E}$ .

We will say that a subspace  $D$  of a locally convex space  $E$  is *topologically complemented* if there exists a subspace  $S$  of  $E$  such that  $S \times D \xrightarrow{+} E$  is a linear homeomorphism or, equivalently, if there exists a continuous linear projection of  $E$  onto

*D.*

1. A (non-archimedean) seminorm  $p$  on a vector space  $E$  is called *of finite type* if  $E_p := E/\text{Ker } p$  is finite-dimensional. A locally convex space  $E$  is called *of finite type* if each continuous seminorm is of finite type or, equivalently, if there is a generating set of seminorms of finite type.

One verifies without effort that the class of locally convex spaces of finite type is closed for the formation of products, subspaces and continuous linear images (in particular, quotients). If a Hausdorff locally convex space is of finite type then so is its completion.

Recall ([4]) that a subset  $X$  of a locally convex space  $E$  is a *local compactoid* in  $E$  if for every zero neighbourhood  $U$  in  $E$  there exists a finite-dimensional subspace  $D$  of  $E$  such that  $X \subset U + D$ .

**2. THEOREM.** *For a locally convex space  $E = (E, \tau)$  the following are equivalent.*

( $\alpha$ )  *$E$  is of finite type.*

( $\beta$ )  *$\tau$  is a weak topology.*

( $\gamma$ )  *$E$  is a local compactoid in  $E$ .*

*Proof.* ( $\alpha$ )  $\Rightarrow$  ( $\beta$ ). We prove that  $\tau$  is equal to the weak topology  $\sigma := \sigma(E, E')$ . Obviously,  $\sigma$  is weaker than  $\tau$ . Conversely, let  $p$  be a  $\tau$ -continuous seminorm on  $E$ . By finite-dimensionality of  $E_p$  there exists an  $n \in \mathbb{N}$  and  $f_1, \dots, f_n \in E^*$  such that  $q : x \mapsto \max |f_n(x)|$  is equivalent to  $p$ . Then  $f_1, \dots, f_n$  are  $\tau$ -continuous, hence  $\sigma$ -continuous, so is  $q$  and is  $p$ . It follows that  $\tau$  is weaker than  $\sigma$ .

( $\beta$ )  $\Rightarrow$  ( $\gamma$ ). Let  $U$  be a zero neighbourhood in  $(E, \tau)$ . By ( $\beta$ ) there exist an  $\varepsilon > 0$  and  $f_1, \dots, f_n \in E'$  such that  $U \supset \bigcap_i \{x \in E : |f_i(x)| < \varepsilon\}$ . So,  $U$  contains  $H := \bigcap_i \text{Ker } f_i$ , a space of finite codimension. Let  $D$  be a complement of  $H$ . Then  $D$  is finite-dimensional and  $E \subset H + D \subset U + D$ .

( $\gamma$ )  $\Rightarrow$  ( $\alpha$ ). Let  $p$  be a  $\tau$ -continuous seminorm on  $E$ , let  $D$  be a finite-dimensional subspace of  $E$  such that  $E \subset \{x \in E : p(x) < 1\} + D$ . The seminorm  $x \mapsto \inf_{d \in D} p(x-d)$  is smaller than  $p$ , hence is  $\tau$ -continuous, it is bounded on  $E$ , hence it is identically zero i.e.  $E$  is in the closure of  $D$  with respect to the topology induced by  $p$ . Now  $D + \text{Ker } p$  is closed in that topology, so  $E \subset D + \text{Ker } p$  so that  $E_p = E/\text{Ker } p$  is finite-dimensional. ■

**3. COROLLARY.** *If every countably generated subspace of a locally convex space  $E = (E, \tau)$  is of finite type then so is  $E$ .*

*Proof.* Let  $p$  be a continuous seminorm on  $E$ , let  $X$  be a countably infinite set in  $E_p$ , let  $\pi : E \rightarrow E_p$  be the quotient map. Select a countable set  $Y \subset E$  with  $\pi(Y) = X$ . By assumption  $[Y]$  is of finite type so there is a non-trivial linear combination of elements of  $Y$  in  $\text{Ker } p$ . Applying  $\pi$  we can conclude that  $X$  is not linearly independent. It follows that  $E_p$  is finite-dimensional. ■

4. If the topology  $\tau$  of Theorem 2 is Hausdorff we may add the following equivalent statement to  $(\alpha) - (\gamma)$ .

$(\delta)$   $E$  is linearly homeomorphic to a subspace of  $K^I$  for some set  $I$ .

Indeed, the map  $x \mapsto (f(x))_{f \in E'}$  is an injection  $(E, \tau) \rightarrow K^{E'}$ . If  $(\beta)$  holds it is a topological embedding and we have  $(\beta) \implies (\delta)$ . To arrive at  $(\delta) \implies (\alpha)$  observe that products and subspaces of spaces of finite type are of finite type. ■

5. For an index set  $I$ , the spaces  $K^I$  (with the product topology) and  $K^{(I)} := \{x \in K^I : x_i \neq 0 \text{ only for finitely many } i \in I\}$  (with the strongest locally convex topology) are easily seen to be strong duals of one another via the pairing  $(x, y) = \sum_{i \in I} x_i y_i$  ( $x \in K^{(I)}, y \in K^I$ ). Both spaces are complete. Let  $D$  be a closed subspace of  $K^I$ . Since  $K^I$  is strongly polar ([3] 4.5(iv)) there is a subspace  $S$  of  $K^{(I)}$  such that  $D = S^0 := \{y \in K^I : (x, y) = 0 \text{ for all } x \in S\}$ . This  $S$  is automatically closed and isomorphic to  $K^{(J_1)}$  when  $J_1$  is the cardinality of an algebraic base of  $S$ .  $S$  has an algebraic complement  $T$  which is again closed and isomorphic to  $K^{(J_2)}$  for some index set  $J_2$ . Then,  $S$  is topologically complemented and from  $K^{(I)} \sim S \times T$  one deduces  $S^0 \sim T' \sim K^{J_2}$  and  $K^I \sim S^0 \times T^0$ . We have found the following.

**PROPOSITION.** *Closed subspaces of  $K^I$  are again linearly homeomorphic to a power of  $K$ , and are topologically complemented. Quotients of  $K^I$  are again linearly homeomorphic to a power of  $K$ . ■*

6. A Hausdorff locally convex space  $(E, \tau)$  is said to be *minimal* if for every Hausdorff locally convex topology  $\tau'$  weaker than  $\tau$  we have  $\tau' = \tau$ . Our aim is to prove Theorem 7 below.

6.1. *A minimal space is complete. Proof.* Let  $(E^\wedge, \tau^\wedge)$  be the completion of a minimal space  $(E, \tau)$ . Suppose there exists an  $x \in E^\wedge \setminus E$ ; we derive a contradiction. The space  $Kx$  is closed so the quotient topology on  $E^\wedge/Kx$  is Hausdorff, let  $\pi : E^\wedge \rightarrow E^\wedge/Kx$  be the quotient map.  $\pi$  is injective on  $E$ . Now let  $i \mapsto x_i$  be a net in  $E$  converging to  $x$

with respect to  $\tau^\wedge$ . Then it does not converge in  $(E, \tau)$  while  $\pi(x_i) \rightarrow 0$ . Hence,  $\pi|E$  is not a homeomorphism conflicting the minimality of  $E$ . ■

**6.2.** *A closed subspace of a minimal locally convex space is minimal.*

*Proof.* It suffices to prove the following. Let  $D$  be a closed subspace of a Hausdorff locally convex space  $(E, \tau)$ , let  $\nu$  be a locally convex Hausdorff topology on  $D$  such that  $\nu \leq \tau|D$ . Then there exists a Hausdorff locally convex topology  $\nu_1$  on  $E$  such that  $\nu_1|D = \nu$  and  $\nu_1 \leq \tau$ . To this end, let  $\nu_1$  be the topology generated by all  $\tau$ -continuous seminorms  $p$  on  $E$  for which  $p|D$  is  $\nu$ -continuous. Then obviously  $\nu_1 \leq \tau$  and  $\nu_1|D \leq \nu$ . Every  $\nu$ -continuous seminorm on  $D$  can in a standard way be extended to a  $\tau$ -continuous seminorm on  $E$  so we have also  $\nu_1|D \geq \nu$ . Finally, to see that  $\nu_1$  is Hausdorff, let  $a \in E \setminus D$ ; it is enough to find a  $\nu_1$ -continuous seminorm separating  $\{0\}$  and  $\{a\}$ . Now  $D$  is  $\tau$ -closed so there is a  $\tau$ -continuous seminorm  $p$  such that  $\inf\{p(a - d) : d \in D\} > 0$ . Then the formula

$$q(x) = \inf\{p(x - d) : d \in D\}$$

defines a  $\tau$ -continuous seminorm  $q$  for which  $q(a) > 0$ . This  $q$  is automatically  $\nu_1$ -continuous since  $q|D = 0$ . ■

**6.3.** *A minimal topology is of finite type.* Let  $(E, \tau)$  be minimal. By Corollary 3 it suffices to show that, for any countable  $X \subset E$ , the space  $D := \overline{[X]}$  is of finite type. From 6.2 we obtain that  $(D, \tau|D)$  is minimal. Now  $D$  is of countable type hence is a (strongly) polar space ([3] 4.4) so that its weak topology is Hausdorff. By minimality  $\tau|D$  equals this weak topology and therefore is of finite type (Theorem 2). ■

We can now prove our main Theorem.

**7. THEOREM.** *For a Hausdorff locally convex space  $E = (E, \tau)$  the following are equivalent.*

- ( $\alpha$ )  $E$  is minimal.
- ( $\beta$ ) For every Hausdorff locally convex space  $X$  and for every continuous linear map  $T : E \rightarrow X$  the image  $TE$  is closed (complete).
- ( $\gamma$ ) For every Hausdorff locally convex space  $X$ , every surjective continuous linear map  $T : E \rightarrow X$  is open.
- ( $\delta$ )  $E$  is complete and of finite type.
- ( $\varepsilon$ )  $E$  is a complete local compactoid.
- ( $\zeta$ )  $E$  is linearly homeomorphic to a closed subspace of some power of  $K$ .

( $\eta$ )  $E$  is linearly homeomorphic to a power of  $K$ .

*Proof.* We have  $(\alpha) \Rightarrow (\delta)$  by 6.1 and 6.3,  $(\delta) \Rightarrow (\varepsilon)$  by Theorem 2,  $(\varepsilon) \Rightarrow (\zeta)$  from 4, and  $(\zeta) \Rightarrow (\eta)$  from 5. Next, we prove  $(\eta) \Rightarrow (\alpha)$ . Let  $\tau$  be a Hausdorff locally convex topology on  $K^I$ , weaker than the product topology. Then  $(K^I, \tau)'$  is a subspace of  $F := (K^I)' \sim K^{(I)}$ . As  $\tau$  is Hausdorff,  $(K^I, \tau)'$  is  $\sigma(F, K^I)$ -dense in  $F$ . But every subspace of  $F$  is  $\sigma(F, K^I)$ -closed so that  $(K^I, \tau)' = F$  i.e.  $(K^I, \tau)$  and  $K^I$  have the same dual space. Since both  $\tau$  and the product topology are weak topologies (Theorem 2) it follows that  $\tau$  equals the product topology. We conclude that the product topology is minimal.

At this stage we have proved the equivalence of  $(\alpha), (\delta), (\varepsilon), (\zeta), (\eta)$ . To prove  $(\alpha) \Rightarrow (\beta)$  and  $(\alpha) \Rightarrow (\gamma)$  we factorize  $T$  in the obvious manner:

$$\begin{array}{ccc} E & \xrightarrow{T} & TE \\ \searrow \pi & & \nearrow T_1 \\ & E/\text{Ker } T & \end{array}$$

From the equivalence  $(\alpha) \Leftrightarrow (\eta)$  and 5 it follows that the quotient  $E/\text{Ker } T$  is also minimal, so  $T_1$  is not only continuous but also open implying that  $T$  is open and that  $TE \sim E/\text{Ker } T$  is complete. Obviously,  $(\gamma) \Rightarrow (\alpha)$ . Finally we prove  $(\beta) \Rightarrow (\delta)$ . It suffices to prove that  $E$  is of finite type. Suppose  $E$  is not. Then we could find a continuous seminorm  $p$  such that  $E_p$  is infinite-dimensional. By  $(\beta)$ ,  $E_p$  is a Banach space with respect to the norm  $\bar{p}$  induced by  $p$ . Now we can find a seminorm  $q \leq \bar{p}$  on  $E_p$  that is not equivalent to  $\bar{p}$  (for example, let  $e_1, e_2, \dots \in E_p$  be a  $\frac{1}{2}$ -orthogonal set with respect to  $\bar{p}$ , bounded away from zero. Then the formula

$$q_1 \left( \sum_{n=1}^{\infty} \lambda_n e_n \right) = \frac{1}{2} \max \left\{ \frac{|\lambda_n|}{n} \bar{p}(e_n) : n \in \mathbf{N} \right\}$$

defines a nonequivalent norm  $q_1$  on the  $\bar{p}$ -closure  $D$  of  $[e_1, e_2, \dots]$ ; next extend  $q_1$  to a norm  $q$  on  $E_p$  by the formula  $q(x) = \inf \{ \max(q_1(d), \bar{p}(x-d)) : d \in D \}$ . Then by Banach's Open Mapping Theorem the space  $(E_p, q)$  is not complete and the image of the map  $E \rightarrow (E_p, \bar{p}) \rightarrow (E_p, q) \rightarrow (E_p, q)^\wedge$  is not closed, a contradiction. ■

**REMARK.** If  $\tau$  is a Hausdorff vector topology on  $K^I$  (not necessarily locally convex) weaker than the product topology then these topologies coincide. Thus,  $K^I$  is also "minimal in the category of topological vector spaces". The above proof of  $(\eta) \Rightarrow (\alpha)$  applies with only obvious modifications.

8. Let  $K$  be spherically complete. Then we may add the following equivalent condition to  $(\alpha) - (\eta)$  of Theorem 7.

$(\theta)$   $E$  is  $c$ -compact.

Indeed, we have obviously  $(\eta) \Rightarrow (\theta)$  and  $(\theta) \Rightarrow (\beta)$ . ■

9. If  $K$  is not spherically complete the only space satisfying  $(\theta)$  is  $\{0\}$ . On the other hand we have seen in Theorem 7 that minimal spaces still behave very much  $c$ -compact-like. The next Corollary further confirms this idea.

**9.1. COROLLARY.**

- (i) *The product of a collection of minimal spaces is minimal.*
- (ii) *A closed linear subspace of a minimal space is minimal.*
- (iii) *A Hausdorff image of a minimal space under a continuous linear map is minimal.*
- (iv) *Let a minimal space  $E$  be a subspace of some locally convex space  $X$  such that  $X'$  separates the points of  $X$ . Then  $E$  is topologically complemented in  $X$ .*

*Proof.* (i), (ii) and (iii) are obvious consequences of 7 and 5. To prove (iv), let  $\sigma$  be the weak topology  $\sigma(X, X')$ , which is Hausdorff by assumption. The inclusion map  $E \xrightarrow{i} (X, \sigma)$  is continuous, hence, by minimality, a homeomorphism onto  $i(E)$ . Now  $X^\wedge := (X, \sigma)^\wedge$  is complete and of finite type hence, by Theorem 7 and 5, there is a continuous projection  $P : X^\wedge \rightarrow E$  whose restriction to  $X$  is the required one. ■

Finally we prove a characterization of minimal spaces that resembles very much the definition of  $c$ -compactness (also compare [5] §5).

**9.2. THEOREM.** *For a Hausdorff locally convex space  $E = (E, \tau)$  the following are equivalent.*

- $(\alpha)$   $E$  is minimal.
- $(\beta)$  *If  $\{C_i : i \in I\}$  is a collection of closed linear manifolds in  $E$  with the finite intersection property then  $\bigcap_i C_i \neq \emptyset$ .*

*Proof.*  $(\alpha) \Rightarrow (\beta)$ . We may assume that  $E = K^I$  for some set  $I$ . Using the duality of 5 we may write  $C_i = f_i + D_i^0$  where, for each  $i \in I$ ,  $f_i \in K^I = (K^{(I)})'$  and  $D_i$  is a subspace of  $K^{(I)}$ . The finite intersection property ensures that (we may assume that) the collection of the  $D_i$ 's is closed with respect to finite sums. The formula

$$f(x) = f_i(x) \text{ if } i \in I, x \in D_i$$

is easily seen to define a map on  $\bigcup D_i = \sum D_i$  which is linear and can be extended to a linear function  $f$  on  $K^{(I)}$ . Then  $f \in E$  and, for each  $i \in I$ ,  $f = f_i$  on  $D_i$  so that  $f \in C_i$ . Hence  $f \in \bigcap_i C_i$  and we have  $(\beta)$ .

$(\beta) \Rightarrow (\alpha)$ . First, let  $E$  be a normed space satisfying  $(\beta)$ ; we prove that  $E$  is finite-dimensional. In fact, if  $E$  were infinite-dimensional we could find a  $\frac{1}{2}$ -orthogonal sequence  $e_1, e_2, \dots$  in  $E$  with  $\inf\{\|e_n\| : n \in \mathbb{N}\} > 0$ . Set  $C_n := e_1 + \dots + e_n + \overline{[e_m : m > n]}$ . Then each  $C_n$  is a closed linear manifold,  $C_1 \supset C_2 \supset \dots$ , but one proves easily that  $\bigcap_n C_n = \emptyset$ .

Now let  $E$  be an arbitrary space satisfying  $(\beta)$ ; we shall prove that  $E$  is of finite type and complete (than we are done by Theorem 7). If  $p$  is a continuous seminorm then the normed space  $E_p$  is easily seen also to satisfy  $(\beta)$  so, by the above,  $E_p$  is finite-dimensional implying that  $E$  is of finite type. To prove completeness, assume that  $E$  is infinite-dimensional, and let  $x \in E^\wedge$ . For every convex neighbourhood  $U$  of  $x$  in  $E^\wedge$ , let  $H_U$  be the largest linear manifold in  $U$  that contains  $x$ . Then  $H_U$  is closed. Since  $E^\wedge$  is of finite type each  $H_U$  has finite codimension. Then so has  $H_U \cap E$ , in particular  $H_U \cap E \neq \emptyset$  for each convex neighbourhood  $U$  of  $x$  in  $E^\wedge$ . If  $U, V$  are convex neighbourhoods of  $x$  in  $E^\wedge$  then  $H_{U \cap V} = H_U \cap H_V$ . Thus,  $\{H_U \cap E : U \text{ is convex neighbourhood of } x \text{ in } E^\wedge\}$  has the finite intersection property, so its intersection is not empty by  $(\beta)$ . From

$$\emptyset \neq \bigcap_U (H_U \cap E) \subset \bigcap_U H_U = \{x\}$$

we infer that  $x \in E$ . It follows that  $E$  is complete. ■.

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