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THE MACKEY-ARENS AND HAHN-BANACH THEOREMS

FOR SPACES OVER VALUED FIELDS

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Abstract. Characterizations of the spherical completeness of a non-archimedean complete non-trivially valued field in terms of classical theorems of Functional Analysis are obtained.

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Spherical completeness

Throughout this paper $K = (K, | \cdot |)$ will denote a non-archimedean complete valued field with a non-trivial valuation $| \cdot |$. It is well-known that the absolute value function $| \cdot |$ of the field of the real numbers \mathbb{R} or the complex numbers \mathbb{C} satisfies the following properties :

- (i) $0 \leq |x|$, $|x| = 0$ iff $x = 0$,
- (ii) $|x + y| \leq |x| + |y|$,
- (iii) $|xy| = |x||y|$, $x, y \in \mathbb{R}$ or $x, y \in \mathbb{C}$.

If K is a field, then by a *valuation* on K we will mean a map $| \cdot |$ of K into \mathbb{R} satisfying the above properties; in this case $(K, | \cdot |)$ will be called a *valued field*. We will assume that K is complete with respect to the natural metric of K .

It turns out that if K is not isomorphic to \mathbb{R} or \mathbb{C} , then its valuation satisfies the following *strong triangle inequality*, cf. e.g. [12],

- (ii') $|x + y| \leq \max \{|x|, |y|\}$, $x, y \in K$.

A valued field K whose valuation satisfies (ii') will be called *non-archimedean* and its valuation *non-archimedean*.

Let us first recall the following well-known result of Cantor

Theorem 0 *Let (X, ρ) be a metric space. Then it is complete iff every shrinking sequence of closed balls whose radii tend to zero has non-empty intersection.*

Consider the set \mathbb{N} of the natural numbers endowed with the following metric ρ defined by $\rho(m, n) = 0$ if $m = n$ and $1 + \max(\frac{1}{m}, \frac{1}{n})$ if $m \neq n$.

Then the metric ρ is non-archimedean, i.e. $\rho(m, n) = 0$ iff either $m = n$, or $\rho(m, n) \leq \max\{\rho(m, k), \rho(k, n)\}$, for all $m, n, k \in \mathbb{N}$.

It is easy to see that every shrinking sequence of balls in \mathbb{N} whose radii tend to zero has non-empty intersection; note that every ball whose radius is smaller than 1 contains exactly one point. On the other hand, the balls $B_{1+\frac{1}{2}}(1), B_{1+\frac{1}{3}}(2), \dots$, form a decreasing sequence and their intersection is empty. This suggests the following, see Ingleton [3] :

A non-archimedean metric space (X, ρ) will be said to be *spherically complete* if the intersection of every shrinking sequence of its balls is non-empty.

Clearly spherical completeness implies completeness; the converse fails : The space (\mathbb{N}, ρ) is complete but not spherically complete. We refer to [11] and [12] for more information concerning this property.

Theorem 1 *Let (X, ρ) be a non-archimedean metric space. Then (X, ρ) is spherically complete iff given an arbitrary family \mathcal{B} of balls in X , no two of which are disjoint, then the intersection of the elements of \mathcal{B} is non-empty.*

The aim of this note is to collect a few characterizations of the spherical completeness of K in terms of the Mackey-Arens, Hahn-Banach and weak Schauder basis theorems, respectively, see [5], [6], [7], [12].

The Mackey-Arens and Hahn-Banach theorems

The terms "*K-space*", "*topology*", "*seminorm or norm*" will mean a Hausdorff locally convex space (lcs) over K , a locally convex topology (in the sense of Monna) and a non-archimedean seminorm (norm), respectively. A seminorm on a vector space E over K is *non-archimedean* if it satisfies condition (ii'). Clearly the topology τ generated by a norm is *locally convex*. Recall that a *topological vector space* (tvs) $E = (E, \tau)$ over K is *locally convex* [10] if τ has a basis of absolutely convex neighbourhoods of zero. A subset U of E is *absolutely convex* (in the sense of Monna [10]) if $\alpha x + \beta y \in U$, whenever $x, y \in U$, $\alpha, \beta \in K$, $|\alpha| \leq 1, |\beta| \leq 1$. For the basic notions and properties concerning tvs and lcs over K we refer to [10], [11], [13].

A locally convex (lc) topology γ on (E, τ) is called *compatible* with τ , if τ and γ have the same continuous linear functionals; $(E, \tau)^* = (E, \gamma)^*$. (E, τ) is *dual-separating* if $(E, \tau)^*$ separates points of E . If G is a vector subspace of E , $\tau|G$ and τ/G denote the topology τ restricted to G and the quotient topology of the quotient space E/G , respectively. If α is a finer l.c. topology on E/G , we denote by $\gamma := \tau \vee \alpha$ the weakest l.c. topology on E such that $\tau \leq \gamma$, $\gamma/G = \alpha$, $\gamma|G = \tau|G$, cf. e.g. [1]. The sets $U \cap q^{-1}(V)$ compose a basis of neighbourhoods of zero for γ , where U, V run over bases of neighbourhoods of zero for τ and α , respectively, $q := EE/G$ is the quotient map. By $\sup\{\tau, \alpha\}$ we denote the weakest l.c. topology on E which is finer than τ and α .

By the *Mackey topology* $\mu(E, E^*)$ associated with a lcs $E = (E, \tau)$ we mean the finest locally convex topology on E compatible with τ . In [14] Van Tiel showed that every lcs over spherically complete K admits the Mackey topology.

In [3] Ingleton obtained a non-archimedean variant of the Hahn-Banach theorem for normed spaces, where K is spherically complete.

Theorem 2 *If $E = (E, \|\cdot\|)$ is a normed space over K and K is spherically complete and D is a subspace of E , then for every continuous linear functional $g \in D^*$ there exists a continuous linear extension $f \in E^*$ of g such that $\|g\| = \|f\|$.*

This suggests the following : A lcs E will be said to have the *Hahn-Banach Extension Property* (HBEP) [9] if for every subspace D every $g \in D^*$ can be extended to $f \in E^*$. It is known that every lcs over spherically complete K has the HBEP, cf. e.g. [11].

The following theorem characterizes the spherical completeness of K in terms of classical theorems of Functional Analysis; cf. also [5], [6] and [12], Theorem 4.15. The proof of our Theorem 3 uses some ideas of [4] extended to the non-archimedean case.

l^∞ (resp. c_0) denotes the space of the bounded sequences (resp. the sequences of limit 0) with coefficients in K .

Theorem 3 *The following conditions on K are equivalent :*

- (i) *K is spherically complete.*
- (ii) *There exists $g \in (l^\infty)^*$ such that $g(x) = \sum_n x_n$ for every $x \in c_0$.*
- (iii) *$(l^\infty/c_0)^* \neq 0$.*
- (iv) *Every lcs over K admits the Mackey topology.*
- (v) *Every lcs over K (resp. K -normed space) has the HBEP.*
- (vi) *The completion of a dual-separating lcs over K (resp. K -normed space) is dual-separating.*
- (vii) *Every closed subspace of a dual-separating lcs over K (resp. K -normed space) is weakly closed.*
- (viii) *For every lcs over K (resp. K -normed space) every weakly convergent sequence is convergent.*
- (ix) *Every weak Schauder basis in a lcs over K (resp. K -normed space) is a Schauder basis.*

Proof By Theorem 4.15 of [12] conditions (i), (ii), (iii) are equivalent. (i) *implies* (iv) : [14], Theorem 4.17. (i) *implies* (v) : [3], [11]. The implications (v) *implies* (vi), (v) *implies* (vii) are obvious. (i) *implies* (viii) : see [7]; Theorem 3, [2], Proposition 4.3. (viii) *implies* (ix) is obvious.

(iv) *implies* (i) : Assume that K is not spherically complete and consider the space l^∞ of K -valued bounded sequences endowed with the topology τ generated by the norm $\|x\| = \sup_n |x_n|$, $x = (x_n) \in l^\infty$. Let f be a non-zero linear function on l^∞ with $f|_{c_0} = 0$. Set $E := l^\infty$ and $F := c_0$. Define a linear functional h on the quotient space E/F by $h(q(x)) = f(x)$, where $q : E \rightarrow E/F$ is the quotient map. Let α be the quotient topology

of E/F . Since $(E/F, \alpha)^* = 0$, see (iii) *implies* (i), F is dense in the weak topology $\sigma(E, E^*)$ (recall that $E^* = F$, [12], Theorem 4.17). Observe that on E/F there exists a K -normed topology β such that $(E/F, \alpha)$ and $(E/F, \beta)$ are isomorphic and h is continuous in the topology $\sup\{\alpha, \beta\}$. Indeed, choose $x_0 \in E/F$ such that $h(x_0) = 2$ and define a linear map $T : E/F \rightarrow E/F$ by $T(x) := x - h(x)x_0$, $x \in E/F$. Then $T^2 = id$. Define $\beta := T(\alpha)$ (the image topology). Then h is continuous in the topology $\sup\{\alpha, \beta\}$.

Set $\gamma_\alpha := \sigma(E, E^*) \vee \alpha$, $\gamma_\beta := \sigma(E, E^*) \vee \beta$. Then γ_α and γ_β are compatible with $\sigma(E, E^*)$, hence with τ . Assume that E admits the finest locally convex topology μ compatible with τ . Then $\sigma(E, E^*) \leq \sup\{\gamma_\alpha, \gamma_\beta\} \leq \mu$.

On the other hand $\sup\{\gamma_\alpha, \gamma_\beta\}/F = \sup\{\alpha, \beta\}$. Therefore f is continuous in $\sup\{\gamma_\alpha, \gamma_\beta\}$. Since f is not continuous in $\sigma(E, E^*)$ we get a contradiction. The proof is complete.

(vi) *implies* (i) : Assume that K is not spherically complete. By the Baire category theorem we find a dense subspace G of E with $\dim(E/G) = \dim(E/F)$, where E and F are defined as above. Indeed, let $\{x_s\}_{s \in S}$ be a Hamel basis of E and (S_n) a partition of S such that $S = \bigcup_{n \in \mathbb{N}} S_n$ and $\text{card } S_n = \text{card } S$, $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, we denote by G_n the vector space generated by the elements x_s , when s runs in $\bigcup_{k=1}^n S_k$. Then we have $E = \bigcup_{n \in \mathbb{N}} G_n$ and $\dim G_n = \dim(E/G_n) = \dim E$, $n \in \mathbb{N}$.

Then there exists $m \in \mathbb{N}$ such that G_m is dense in E . Hence we obtain a subspace G as required. Let α be a K -normed topology on E/G such that the spaces $(E/G, \alpha)$ and $(E/F, \tau/F)$ are isomorphic. Then the topology $\gamma := \tau \vee \alpha$ is compatible with τ and strictly finer than τ . Let E_0 be the completion of the dual-separating K -normed space (E, γ) . Choose $x \in E_0 \setminus E$. There exists a sequence (x_n) in E and $y \in E$ such that $x_n \rightarrow x$ in E_0 and $x_n \rightarrow y$ in (E, τ) . Then $f(x - y) = 0$ for all $f \in E_0^*$ but $x - y \neq 0$. This completes the proof.

(vii) *implies* (i) : Assume that K is not spherically complete. The space G constructed in the previous case is closed in (E, γ) and dense in $(E, \sigma(E, E^*))$, where $E^* := (E, \gamma)^*$.

(v) *implies* (i) : Assume that K is not spherically complete. Let (e_n) be the sequence of the unit vectors in E , where E is as above. Then $e_n \rightarrow 0$ in $\sigma(E, E^*)$, [13]. Clearly (e_n) is a normalized Schauder basis in F . If $x = (x_n) \in F$, then $x = \sum_n x_n e_n$. Set $g(x) := \sum_n x_n$. Then g is a well-defined continuous linear functional on F . Suppose that g has a continuous linear extension f to the whole space E . Then $f(e_n) \rightarrow 0$ but $g(e_n) = 1$ for all $n \in \mathbb{N}$, a contradiction.

(viii) *implies* (i) : See the proof of the previous implication.

(ix) *implies* (i) : Assume that K is not spherically complete. The sequence (e_n) is a Schauder basis in $(E, \sigma(E, E^*))$ but it is not a Schauder basis in the original topology of E . The second part of this sentence follows from the fact that E is not of countable type, cf. e.g. [12]. On the other hand, by Theorem 4.17 of [12] (and its proof) the space E is reflexive and for every $g \in E^*$ there exists $(a_n) \in F$ such that $g(x) = \sum_n x_n a_n$ for every

$x = (x_n) \in E$. Since $(E, \sigma(E, E^*))$ is a sequentially complete lcs [12], Theorem 9.6, then $\sum_{k=1}^n x_k e_k$ weakly converges to $x = (x_n)$.

Remark In [9] Martinez-Maurica and Perez-Garcia proved that whenever K is spherically complete, then the local convexity is a *three space property*, i.e. if E is an A-Banach tvs over K and F its subspace such that F and E/F are locally convex, then E is locally convex. Is the converse also true?

By $L(E, F)$ we denote the space of all continuous linear maps between lcs E and F . A topology α on E will be called *compatible* with the pair $(E, L(E, F))$ if $L((E, \alpha), F) = L(E, F)$; if $F = K$, as usual we shall say that α is compatible with the dual pair (E, E^*) , where $E^* := L(E, K)$.

A lcs space F will be said to have the *Mackey-Arens property* (MA-property) if for every lcs space E the finest topology $\mu(E, L(E, F))$ compatible with $(E, L(E, F))$ exists, [7].

As we have already mentioned Van Tiel [14] proved that if K is spherically complete, then K has the MA-property, i.e. every K -space E over spherically complete K admits the finest topology $\mu(E, E^*)$ compatible with the dual pair (E, E^*) . We have already proved the converse : If K is not spherically complete, then ℓ^∞ does not admit the Mackey topology $\mu(\ell^\infty, (\ell^\infty)^*)$. Hence

Corollary K is spherically complete iff it has the MA-property.

On the other hand one has the following

Theorem 4 Every spherically complete normed K -space $F = (F, \|\cdot\|)$ has the MA-property.

We shall need the following

Lemma 1 Let E, F be two vector spaces over K , where F is endowed with a norm $\|\cdot\|$ and p, q are seminorms on E . Let $T : E \rightarrow F$ be a linear map such that $\|(T(x))\| \leq \max\{p(x), q(x)\}$. If F is spherically complete, then there exists two linear maps $T_i : E \rightarrow F$, $i = 1, 2$, such that $T = T_1 + T_2$ and $\|(T_1(x))\| \leq p(x)$, $\|(T_2(x))\| \leq q(x)$, $x \in E$.

Proof Set $P(x, x) = T(x)$, $U(x, y) = \max\{p(x), q(y)\}$, $x, y \in E$. Then $U(x, y)$ is a seminorm on $E \times E$ and $\|(P(x, x))\| = \|(T(x))\| \leq \max\{p(x), q(x)\} = U(x, x)$. Since F is spherically complete, then by Ingleton theorem, cf. e.g. [6], Theorem 4.18, there exists a linear map $P_0 : E \times E \rightarrow F$ extending P such that $\|(P_0(x, y))\| \leq U(x, y)$, $x, y \in E$. To complete the proof it is enough to put $T_1(x) = P_0(x, 0)$, $T_2(x) = P_0(0, x)$.

We shall also need the following lemma. Its proof uses some ideas of [1] and [4].

Lemma 2 Let E, F be two dual-separating K -spaces over non-spherically complete K and such that F is complete and E is an infinite dimensional metrizable and complete. Then E admits two topologies τ_1 and τ_2 strictly finer than the original one of E and compatible with the pair $(E, L(E, F))$ and such that the topology $\sup\{\tau_1, \tau_2\}$ is not compatible with $(E, L(E, F))$.

Proof : Observe that E contains a dense subspace G with $\dim(E/G)=\dim(\ell^\infty/c_0)$. Let h be a non-zero linear functional on E vanishing on G . As above we construct on E two topologies τ_1 and τ_2 strictly finer than the original one τ of E such that $\tau_j|G = \tau|G$ and $(E/G, \tau_j/G)$ is isomorphic to the quotient space ℓ^∞/c_0 , $j = 1, 2$, and h is continuous in $\sup\{\tau_1, \tau_2\}$. We show that the topologies τ_j , $j = 1, 2$, are compatible with the pair $(E, L(E, F))$. Fix $j \in \{1, 2\}$ and non-zero $T \in L((E, \tau_j), F)$. There exists $x_0 \in E$ and $f \in F^*$ such that $f(T(x_0)) \neq 0$. Suppose that $T|G = \{0\}$. Then the map $q(x) \rightarrow f(Tx)$ defines a non-zero continuous linear functional on $(E/G, \tau_j/G)$, $q : E \rightarrow E/G$ is the quotient map. Since $(\ell^\infty/c_0)^* = \{0\}$, [12], Corollary 4.3, we get a contradiction. Hence $T|G$ is non-zero. Since G is dense in E and τ and τ_j coincide on G , there exists a continuous linear extension W of T to E . It is easy to see that $T = W$. Hence $T \in L(E, F)$. Finally the map $x \rightarrow h(x)y$, for fixed $y \in F$, defines a τ -discontinuous linear map H of E into F such that $H \in L((E, \sup \tau_1, \tau_2), F)$.

Proof of Theorem 4 Let $E = (E, \tau)$ be a lcs and \mathcal{F} the family of all topologies on E compatible with $(E, L(E, F))$. It is enough to show that the topology $\mu := \sup \mathcal{F}$ belongs to \mathcal{F} . Let $T : (E, \mu) \rightarrow F$ be a continuous linear map. There exist seminorms p_j on E , $j = 1, \dots, n$, continuous in topologies γ_j ($\gamma_j \in \mathcal{F}$), respectively, and $M > 0$ such that $\|T(x)\| \leq M \max_{1 \leq j \leq n} p_j(x)$ for every $x \in E$. Using Lemma 1 one shows that T is τ -continuous.

Remarks (1) There exist complete normed K -spaces having the MA-property which are not spherically complete. In fact, assume that K is spherically complete; then ℓ^∞ is spherically complete [12], p. 97; hence ℓ^∞ has the MA-property (by our Theorem 4). On the other hand there exists on the space ℓ^∞ another norm ν which is equivalent with the usual norm, such that (ℓ^∞, ν) is not spherically complete [12], p. 50 and p. 98. On the other hand the space (ℓ^∞, ν) has the MA-property.

(2) Let E be an infinite dimensional normed and complete K -space. Since $F := \prod_n E_n / \bigoplus_n E_n$, where $E_n = E$ for every $n \in \mathbb{N}$, is spherically complete for any K [12], Theorem 4.1, then by our Theorem 4 the space F has the MA-property. For concrete spaces put $E = \ell^\infty$; then $F = \ell^\infty/c_0$. If K is not spherically complete, then by Lemma 2 the space ℓ^∞ does not admit the Mackey topology $\mu(\ell^\infty, (\ell^\infty)^*)$ but ℓ^∞/c_0 has the MA-property. In particular there exists on ℓ^∞ the finest topology μ compatible with $(\ell^\infty, L(\ell^\infty, \ell^\infty/c_0))$.

(3) Let E and F be K -spaces and assume that E admits the Mackey topology $\mu = \mu(E, E^*)$. Then the finest topology on E compatible with $((E, \mu), L((E, \mu), F))$ exists and equals μ .

(4) In [13], Corollary 7.9, Schikhof proved that for polarly barrelled or polarly bornological K -spaces (E, τ) where K is not spherically complete, the finest polar topology $\mu(E, E^*)$ compatible with (E, E^*) exists and equals τ .

REFERENCES

- [1] Dierolf S. *A note on lifting of linear and locally convex topologies on a quotient space* Collectanea Math. vol 31 1980 p 193–198
- [2] De Grande-De Kimpe N. and Perez-Garcia C. *Weakly closed subspaces and the Hahn-Banach extension property in p -adic analysis* Indag. Math. vol 50 1988 p 253–261
- [3] Ingleton W. *The Hahn-Banach Theorem for non-archimedean valued fields* Proc. Cambridge Phil. Soc. vol 48 1952 p 41–45
- [4] Kąkol J. *The Mackey-Arens theorem for non-locally convex spaces* Collectanea Math. vol 41 1990 p 129–132
- [5] Kąkol *Remarks on spherical completeness of non-archimedean valued fields* Indag. Math.
- [6] Kąkol J. *The weak basis theorem for K -Banach spaces* Bull. Soc. Belg. vol 45 1993 p 1–4
- [7] Kąkol J. *The Mackey-Arens property for spaces over valued fields* Bull. Acad. Polon Sci. to appear
- [8] Martinez-Maurica J. and Perez-Garcia C. *The Hahn-Banach extension property in a class of normed spaces* Quaestiones Math. vol 8 1986 p 335–341
- [9] Martinez-Maurica J. and Perez-Garcia C. *The three-space problem for a class of normed spaces* Bull. Math. Soc. Belg. vol 39 1987 p 209–214
- [10] Monna A.F. *Analyse non-archimédienne* Lecture Notes in Math. 1970 Springer Berlin
- [11] Prolla J. B. *Topics in Functional Analysis over valued Division Rings* Math. Studies 1982 Amsterdam North-Holland
- [12] Van Rooij A.C.M. *Non-Archimedean Functional Analysis* Marcel Dekker 1978 New York
- [13] Schikhof W.H. *Locally convex spaces over non-spherically complete valued fields* Bull. Soc. Math. Belg. vol 38 1986 p 187–224
- [14] Van Tiel I. *Espaces localement K -convexes* Indag. Math. vol 27 1965 p 249–289

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