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ON UNIFORM EXponential N-DICHOTOMY

M. MEGAN and D.R. LATCU

The problem of uniform exponential $N$-dichotomy of evolutionary processes in Banach spaces is discussed. Generalizations of some well-known results of R. Datko, Z. Zabczyk, S. Rollewicz and A. Ichikawa are obtained. The results are applicable for a large class of nonlinear differential equations.

I - INTRODUCTION.

Let $X$ be a real or complex Banach space with the norm $\|\cdot\|$. Let $T$ be the set defined by

$$T = \{ (t, t_0) : 0 \leq t_0 \leq t < \infty \}$$

Let $\Phi(t, t_0)$ with $(t, t_0) \in T$ be a family of operators with domain $X_{t_0} \subset X$.

**Definition 1.1**

The family $\Phi(t, t_0)$ with $(t, t_0) \in T$ is called an evolutionary process if:

1. $\Phi(t, t_0)x_0 \in X_t$ for all $(t, t_0)$ and $x_0 \in X_{t_0}$;
2. $\Phi(t, t_1)\Phi(t_1, t_0)x_0 = \Phi(t, t_0)x_0$ for $(t, t_1), (t_1, t_0) \in T$ and $x_0 \in X_{t_0}$;
3. $\Phi(t, t)x = x$ for all $t \geq 0$ and $x \in x_t$;
4. for each $t_0 \geq 0$ and $x_0 \in X_{t_0}$ the function $t \mapsto \Phi(t, t_0)x_0$ is continuous on $[t_0, \infty]$;
5. there is a positive nondecreasing function $\varphi : (0, \infty) \to (0, \infty)$ such that $\|\Phi(t, t_0)x_0\| \leq \varphi(t - t_0)\|x_0\|$ for all $(t, t_0) \in T$ and $x_0 \in X_{t_0}$.

Throughout in this paper for each $t_0 \geq 0$ we denote by

$$X_{t_0}^1 = \{ x_0 \in X_{t_0} : \Phi(\cdot, t_0)x_0 \in L_{t_0}^\infty(X) \} \quad \text{and} \quad X_{t_0}^2 = X_{t_0}^2 = X_{t_0} \setminus X_{t_0}^1$$

where $L_{t_0}^\infty(X)$ is the Banach space of $X$-valued function $f$ defined a.e. on $[t_0, \infty)$, such that $f$ is strongly measurable and essentially bounded.
Remark 1.1. If \( x_0 \in X_{t_0}^1 \) and \( t \geq t_0 \) then \( \Phi(t, t_0)x_0 \in X_t^1 \).

Indeed, if \( x_0 \in X_{t_0}^1 \) then
\[
\Phi(., t)\Phi(t, t_0)x_0 = \Phi(., t_0)x_0 \in L_{t_0}^\infty(X) \subset L_t^\infty(X)
\]
and hence \( \phi(t, t_0)x_0 \in X_t^1 \).

Let \( \mathcal{N} \) be the set of strictly increasing real functions \( N \) defined on \([0, \infty)\) which satisfies:
\[
\lim_{t \to 0} N(t) = 0 \quad \text{and} \quad N(t, t_0) \leq N(t)N(t_0)
\]
for all \( t, t_0 \geq 0 \).

Remark 1.2. It is easy to see that if \( N \in \mathcal{N} \) then
i) \( N(t) > 0 \) for every \( t > 0 \);
ii) \( N(0) = 0 \) and \( N(1) \geq 1 \);
iii) \( \lim_{t \to \infty} N(t) = \infty \).

Definition 1.2. Let \( N \in \mathcal{N} \). The evolutionary process \( \Phi(., .) \) is said to be uniformly exponentially \( N \)-dichotomic (and we write u.e.-\( N \)-d.) if there are \( M_1, M_2, \nu_1, \nu_2 > 0 \) such that for all \( t \leq s \leq t_0 \leq 0 \) and \( x_1 \in X_{t_0}^1, x_2 \in X_{t_0}^2 \) we have:
\[
N_{d_1} \quad N(\|\Phi(t, t_0)x_1\|) \leq M_1 e^{-\nu_1(t-s)}N(\|\Phi(s, t_0)x_1\|), \text{ and}
\]
\[
N_{d_2} \quad N(\|\Phi(t, t_0)x_2\|) \leq M_2 e^{-\nu_2(t-s)}N(\|\Phi(s, t_0)x_2\|).
\]

Particularly, for \( N(t) = t \), if \( \Phi(., .) \) is u.e.-N-d. then \( \Phi(., .) \) is called an uniform exponential dichotomic (and we write u.e.d.) process. If \( \Phi(., .) \) is u.e-N-d. (respectively u.e.d.) and \( X_{t_0}^1 = X_{t_0}^2 \) for every \( t_0 \leq 0 \) then \( \Phi(., .) \) is called an uniform exponential -\( N \)-stable (respectively uniform exponential stable) process.

Remark 1.3. \( \Phi(., .) \) is u.e-N-d. if and only if the inequalities \( d_1 \) and \( d_2 \) from Definition 1.2. hold for all \( t \leq s + 1 > s \leq t_0 \leq 0 \).

Indeed, if \( t_0 \geq s \geq t \geq s + 1, x_1 \in X_{t_0}^1 \) and \( T_2 \in X_{t_0}^2 \) then
\[
N(\|\Phi(t, t_0)x_1\|) \leq N(\varphi(t-s))N(\|\Phi(s, t_0)x_1\|) \geq N(\varphi(1))N(\|\Phi(s, t_0)x_1\|) \geq
\]
\[
\leq N(\varphi(1)).e^{-\nu_1(t-s)}N(\|\Phi(s, t_0)x_1\|)
\]
and
\[
M_2 e^{-\nu_2} N(\|\Phi(s, t_0)x_2\|) \leq N(\|\Phi(s + 1, t_0)x_2\|) \leq
\]
\[
\leq N(\varphi(s + 1 - t)).N(\|\Phi(t, t_0)x_2\|) \leq
\]
\[
\leq N(\varphi(1)).e^{-\nu_2}(t-s)N(\|\Phi(t, t_0)x_2\|).
A necessary and sufficient condition for the uniform exponential stability of a linear evolutionary process in a Banach space has been proved by Dakto in [1]. The extension of Datko's theorem for uniform exponential dichotomy has been obtained by Preda and Megan in [3].

The case of uniform exponential-N-stable processes has been considered by Ichikawa in [2]. The particular case when the process is a strongly continuous semigroup of bounded linear operators has been studied by Zabczyk in [5] and Rolewicz in [4].

In this paper we shall extend these results in two directions. First, we shall give a characterization of u.e.-N-dichotomy, which can be considered as a generalization of Datko's theorem. Second, we shall not assume the linearity and boundedness of the process \( \Phi(.,.) \). The obtained results are applicable for a large class of nonlinear differential equations described in [2].

\[ \text{II - PRELIMINARY RESULTS} \]

An useful characterization of the uniform exponential-N-dichotomy property is given by

**Proposition 2.1**

The evolutionary process \( \Phi(.,.) \) is u.e.-N-d. if and only if there are two continuous functions \( \varphi_1, \varphi_2 : [0, \infty) \to (0, \infty) \) with the properties:

\[ \begin{align*}
Nd_1' & \quad N(\|\phi(t,t_0)x_1\|) \leq \varphi_1(t-s)N(\|\Phi(s,t_0)x_1\|) \\
Nd_2' & \quad N(\|\phi(t,t_0)x_2\|) \leq \varphi_2(t-s)N(\|\Phi(s,t_0)x_2\|) \\
Nd_3' & \quad \lim_{t \to \infty} \varphi_1(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \varphi_2(t) = \infty \quad \text{for all} \quad t \geq s \geq t_0 \geq 0,
\end{align*} \]

\( x_1 \in X^1_{t_0} \quad \text{and} \quad x_2 \in X^2_{t_0} \).

**Proof.**

The necessity is obvious from Definition 1.2 for \( \varphi_1(t) = M_1e^{-\nu_1t} \) and \( \varphi_2(t) = M_2e^{\nu_2t} \).

The sufficiency. From \( (Nd_3') \) it follows that there are \( s_1, s_2 > 0 \) such that \( \varphi_1(s_1) < 1 \) and \( \varphi_2(s_2) > 1 \). Then for all \( t \geq s \geq t_0 \) there are two natural numbers \( n_1 \) and \( n_2 \) such that

\[ t - s = n_1s_1 + r_1 = n_2s_2 + r_2, \quad \text{where} \quad r_1 \in [0, s_2]. \]

From \( (e_3') \) and \( (Nd_1') \) it results that if \( t \geq s \geq t_0 \geq 0 \) and \( x_1 \in X^1_{t_0} \) then

\[ N(\|\Phi(t,t_0)x_1\|) \leq N(\varphi(r_1))N(\|\Phi(s+n_1x_1,t_0)x_1\|) \leq N(\varphi(s_1))\varphi_1(s_1). \]
\[ N(\|\Phi(s, t_0)x_1\|) \leq M_1 e^{-v_1(t-s)} N(\|\Phi(s, t_0)x_1\|) \]

where \( M_1 = N(\varphi(s_1)) e^{v_1 s_1} = \frac{N(\varphi(s_1))}{\varphi_1(x_1)} \) and \( v_1 = \frac{-\ell n \varphi_1(s_1)}{x_1} \).

Similarly, if \( t \geq s \geq t_0 \geq 0 \) and \( x_2 \in X_1^2 \) then
\[
N(\|\Phi(t, t_0)x_2\|) \geq \varphi_2(r_2) N(\|(s + n_2 s_2, t_0)x_2\|) \geq \varphi_2(r_2) \varphi_2(s_2)^2.
\]
\[
N(\|\Phi(s, t_0)x_2\|) \geq M_2 e^{v_2 s_2} N(\|\Phi(s, t_0)x_2\|)
\]
\[
M_2 e^{v_2(t-s)} N(\|\Phi(s, t_0)x_2\|),
\]

where \( m_2 = \inf_{0 \leq t \leq s_2} \varphi_2(t) \), \( M_2 = \frac{m_2}{\varphi_2(s_2)} \) and \( v_2 = \frac{\ell n \varphi_2(s_2)}{s_2} \).

In virtue of Definition 1.2 it follows that \( \Phi(., .) \) is u.e-N-d.

**Corollary 2.1.**

The evolutionary process \( \Phi(., .) \) is u.e.d. if and only if there are two continuous functions \( \varphi_1, \varphi_2 : (0, \infty) \to (0, \infty) \) with the properties:

\[ d'1) \|\Phi(t, t_0)x_1\| \leq \varphi_1(t - s).\|\Phi(s, t_0)x_1\|, \]
\[ d'2) \|\Phi(t, t_0)x_2\| \geq \varphi_2(t - s).\|\Phi(s, t_0)x_2\|, \]
\[ Nd'3 \lim_{t \to \infty} \varphi_1(t) = 0 \quad \text{and} \quad \lim_{t \to -\infty} \varphi_2(t) = \infty. \]

for all \( t \geq s \geq t_0 \geq 0 \), \( x_1 \in X_{t_0}^1 \) and \( x_2 \in S_{t_0}^2 \).

**Proof.** : Is obvious from Proposition 2.1 for \( N(t) = t \).

The relation between u. e-N-d. and u.e.d. properties is given by

**Proposition 2.2 :**

The evolutionary process \( \Phi(., .) \) is u.e.d. if and only if there is \( N \in \mathcal{N} \) such that \( \Phi(., .) \) is u.e-N-d.

**Proof :**

The necessity is obvious from Definition 1.2.

The sufficiency. Suppose that there is \( N \in \mathcal{N} \) such that \( \Phi(., .) \) satisfies the condition \( (Nd_1) \) and \( (Nd_2) \) from Definition 1.2.

Let \( s_1, s_2, s_3 > 0 \) such that \( M_1 N(2) < e^{v_1 s_1} N(2) < M_2 e^{v_2 s_2} \) and \( N(s_3) < M_2 \). If \( t \geq s \geq t_0 \) then there are two natural numbers \( n_1 \) and \( n_2 \) such that \( t - s = n_1 s_1 + r_1 = n_2 s_2 + r_2 \), where \( r_1 \in (0, s_1) \) and \( r_1 \in (0, s_2) \). Then for \( s \geq t_0 \geq 0 \) and \( x_1 \in X_{t_0}^1 \) we have
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\[ N(||\Phi(s, t_0)x_1||) \geq \frac{e^{\nu_1 s_1}}{M_1} N(||\Phi(s + s, t_0)x_1||) \]
\[ \geq N(2)N(||\Phi(s + s_1, t_0)x_1||) \geq N(2||\Phi(s + s_1, t_0)x_1||) \]

and hence (because $N$ is nondecreasing)

\[ ||\Phi(s, t_0)x_1|| \geq 2^i ||(s + s_1, t_0)x_1|| \quad \text{and (by induction)} \]
\[ ||\Phi(s, t_0)x_1|| \geq 2^n ||\Phi(s + ns_1, t_0)x_1|| \quad \text{for every natural number } n. \]

Therefore for $t \geq s \geq t_0 \geq 0$ and $x_1 \in X_{t_0}^1$, we obtain that

\[ ||\phi(t, t_0)x_1|| \leq \varphi(r_1)||\Phi(s + n_1s_1, t_0)x_1|| \leq \frac{\varphi(s)}{2^n} ||\Phi(s, t_0)x_1|| \]

and hence

\[ (2.1) \quad ||\phi(t, t_0)x_1|| \leq \varphi_1(t - s)||\Phi(s, t_0)x_1|| \quad \text{for } t \geq s \geq t_0 \text{ and } x_1 \in X_{t_0}^1, \text{ where } \varphi_1(u) = \frac{\varphi(s)}{2^u/s_1}. \]

On the other hand, for $s \geq t_0 \geq 0$ and $x_2 \in X_{t_0}^2$ we have

\[ ||\phi(t, t_0)x_2|| \geq M_2 e^{\nu_2 s_2} N(||\Phi(s, t_0)x_2||) \geq N(2)N(||\phi(s, t_0)x_2||) \]
\[ \geq N(2||\Phi(s, t_0)x_2||) \]

and hence

\[ ||\phi(s + s_2, t_0)x_2|| \geq 2^n ||\phi(s, t_0)x_2|| \quad \text{and (by induction)} \]
\[ ||\phi(s + ns_2, t_0)x_2|| \geq 2^n ||\Phi(s, t_0)x_2|| \quad \text{for all } s \geq t_0 \geq 0, \text{ } x_2 \in X_{t_0}^2 \text{ and every natural number } n. \]

hence, if $t \geq s \geq t_0 \geq 0$ and $x_2 \in X_{t_0}^2$ then

\[ N(||\Phi(t, t_0)x_2||) = N(||\Phi(s + n_2s_2 + r_2, t_0)x_2||)M_2 e^{\nu_2 r_2} N(||\Phi(s + n_2s_2, t_0)x_2||) \]
\[ \geq N(s_3 ||\Phi(s + n_2s_2, t_0)x_2||), \]

which implies

\[ ||\Phi(t, t_0)x_2|| \geq s_3 ||\Phi(s + n_2s_2, t_0)x_2|| \geq 2^{n_2} s_3 ||\Phi(s, t_0)x_2|| \]

and hence

\[ (2.2) \quad ||\phi(t, t_0)x_2|| \geq \varphi_2(t - s)||\Phi(s, t_0)x_2|| \quad \text{for } t \geq s \geq 0 \text{ and } x_2 \in X_{t_0}^2, \text{ where } \varphi_2(u) = \frac{s_3}{2^{u/s_2}}. \]
From (2.1), (2.2) and Corollary 2.1 it follows that \( \Phi(\cdot, \cdot) \) is u.e.d.

3 - THE MAIN RESULTS.

The following theorem is an extension of Datko’s theorem ([1]) to the general case of uniform exponential-N-dichotomy.

Theorem 3.1.

The evolutionary process \( \Phi(\cdot, \cdot) \) is u.e.N-d. if and only if there are \( M, m > 0 \) such that

\[
\begin{align*}
(Nd_1') & \quad \int_t^{\infty} N(\|\Phi(s, t_0)x_1\|) ds \leq M.N(\|\Phi(t, t_0)x_1\|), \\
(Nd_2') & \quad \int_t^{\infty} N(\|\Phi(s, t_0)x_2\|) ds M.N(\|\Phi(t, t_0)x_2\|), \\
(Nd_3') & \quad N(\|\Phi(t + 1, t_0)x_2\|) m.N(\|\Phi(t, t_0)x_2\|)
\end{align*}
\]

for all \( t \geq t_0 \geq 0, x_1 \in X_{t_0}^1 \) and \( x_2 \in X_{t_0}^2 \).

Proof. The necessity is simply verified. Now we prove the sufficiency part.

Let \( s \geq t_0 \geq 0, x_1 \in X_{t_0}^1 \) and \( \frac{1}{M_0} = \int_0^1 \frac{dt}{\psi(t)} \), where \( \psi = N.\phi \).

If \( t \geq s + 1 \) then

\[
\begin{align*}
\frac{N(\|\Phi(t, t_0)x_1\|)}{M_0} & = \int_0^1 \frac{N(\|\Phi(t, t_0)x_1\|)}{\psi(r)} dr \leq \int_s^1 \frac{N(\|\Phi(t, t_0)x_1\|)}{\psi(t - v)} dr \\
& \leq \int_s^t N(\|\Phi(v, t_0)x_1\|) dv \leq \int_s^\infty N(\|\Phi(t, t_0)x_1\|) dv \leq M.N(\|\Phi(s, t_0)x_1\|)
\end{align*}
\]

and hence

\[
N(\|\Phi(t, t_0)x_1\|) \leq M.M_0 N(\|\Phi(s, t_0)x_1\|),
\]

for all \( t \geq s + 1 \geq t_0 \geq 0 \) and \( x_1 \in X_{t_0}^1 \).

Therefore

\[
(t - s - 1)N(\|\Phi(t, t_0)x_1\|) = \int_s^{t-1} N(\|\Phi(t, t_0)x_1\|) ds \leq M.M_0 \int_s^{\infty} N(\|\Phi(t, t_0)x_1\|) dv \leq M^2.M_0.N(\|\Phi(s, t_0)x_1\|),
\]

which implies

\( (3.1) \ N(\|\Phi(t, t_0)x_1\|) \leq \varphi_1(t - s)N(\|\Phi(s, t_0)x_1\|), \)
for all \( t \geq s + 1 \geq s \geq t_0 \geq 0 \) and \( x_1 \in X^1_{t_0} \), where

\[
\varphi_1(v) = \frac{M M_0 (1 + M)}{1 + v}
\]

Let \( t_0 \geq 0 \), \( x_2 \in X^2_{t_0} \) and \( s \geq t_0 + 1 \). Then

\[
\frac{N(||\Phi(s, t_0)x_2||)}{M_0} \leq N(||\Phi(s, t_0)x_2||) \int_{t_0}^{s} \frac{dv}{\psi(s - v)} \leq \int_{t_0}^{s} N(||\Phi(v, t_0)x_2||) dv \leq \int_{t_0}^{t} N(||\Phi(v, t_0)x_2||) dv \leq M N(||\Phi(t, t_0)x_2||)
\]

and hence

\[
N(||\Phi(t, t_0)x_2||) \geq \frac{N(||\Phi(s, t_0)x_2||)}{M M_0} \quad \text{for all } t \geq s \geq t_0 + 1 \text{ and } x_2 \in X^2_{t_0}.
\]

If \( t \geq s + 1 \geq s \geq t_0 \) then (by preceding inequality and Nd'')

\[
N(||\Phi(y, t_0)x_2||) \geq \frac{N(||\Phi(s + 1, t_0)x_2||)}{M M_0} \geq \frac{m N(||\Phi(s, t_0)x_2||)}{M M_0} \geq \frac{N(||\Phi(s, t_0)x_2||)}{M_2}
\]

for all \( x_2 \in X^2_{t_0} \), where \( \frac{1}{M_2} = \min\{\frac{1}{M M_0}, \frac{m}{M M_0}\} \).

Therefore

\[
(t - s - 1) N(||\Phi(t, t_0)x_2||) \leq M_2 \int_{s+1}^{t} N(||\Phi(v, t_0)x_2||) dv \leq M_2 \int_{t_0}^{t} N(||\Phi(v, t_0)x_1||) dv \leq M M_2 N(||\Phi(t, t_0)x_2||),
\]

which implies

\[
(3.2) \quad N(||\Phi(t, t_0)x_2||) \geq \varphi_2(t - s) N(||\Phi(s, t_0)x_2||)
\]

for all \( t \geq s + 1 \geq s \geq t_0 \geq 0 \) and \( x_2 \in X^2_{t_0} \), where \( \varphi_2 = \frac{v + 1}{M_2 (M + 1)} \).

From (3.1), (3.2) and Proposition 2.1 it follows that \( \Phi(., .) \) is u.e-N-d.

As a particular case we obtain
Corollary 3.1

The evolutionary process $\Phi(\cdot, \cdot)$ is u. e. d. if and only if there are two positive constants $M$ and $m$ such that

\[(d_1') \quad \int_t^\infty \|\Phi(s, t_0)x_1\| ds \leq M\|\Phi(t, t_0)x_1\|,\]

\[(d_2') \quad \int_{t_0}^t \|\Phi(s, t_0)x_2\| ds \leq M\|\Phi(t, t_0)x_2\|,\]

\[(d_3') \quad \|\Phi(t + 1, t_0)x_2\| ds \geq m\|\Phi(t, t_0)x_2\|,
\]

for all $t \geq t_0 \geq 0$, $x_1 \in X_{t_0}^1$ and $x_2 \in X_{t_0}^2$.

**Proof.** Is obvious from Theorem 3.1 for $N(t) = t$.

**Remark 3.1** Corollary 3.1 is a nonlinear version of Theorem 3.2 form [3]. It is an extension of Theorem 2.1 from [2] from the general case of uniform exponential dichotomy.

**Remark 3.2.** Corollary 3.1 remains valid if the power 1 from $(d_1')$ and $(d_2')$ is replaced by any $p \in (1, \infty)$, i.e. the inequalities $(d_1')$ and $(d_2')$ can be replaced, respectively, by

\[(d_1'') \quad \int_t^\infty \|\Phi(s, t_0)x_1\|^p ds \leq M\|\Phi(t, t_0)x_1\|^p\]

and

\[(d_2'') \quad \int_{t_0}^t \|\Phi(s, t_0)x_2\|^p ds \leq M\|\Phi(t, t_0)x_2\|^p.\]

The proof follows almost verbatim from those given in the case $p = 1$ for $N(t) = t$. 
REFERENCES


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