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Annales mathématiques Blaise Pascal, tome 1, n° 2 (1994), p. 1-6

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EXPONENTIAL-BESSEL PARTIAL DIFFERENTIAL EQUATION AND FOX'S H -FUNCTION

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ABSTRACT : In this paper, we present and solve a two dimensional Exponential-Bessel partial differential equation, and obtain a particular solution of it involving Fox's H -function.

1. - INTRODUCTION . The object of this paper is to formulate a two dimensional Exponential-Bessel partial differential equation and obtain its double series solution. We further present a particular solution of our Exponential-Bessel equation involving Fox's H -function. It is interesting to note that the particular solution also yields a new two dimensional series expansion for Fox's H -function involving exponential functions and Bessel functions.

The H -function introduced by Fox [5, p. 408], will be represented as follows :

$$(1.1) \quad H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_1, e_1), \dots, (a_p, e_p) \\ (b_1, f_1), \dots, (b_q, f_q) \end{matrix} \right. \right] \equiv H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right. \right].$$

In what follows for sake of brevity :

$$\sum_{j=1}^p e_j - \sum_{j=1}^q f_j \equiv A, \sum_{j=1}^n e_j - \sum_{j=n+1}^p e_j + \sum_{j=1}^m f_j - \sum_{j=m+1}^q f_j \equiv B.$$

The following formulae are required in the proof :

The integral [2, p. 704, (2.2)] :

$$(1.2) \quad \int_0^\Pi \cos 2ux \left(\sin \frac{x}{2} \right)^{-2w_1} H_{p,q}^{m,n} \left[z \left(\sin \frac{x}{2} \right)^{-2h} \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx$$

$$= \sqrt{(\Pi)} H_{p+2, q+2}^{m+1, n+1} \left[z \middle| \begin{matrix} (1-w_1-2u, h), (a_p, e_p), (1-w_1+2u, h) \\ (1/2-w_1, h), (b_q, f_q), (1-w_1, h) \end{matrix} \right],$$

where $h > 0$, $\sum_{j=1}^p e_j - \sum_{j=1}^q f_j \equiv A \leq 0$, $\sum_{j=1}^n e_j - \sum_{j=n+1}^1 e_j + \sum_{j=1}^m e_j - \sum_{j=m+1}^q f_j \equiv B > 0$,
 $|\arg z| < 1/2B\Pi$, $\operatorname{Re}(1-2w_1) - 2h \max_{1 \leq j \leq n} [\operatorname{Re}(a_j - 1)/e_j] > 0$.

The integral [7, p. 94, (2.2)] :

$$(1.3) \quad \int_0^\infty y^{w_2-1} \sin y J_\nu(y) H_{p,q}^{m,n} \left[zy^{2k} \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dy$$

$$= 2^{w_2-1} \sqrt{\Pi} H_{p+4, q+1}^{m+1, n+1} \left[\begin{matrix} 2^{2k} z & \middle| & \begin{matrix} \left(\frac{1-w_2-v}{2}, k \right) & , & (a_p, e_p) & , & \left(1 + \frac{v-w_2}{2}, k \right) \\ \left(1 - \frac{v+w_2}{2}, k \right) & , & \left(\frac{1+v-w_2}{2}, k \right) \\ \left(\frac{1}{2} - w_2, 2k \right) & , & (b_q, f_q) \end{matrix} \end{matrix} \right],$$

where $k > 0$, $A \leq 0$, $B > 0$, $|\arg z| < 1/2B\Pi$, $\operatorname{Re}(w_2 + v) + 2k \min_{1 \leq j \leq m} [\operatorname{Re} b_j / f_j] > 0$.

The orthogonality property of the Bessel functions [6, p. 291, (6)] :

$$(1.4) \quad \int_0^\infty x^{-1} J_{a+2n+1}(x) J_{a+2m+1}(x) dx$$

$$= \begin{cases} 0, & \text{if } m \neq n; \\ (4n+2a+2)^{-1}, & \text{if } m = n, \operatorname{Re} a + m + n > -1. \end{cases}$$

The following orthogonality property :

$$(1.5) \quad \int_0^\Pi e^{2imx} \cos 2nx dx = \begin{cases} 0, & m \neq n \\ \Pi/2, & m = n \neq 0 \\ \Pi, & m = n = 0. \end{cases}$$

2. TWO DIMENSIONAL EXPONENTIAL-BESSEL PARTIAL DIFFERENTIAL EQUATION

Let us consider

$$(2.1) \quad \frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} + y^2 u,$$

where $u \equiv u(x, y, t)$ and $u(x, y, 0) = f(x, y)$.

To solve (2.1), we assume that (2.1) has a solution of the form :

$$(2.2) \quad u(x, y, t) = e^{4cr^2t + (v+2s+1)^2t} X(ix)Y(y).$$

The substitution of (2.2) into (2.1) yields :

$$(2.3) \quad -c[X'' + 4r^2X]Y + X[y^2Y'' + yY' + \{y^2 - (v+2s+1)^2\}Y] = 0.$$

We see that $X'' + 4r^2X = 0$ has a solution $X = e^{2rix}$ and $y^2Y'' + yY' + \{y^2 - (v+2s+1)^2\}Y = 0$ is Bessel equation [1, p. 200, (6.25)], with solution $Y = J_{v+2s+1}(y)$. Therefore the solution of (2.1) is of the form :

$$(2.4) \quad u(x, y, t) = e^{4cr^2t + (v+2s+1)^2t} e^{2rix} J_{v+2s+1}(y).$$

In view of the principle of superposition, the general solution of (2.1) is given by

$$(2.5) \quad u(x, y, t) = \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s} e^{4cr^2t + (v+2s+1)^2t + 2rix} J_{v+2s+1}(y).$$

In (2.5), putting $t = 0$, we get

$$(2.6) \quad f(x, y) = \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s} e^{2rix} J_{v+2s+1}(y).$$

Multiplying both sides of (2.6) by $y^{-1} \cos 2ux J_{v+2w+1}(y)$, integrating with respect to y from 0 to ∞ and with respect to x from 0 to Π , then using (1.4) and (1.5), the Fourier Exponential-Bessel coefficients are given by

$$(2.7) \quad A_{r,s} = \frac{4}{\Pi} (v+2s+1) \times \int_0^{\Pi} \int_0^{\infty} f(x, y) y^{-1} \cos 2rx J_{v+2s+1}(y) dy dx.$$

In view of the theory of double and multiple Fourier series given by Carslaw and Jaeger [3, pp. 180-183], and many other references, such as Erdélyi [4, pp. 64-65] etc..., the double series (2.6) is convergent, provided the function $f(x, y)$ is defined in the region $0 < x < \pi, 0 < y < \infty$. In brief, the double series (2.6) converges, if the double integral on the right hand side of (2.7) exists.

In the subsequent section, we take $f(x, y)$ as Fox's H -function and present another method to obtain Fourier exponential-Bessel coefficients $A_{r,s}$.

3. PARTICULAR SOLUTION INVOLVING FOX'S H -FUNCTION

The particular solution to be obtained is

$$(3.1) \quad u(x, y, t) = 2^{W_2+1} \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} e^{4cr^2t + (v+2s+1)^2t + 2riz} (v+2+s+1) j_{v+2s+1}(y)$$

$$\times H_{p+6, q+3}^{m+2, n+2} \left[\begin{array}{c} | \\ 2^{2k} z \left| \begin{array}{l} (1-w_1-2r, h), \left(-\frac{w_2+v+2s}{2}, k\right), (a_p, e_p), \\ (1-w_1+2r, h), \left(1+\frac{v+2s+1-w_2}{2}, k\right), \\ \left(1+\frac{v+2s+1+w_2}{2}, k\right), \left(1+\frac{v+2s-w_2}{2}, k\right) \\ \left(\frac{1}{2}-w, h\right), \left(\frac{1}{2}-w_2, k\right), (b_q, f_q), (1-w_1, h) \end{array} \right. \end{array} \right],$$

valid under the conditions of (1.2), (1.3) and (1;4).

Proof. Let

$$(3.2) \quad f(x, y) = \left(\sin \frac{x}{2}\right)^{-2w_1} y^{w_2} \sin y H_{p,q}^{m,n} \left[\begin{array}{c} | \\ z \left(\sin \frac{x}{2}\right)^{-2h} y^{2k} \left| \begin{array}{l} (a_p, e_p) \\ (b_q, f_q) \end{array} \right. \end{array} \right]$$

$$= \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s} e^{2irx} J_{v+2s+1}(y).$$

Equation (3.2) is valid, since $f(x, y)$ is defined in the region $0 < x < \pi, 0 < y < \infty$.

Multiplying both sides of (3.2) by $y^{-1} J_{v+2w+1}(y)$ and integrating with respect to y from 0 to ∞ , then using (1.3) and (1;4). Now multiplying both sides of the resulting

expression by $\cos 2ux$ and integrating with respect to x from 0 to Π , then using (1.2) and (1.5), we obtain the value of $A_{r,s}$. Substituting this value of $A_{r,s}$ in (2.5), the expansion (3.1) is obtained.

NOTE 1 : The value of $A_{0,s}$ is one-half the value of $A_{r,s}$.

NOTE 2 : If we put $t = 0$ in (3.1), it reduces to a new two dimensional series expansion for Fox's H -function involving exponential functions and Bessel functions.

Since on specializing the parameters Fox's H -function yields almost all special functions appearing in applied mathematics and physical sciences. Therefore, the result (3.1) presented in this paper is of a general character and hence may encompass several cases of interest.

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