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for  $p$ -adic  $C^n$ -functions**

*Annales mathématiques Blaise Pascal*, tome 1, n° 1 (1994), p. 61-74

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## THE WEIERSTRASS-STONE APPROXIMATION THEOREM FOR $p$ -ADIC $C^n$ -FUNCTIONS

J. Araujo and Wim H. Schikhof

### Abstract.

Let  $K$  be a non-Archimedean valued field. Then, on compact subsets of  $K$ , every  $K$ -valued  $C^n$ -function can be approximated in the  $C^n$ -topology by polynomial functions (Theorem 1.4). This result is extended to a Weierstrass-Stone type theorem (Theorem 2.10).

### INTRODUCTION

The non-archimedean version of the classical Weierstrass Approximation Theorem - the case  $n = 0$  of the Abstract - is well known and named after Kaplansky ([1], 5.28). To investigate the case  $n = 1$  first let us return to the Archimedean case and consider a real-valued  $C^1$ -function  $f$  on the unit interval. To find a polynomial function  $P$  such that both  $|f - P|$  and  $|f' - P'|$  are smaller or equal than a prescribed  $\varepsilon > 0$  one simply can apply the standard Weierstrass Theorem to  $f'$  obtaining a polynomial function  $Q$  for which  $|f' - Q| \leq \varepsilon$ . Then  $x \mapsto P(x) := f(0) + \int_0^x Q(t)dt$  solves the problem.

Now let  $f : X \rightarrow K$  be a  $C^1$ -function where  $K$  is a non-archimedean valued field and  $X \subset K$  is compact.

Lacking an indefinite integral the above method no longer works. There do exist continuous linear antiderivations ([3], §64) but they do not map polynomials into polynomials ([3], Ex. 30.C). A further complicating factor is that the natural norm for  $C^1$ -functions on  $X$  is given by

$$f \mapsto \max\{|f(x)| : x \in X\} \vee \max\left\{\left|\frac{f(x)-f(y)}{x-y}\right| : x, y \in X, x \neq y\right\}$$

rather than the more classical formula

$$f \mapsto \max\{|f(x)| : x \in X\} \vee \max\{|f'(x)| : x \in X\}.$$

(Observe that in the real case both formulas lead to the same norm thanks to the Mean Value Theorem, see [3], §§26,27 for further discussions.)

Thus, to obtain non-archimedean  $C^n$ -Weierstrass-Stone Theorems for  $n \in \{1, 2, \dots\}$  our methods will necessarily deviate from the 'classical' ones.

## 0. PRELIMINARIES

1. Throughout  $K$  is a non-archimedean complete valued field whose valuation  $|\cdot|$  is not trivial. For  $a \in K$ ,  $r > 0$  we write  $B(a, r) := \{x \in K : |x - a| \leq r\}$ , the 'closed' ball about  $a$  with radius  $r$ . 'Clopen' is an abbreviation for 'closed and open'. The function  $x \mapsto x$  ( $x \in K$ ) is denoted  $\mathcal{X}$ . The  $K$ -valued characteristic function of a subset  $Y$  of  $K$  is written  $\xi_Y$ . For a set  $Z$ , a function  $f : Z \rightarrow K$  and a set  $W \subset Z$  we define  $\|f\|_W := \sup\{|f(z)| : z \in W\}$  (allowing the value  $\infty$ ). The cardinality of a set  $\Gamma$  is  $\#\Gamma$ .  $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ ,  $\mathbb{N} := \{1, 2, 3, \dots\}$ .

We now recall some facts from [2], [3] on  $C^n$ -theory.

2. For a set  $Y \subset K$ ,  $n \in \mathbb{N}$  we set  $\nabla^n Y := \{(y_1, y_2, \dots, y_n) \in Y^n : i \neq j \implies y_i \neq y_j\}$ . For  $f : Y \rightarrow K$ ,  $n \in \mathbb{N}_0$  we define its  $n$ th difference quotient  $\Phi_n f : \nabla^{n+1} Y \rightarrow K$  inductively by  $\Phi_0 f := f$  and the formula

$$\Phi_n f(y_1, \dots, y_{n+1}) = \frac{\Phi_{n-1} f(y_1, y_3, \dots, y_{n+1}) - \Phi_{n-1} f(y_2, y_3, \dots, y_{n+1})}{y_1 - y_2}$$

$f$  is called a  $C^n$ -function if  $\Phi_n f$  can be extended to a continuous function on  $Y^{n+1}$ . The set of all  $C^n$ -functions  $Y \rightarrow K$  is denoted  $C^n(Y \rightarrow K)$ . The function  $f : Y \rightarrow K$  is a  $C^\infty$ -function if it is in  $C^\infty(Y \rightarrow K) := \bigcap_{n=0}^{\infty} C^n(Y \rightarrow K)$ . The space  $C^0(Y \rightarrow K)$ , consisting of all continuous functions  $Y \rightarrow K$  is sometimes written as  $C(Y \rightarrow K)$ .

**FROM NOW ON IN THIS PAPER  $X$  IS A NONEMPTY  
COMPACT SUBSET OF  $K$  WITHOUT ISOLATED POINTS.**

3. Since  $X$  has no isolated points we have for an  $f \in C^n(X \rightarrow K)$  that the continuous extension of  $\Phi_n f$  to  $X^{n+1}$  is unique; we denote this extension by  $\overline{\Phi}_n f$ . Also we write

$$D_n f(a) := \overline{\Phi}_n f(a, a, \dots, a) \quad (a \in X)$$

The following facts are proved in [2] and [3].

### Proposition 0.3.

- (i) For each  $n \in \mathbb{N}_0$  the space  $C^n(X \rightarrow K)$  is a  $K$ -algebra under pointwise operations.
- (ii)  $C^0(X \rightarrow K) \supset C^1(X \rightarrow K) \supset \dots$

- (iii) If  $f \in C^n(X \rightarrow K)$  then  $f$  is  $n$  times differentiable and  $j!D_j f = f^{(j)}$  for each  $j \in \{0, 1, \dots, n\}$ . More generally, if  $i, j \in \{0, 1, \dots, n\}$ ,  $i+j \leq n$  then  $\binom{i+j}{i} D_i D_j f = D_{i+j} f$ .
- (iv) If  $f \in C^n(X \rightarrow K)$  then for  $x, y \in X$  we have Taylor's formula

$$f(x) = f(y) + (x-y)D_1 f(y) + \dots + (x-y)^{n-1} D_{n-1} f(y) + (x-y)^n \rho_1 f(x, y),$$

$$\text{where } \rho_1 f(x, y) = \bar{\Phi}_n f(x, y, y, \dots, y).$$

4. Since  $X$  is compact the difference quotients  $\Phi_i f$  ( $0 \leq i \leq n$ ) are bounded if  $f \in C^n(X \rightarrow K)$ . We set

$$\|f\|_{n,X} := \max\{\|\Phi_i f\|_{\nabla^{i+1}X} : 0 \leq i \leq n\}.$$

Then  $\|f\|_{0,X} = \|f\|_X$ . We quote the following from [2] and [3]. Recall that a function  $f : X \rightarrow K$  is a local polynomial if for every  $a \in X$  there is a neighbourhood  $U$  of  $a$  such that  $f|_{X \cap U}$  is a polynomial function.

**Proposition 0.4.** Let  $n \in \mathbb{N}_0$ .

- (i) The function  $\| \cdot \|_{n,X}$  is a norm on  $C^n(X \rightarrow K)$  making it into a  $K$ -Banach algebra.
- (ii) The local polynomials form a dense subset of  $C^n(X \rightarrow K)$ .
- (iii) The function

$$f \mapsto \|f\|_{n,X}^{\sim} := \max_{0 \leq i \leq n-1} \|D_i f\|_X \vee \|\rho_1 f\|_X$$

(see Proposition 0.3 (iv)) also is a norm on  $C^n(X \rightarrow K)$ . We have

$$\|f\|_{n,X} = \max\{\|D_i f\|_{n-i,X}^{\sim} : 0 \leq i \leq n\} \quad (f \in C^n(X \rightarrow K)).$$

### Remarks

- Proposition 0.4 (ii) will also follow from Proposition 2.8.
- In general  $\| \cdot \|_{n,X}^{\sim}$  is not equivalent to  $\| \cdot \|_{n,X}$  for  $n \geq 3$  (see [3], Example 83.2).

# 1 THE WEIERSTRASS THEOREM FOR $C^n$ -FUNCTIONS

The following product rule for difference quotients is easily proved by induction with respect to  $j$ .

Let  $f, g : X \rightarrow K$ , let  $j \in \mathbb{N}_0$ . Then for all  $(x_1, \dots, x_{j+1}) \in \nabla^{j+1}X$  we have

$$\Phi_j(fg)(x_1, \dots, x_{j+1}) = \sum_{k=0}^j \Phi_k f(x_1, \dots, x_{k+1}) \Phi_{j-k} g(x_{k+1}, \dots, x_{j+1}).$$

Or, less precise,

$$\Phi_j(fg)(x_1, \dots, x_{j+1}) = \sum_{k=0}^j \Phi_k f(z_k) \Phi_{j-k} g(u_{j-k})$$

for certain  $z_k \in \nabla^{k+1}X$ ,  $u_{j-k} \in \nabla^{j-k+1}X$ .

In the sequel we need an extension of this formula to finite products of functions. The proof is straightforward by induction with respect to  $N$ .

**Lemma 1.1. (Product Rule)** *Let  $h_1, \dots, h_N : X \rightarrow K$ , let  $j \in \mathbb{N}_0$ . Then for all  $(x_1, \dots, x_{j+1}) \in \nabla^{j+1}X$  we have*

$$\Phi_j\left(\prod_{s=1}^N h_s\right)(x_1, \dots, x_{j+1}) = \sum_{\sigma} \prod_{s=1}^N \Phi_{j_s} h_s(z_{\sigma,s})$$

where the sum is taken over all  $\sigma := (j_1, \dots, j_N) \in \mathbb{N}_0^N$  for which  $j_1 + \dots + j_N = j$  and where  $z_{\sigma,s} \in \nabla^{j_s+1}X$  for each  $s \in \{1, \dots, N\}$ . (In fact,  $z_{\sigma,1} = (x_1, \dots, x_{j_1+1})$ ,  $z_{\sigma,2} = (x_{j_1+1}, \dots, x_{j_1+j_2+1})$ ,  $\dots$ ,  $z_{\sigma,N} = (x_{j_1+\dots+j_{N-1}+1}, \dots, x_{j+1})$ .)

The following key lemma grew out of [1], 5.28.

**Lemma 1.2.** *Let  $0 < \delta < 1$ ,  $0 < \varepsilon < 1$ , let  $B = B_0 \cup B_1 \cup \dots \cup B_m$  where  $B_0, \dots, B_m$  are pairwise disjoint 'closed' balls in  $K$  of radius  $\delta$ . Then, for each  $n \in \{0, 1, \dots\}$  there exists a polynomial function  $P : K \rightarrow K$  such that  $\|P - \xi_B\|_{n,B} \leq \varepsilon$ .*

**Proof.** We may assume  $0 \in B_0$ . Choose  $c_1 \in B_1, \dots, c_m \in B_m$ ; we may assume that  $|c_1| \leq |c_2| \leq \dots \leq |c_m|$ . Then  $\delta < |c_1|$ . We shall prove the following statement by induction with respect to  $n$ .

Let  $k \in \mathbb{N}$  be such that  $(\delta/|c_1|)^k \leq \varepsilon \delta^n$ ,  $k > n$ . Let  $t_1, t_2, \dots, t_m \in \mathbb{N}$  be such that for all  $\ell \in \{1, \dots, m\}$

$$(1) \quad \left| \frac{c_\ell}{c_1} \right|^{kt_1} \left| \frac{c_\ell}{c_2} \right|^{kt_2} \dots \left| \frac{c_\ell}{c_{\ell-1}} \right|^{kt_{\ell-1}} \left( \frac{\delta}{|c_1|} \right)^{t_\ell} \leq \varepsilon \delta^n$$

(It is easily seen that such  $k, t_1, \dots, t_m$  exist since  $\delta/|c_1| < 1$ .) Then the formula

$$P(x) = \prod_{i=1}^m \left(1 - \left(\frac{x}{c_i}\right)^{t_i}\right)$$

defines a polynomial function  $P : K \rightarrow K$  for which

$$\|P - \xi_{B_0}\|_{n,B} \leq \varepsilon.$$

The case  $n = 0$  is proved in [1], 5.28. To prove the step  $n - 1 \rightarrow n$  we first observe that from the induction hypothesis (with  $\varepsilon$  replaced by  $\varepsilon\delta$ ) it follows that

$$(2) \quad \|P - \xi_{B_0}\|_{n-1,B} \leq \varepsilon\delta$$

So it remains to be shown that

$$(3) \quad |\Phi_n(P - \xi_{B_0})(x_1, \dots, x_{n+1})| \leq \varepsilon$$

for all  $(x_1, \dots, x_{n+1}) \in \nabla^{n+1}B$ . Now, if  $|x_i - x_j| > \delta$  for some  $i, j \in \{1, \dots, n+1\}$  we have, using (2),

$|\Phi_n(P - \xi_{B_0})(x_1, \dots, x_{n+1})| = |x_i - x_j|^{-1} \cdot |\Phi_{n-1}(P - \xi_{B_0})(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}) - \Phi_{n-1}(P - \xi_{B_0})(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})| \leq \delta^{-1} \cdot \varepsilon\delta = \varepsilon$ . So this reduces the proof of (3) to the case where  $|x_i - x_j| \leq \delta$  for all  $i, j \in \{1, \dots, n+1\}$ ; in other words we may assume that  $x_1, \dots, x_{n+1}$  are all in the same  $B_\ell$  for some  $\ell \in \{0, 1, \dots, m\}$ . But then, after observing that  $n \geq 1$ , we have  $\Phi_n \xi_{B_0}(x_1, \dots, x_{n+1}) = 0$  so it suffices to prove the following.

If  $\ell \in \{0, 1, \dots, m\}$  and  $x_1, \dots, x_{n+1} \in B_\ell$  are pairwise distinct then

$$(4) \quad |\Phi_n P(x_1, \dots, x_{n+1})| \leq \varepsilon$$

To prove it we introduce, with  $\ell \in \{1, \dots, m\}$  fixed, the constants  $M_i$  ( $i \in \{1, \dots, n\}$ ) by

$$M_i := \begin{cases} 1 & \text{if } i > \ell \\ \delta/|c_1| & \text{if } i = \ell \\ |c_\ell/c_i|^k & \text{if } i < \ell \end{cases}$$

and use the following three steps.

**Step 1.** For each  $j \in \{0, 1, \dots, n\}$ ,  $i \in \{1, \dots, n\}$  we have

$$\left\| \Phi_j \left(1 - \left(\frac{x}{c_i}\right)^k\right) \right\|_{\nabla^{j+1}B_\ell} \leq \begin{cases} 1 & \text{if } \ell = 0, j = 0 \\ \delta^{-j} \left(\frac{\delta}{|c_1|}\right)^k & \text{if } \ell = 0, j > 0 \\ \delta^{-j} M_i & \text{if } \ell > 0. \end{cases}$$

**Proof.**

a. The case  $j = 0$ . Then for  $x \in B_\ell$  we have

- if  $i > \ell$  then  $|1 - (\frac{x}{c_i})^k| = 1$
- if  $i = \ell$  then  $|1 - (\frac{x}{c_i})^k| = |\frac{c_i - x}{c_i}|^k \leq \frac{\delta^k}{|c_i|^k} \leq \frac{\delta}{|c_1|}$
- if  $i < \ell$  then  $|1 - (\frac{x}{c_i})^k| = |\frac{x}{c_i}|^k = |\frac{c_\ell}{c_i}|^k$

and the statement follows.

b. The case  $j > 0$ . Then  $\Phi_j(1) = 0$  so that

$$\Phi_j(1 - (\frac{\mathcal{X}}{c_i})^k) = \frac{1}{c_i^k} \Phi_j(\mathcal{X}^k)$$

Let  $(x_1, \dots, x_{j+1}) \in \nabla^{j+1} B_\ell$ . By the Product Rule 1.1,  $\Phi_j(\mathcal{X}^k)(x_1, \dots, x_{j+1})$  is a sum of terms of the form  $\prod_{s=1}^k (\Phi_{j_s} \mathcal{X})(z_s)$ . Such a term is 0 if one of the  $j_s$  is  $> 1$ , so we only have to deal with  $j_s = 0$  (then  $\Phi_{j_s} \mathcal{X} = \mathcal{X}$ ) or  $j_s = 1$  (then  $\Phi_{j_s} \mathcal{X} = 1$ ). The latter case occurs  $j$  times (as  $\sum_{s=1}^k j_s = j$ ) and it follows that

$\prod_{s=1}^k (\Phi_{j_s} \mathcal{X})(z_s)$  is a product of  $k-j$  distinct terms taken from  $\{x_1, \dots, x_{j+1}\}$  (observe that, indeed,  $j < k$  since  $j \leq n < k$ ), so its absolute value is  $\leq |c_\ell|^{k-j}$ . It follows that  $\|\Phi_j(1 - (\frac{\mathcal{X}}{c_i})^k)\|_{\nabla^{j+1} B_\ell} \leq |c_\ell|^{k-j}/|c_i|^k$  from which we conclude

- if  $\ell = 0$ :  $|c_\ell|^{k-j}/|c_i|^k \leq \delta^{k-j}/|c_1|^k = \delta^{-j}(\delta/|c_1|)^k$ ,
- if  $i > \ell > 0$ :  $|c_\ell|^{k-j}/|c_i|^k \leq |c_\ell^{-j}| < \delta^{-j} = \delta^{-j} M_i$
- if  $i = \ell > 0$ :  $|c_\ell|^{k-j}/|c_i|^k \leq |c_\ell^{-j}| \leq |c_1^{-j}| = \delta^{-j}(\frac{\delta}{|c_1|})^j \leq \delta^{-j} M_i$
- if  $i < \ell$ :  $|c_\ell|^{k-j}/|c_i|^k \leq |c_\ell|^{-j} |\frac{c_\ell}{c_i}|^k \leq \delta^{-j} M_i$

and step 1 is proved.

**Step 2.** For each  $j \in \{0, 1, \dots, n\}$ ,  $i \in \{1, \dots, n\}$  we have

$$\|\Phi_j(1 - (\frac{\mathcal{X}}{c_i})^k)^{t_i}\|_{\nabla^{j+1} B_\ell} \leq \begin{cases} 1 & \text{if } \ell = 0, j = 0 \\ \delta^{-j}(\frac{\delta}{|c_1|})^k & \text{if } \ell = 0, j > 0 \\ \delta^{-j} M_i^{t_i} & \text{if } \ell > 0 \end{cases}$$

**Proof.** The case  $j = 0$  follows directly from Step 1, part a, so assume  $j > 0$ . By the Product Rule 1.1 applied to  $h_s = 1 - (\frac{\mathcal{X}}{c_i})^k$  for all  $s \in \{1, \dots, t_i\}$  we have for  $(x_1, \dots, x_{j+1}) \in \nabla^{j+1} B_\ell$  that  $\Phi_j(1 - (\frac{\mathcal{X}}{c_i})^k)^{t_i}(x_1, \dots, x_{j+1})$  is a sum of terms of the form

$$(5) \quad \prod_{s=1}^{t_i} \Phi_{j_s}(1 - (\frac{\mathcal{X}}{c_i})^k)(z_s)$$

where  $j_1 + \dots + j_s = j$ . If  $\ell = 0$  it follows from Step 1 that the absolute value of (5) is  $\leq \prod \delta^{-j_s} (\frac{\delta}{|c_1|})^k$  where the product is taken over all  $s$  in the nonempty set  $\Gamma := \{s \in \{1, \dots, t_i\} : j_s > 0\}$ , so the product is  $\leq \delta^{-j} (\frac{\delta}{|c_1|})^{k \cdot \#\Gamma} \leq \delta^{-j} (\frac{\delta}{|c_1|})^k$ . If  $\ell > 0$  it follows from Step 1 that the absolute value of (5) is  $\leq \prod_{s=1}^{t_i} \delta^{-j_s} M_i = \delta^{-j} M_i^{t_i}$ .

The statement of Step 2 follows.

**Step 3.** Proof of (4). Again, the Product Rule 1.1, now applied to  $h_i = (1 - (\frac{\mathcal{X}}{c_i})^k)^{t_i}$  for  $i \in \{1, \dots, m\}$  tells us that for  $(x_1, \dots, x_{n+1}) \in \nabla^{n+1} B_\ell$  the expression  $\Phi_n P(x_1, \dots, x_{n+1})$  is a sum of terms of the form

$$(6) \quad \prod_{i=1}^m \Phi_{n_i} (1 - (\frac{\mathcal{X}}{c_i})^k)^{t_i} (z_s)$$

where  $n_1 + \dots + n_m = n$ . If  $\ell = 0$  we have by Step 2 that the absolute value of (6) is  $\leq \prod \delta^{-n_i} (\frac{\delta}{|c_1|})^k$  where the product is taken over  $i$  in the nonempty set  $\Gamma := \{i : n_i \neq 0\}$ , so the product is  $\leq \delta^{-n} (\frac{\delta}{|c_1|})^{k \cdot \#\Gamma} \leq \delta^{-n} (\frac{\delta}{|c_1|})^k \leq \delta^{-n} \cdot \varepsilon \delta^n = \varepsilon$ , where we used the assumption  $(\delta/|c_1|)^k \leq \varepsilon \delta^n$ . We see that  $|\Phi_n P(x_1, \dots, x_{n+1})| \leq \varepsilon$  if  $(x_1, \dots, x_n) \in B_0$ . Now let  $\ell > 0$ . By Step 2 we have that the absolute value of (6) is  $\leq \prod_{i=1}^m \delta^{-n_i} M_i^{t_i} = \delta^{-n} M_1^{t_1} \dots M_m^{t_m} = \delta^{-n} \cdot |\frac{c_\ell}{c_1}|^{kt_1} \dots |\frac{c_\ell}{c_{t-1}}|^{kt_t} (\frac{\delta}{|c_1|})^{t_t}$  which is  $\leq \delta^{-n} \varepsilon \delta^n$  by (1). This proves (4) and the Lemma.

**Corollary 1.3.** For every locally constant  $f : X \rightarrow K$ , for every  $n \in \mathbb{N}_0$  and  $\varepsilon > 0$  there exists a polynomial function  $P : K \rightarrow K$  such that  $\|f - P\|_{n,X} \leq \varepsilon$ .

**Proof.** There exist a  $\delta \in (0, 1)$ , pairwise disjoint 'closed' balls  $B_1, \dots, B_m$  of radius  $\delta$  covering  $X$  and  $\lambda_1, \dots, \lambda_m \in K$  such that

$$f(x) = \sum_{i=1}^m \lambda_i \xi_{B_i}(x) \quad (x \in X)$$

By Lemma 1.2 there exist polynomials  $P_1, \dots, P_m$  such that  $\|\xi_{B_i} - P_i\|_{n,X} \leq \|\xi_{B_i} - P_i\|_{n, \cup B_i} \leq \varepsilon(|\lambda_i| + 1)^{-1}$  for each  $i \in \{1, \dots, m\}$ . Then  $P := \sum_{i=1}^m \lambda_i P_i$  is a polynomial function and  $\|f - P\|_{n,X} \leq \max_i \|\lambda_i(\xi_{B_i} - P_i)\|_{n,X} \leq \max_i |\lambda_i| \varepsilon(|\lambda_i| + 1)^{-1} \leq \varepsilon$ .

**Theorem 1.4. ( $C^n$ -Weierstrass Theorem)** For each  $n \in \mathbb{N}_0$ ,  $f \in C^n(X \rightarrow K)$  and  $\varepsilon > 0$  there exists a polynomial function  $P : K \rightarrow K$  such that  $\|f - P\|_{n,X} \leq \varepsilon$ .

**Proof.** There is by Proposition 0.4 a local polynomial  $g : K \rightarrow K$  with  $\|f - g\|_{n,X} \leq \varepsilon$ . This  $g$  has the form  $g = \sum_{i=1}^m Q_i h_i$  where  $Q_1, \dots, Q_m$  are polynomials and  $h_1, \dots, h_m$

are locally constant. By Corollary 1.3 we can find polynomials  $P_1, \dots, P_m$  for which  $\|h_i - P_i\|_{n,X} \leq \varepsilon(\|Q_i\|_{n,X} + 1)^{-1}$  for each  $i$ . Then  $P := \sum_{i=1}^m Q_i P_i$  is a polynomial and  $\|g - P\|_{n,X} \leq \varepsilon$ . It follows that  $\|f - P\|_{n,X} \leq \max(\|f - g\|_{n,X}, \|g - P\|_{n,X}) \leq \varepsilon$ .

**Remarks.**

1. In the case where  $X = \mathbb{Z}_p$ ,  $K \supset \mathbb{Q}_p$  the above Theorem 1.4 is not new: The Mahler base  $e_0, e_1, \dots$  of  $C(\mathbb{Z}_p \rightarrow K)$  defined by  $e_m(x) = \binom{x}{m}$  is proved in [3], §54 to be a Schauder base for  $C^n(\mathbb{Z}_p \rightarrow K)$ , for each  $n$ .
2. It follows directly from Theorem 1.4 that the polynomial functions  $X \rightarrow K$  form a dense subset of  $C^\infty(X \rightarrow K)$ .

## 2. A WEIERSTRASS-STONE THEOREM FOR $C^n$ -FUNCTIONS

For this Theorem (2.10) we will need the continuity of  $g \mapsto g \circ f$  in the  $C^n$ -topologies (Proposition 2.5). To prove it we need some technical lemmas that are in the spirit of [3], §77.

Let  $n \in \mathbb{N}$ . For a function  $h : \nabla^n X \rightarrow K$  we define  $\Delta h : \nabla^{n+1} X \rightarrow K$  by the formula

$$\Delta h(x_1, x_2, \dots, x_{n+1}) = \frac{h(x_1, x_3, x_4, \dots, x_{n+1}) - h(x_2, x_3, \dots, x_{n+1})}{x_1 - x_2}$$

We have the following product rule.

**Lemma 2.1. (Product Rule).** *Let  $n \in \mathbb{N}$ , let  $h, t : \nabla^n X \rightarrow K$ . Then for all  $(x_1, x_2, \dots, x_{n+1}) \in \nabla^{n+1} X$  we have  $\Delta(ht)(x_1, x_2, \dots, x_{n+1}) = h(x_2, x_3, \dots, x_{n+1})\Delta t(x_1, x_2, \dots, x_{n+1}) + t(x_1, x_3, \dots, x_{n+1})\Delta h(x_1, x_2, \dots, x_{n+1})$ .*

**Proof.** Straightforward.

**Lemma 2.2.** *Let  $f : X \rightarrow K$ ,  $n \in \mathbb{N}_0$ . Let  $S_n$  be the set of the following functions defined on  $\nabla^{n+1} X$ .*

$$\begin{aligned} (x_1, \dots, x_{n+1}) &\mapsto \Phi_1 f(x_{i_1}, x_{i_2}) & (1 \leq i_1 < i_2 \leq n+1) \\ (x_1, \dots, x_{n+1}) &\mapsto \Phi_2 f(x_{i_1}, x_{i_2}, x_{i_3}) & (1 \leq i_1 < i_2 < i_3 \leq n+1) \\ &\vdots \\ (x_1, \dots, x_{n+1}) &\mapsto \Phi_n f(x_1, \dots, x_{n+1}). \end{aligned}$$

For  $k \in \mathbb{N}$ , let  $R_n^k$  be the additive group generated by  $S_n, S_n^2, \dots, S_n^k$  where, for each  $j \in \{1, \dots, k\}$ ,  $S_n^j$  is the product set  $\{h_1 h_2 \dots h_j : h_i \in S_n \text{ for each } i \in \{1, \dots, j\}\}$ . Then, for all  $k, n \in \mathbb{N}$ ,  $\Delta R_n^k \subset R_{n+1}^k$ .

**Proof.** We use induction with respect to  $k$ . For the case  $k = 1$  it suffices to prove  $h \in S_n \Rightarrow \Delta h \in R_{n+1}^1$ . Then  $h$  has the form

$$(x_1, \dots, x_{n+1}) \mapsto \Phi_j f(x_{i_1}, x_{i_2}, \dots, x_{i_{j+1}})$$

for some  $j \in \{2, 3, \dots, n+1\}$  and so

$$\Delta h(x_1, x_2, \dots, x_{n+1}) = \frac{h(x_1, x_3, \dots, x_{n+2}) - h(x_2, x_3, \dots, x_{n+2})}{x_1 - x_2}$$

vanishes if  $i_1 > 1$  (and then  $\Delta h$  is the null function), while if  $i_1 = 1$  it equals

$$\begin{aligned} & \frac{\Phi_j f(x_1, x_{i_2+1}, \dots, x_{i_{j+1}+1}) - \Phi_j f(x_2, x_{i_2+1}, \dots, x_{i_{j+1}+1})}{x_1 - x_2} = \\ & = \Phi_{j+1} f(x_1, x_2, x_{i_2+1}, \dots, x_{i_{j+1}+1}) \end{aligned}$$

and it follows that  $\Delta h \in S_{n+1} \subset R_{n+1}^1$ . For the induction step assume  $\Delta R_n^{k-1} \subset R_{n+1}^{k-1}$ ; it suffices to prove that  $\Delta S_n^k \subset R_{n+1}^k$ . So let  $h \in S_n^k$  and write  $h = h_1 H$ , where  $h_1 \in S_n$ ,  $H \in S_n^{k-1}$ . By the Product Rule 2.1 we have

$$\begin{aligned} \Delta h(x_1, \dots, x_{n+2}) &= h_1(x_2, x_3, \dots, x_{n+2}) \Delta H(x_1, x_2, \dots, x_{n+2}) + \\ &+ H(x_1, x_3, \dots, x_{n+2}) \Delta h_1(x_1, x_2, \dots, x_{n+2}). \end{aligned}$$

The fact that  $h_1 \in S_n$  makes

$$(x_1, x_2, \dots, x_{n+2}) \mapsto h_1(x_1, x_3, \dots, x_{n+2})$$

into an element of  $S_{n+1}$ . Similarly, since  $H \in S_n^{k-1}$ , the function

$$(x_1, x_2, \dots, x_{n+2}) \mapsto H(x_2, x_3, \dots, x_{n+2})$$

is in  $S_{n+1}^{k-1}$ . By our first induction step,  $\Delta h_1 \in R_{n+1}^1$  and by the induction hypothesis  $\Delta H \in R_{n+1}^{k-1}$ . Hence,

$$\begin{aligned} \Delta h &\in S_{n+1} R_{n+1}^{k-1} + S_{n+1}^{k-1} R_{n+1}^1 \\ &\subset R_{n+1}^1 R_{n+1}^{k-1} + R_{n+1}^{k-1} R_{n+1}^1 \subset R_{n+1}^k. \end{aligned}$$

**Lemma 2.3.** Let  $f, n, S_n, k, R_n^k$  be as in the previous lemma. Let  $f(X) \subset Y \subset K$  where  $Y$  has no isolated points. Let  $g : Y \rightarrow K$  be a  $C^n$ -function. Let  $B_n$  be the set of the following functions defined on  $\nabla^{n+1} X$ .

$$\begin{aligned} (x_1, \dots, x_{n+1}) &\mapsto \overline{\Phi}_1 g(f(x_{i_1}), f(x_{i_2})) & (1 \leq i_1 < i_2 \leq n+1) \\ (x_1, \dots, x_{n+1}) &\mapsto \overline{\Phi}_2 g(f(x_{i_1}), f(x_{i_2}), f(x_{i_3})) & (1 \leq i_1 < i_2 < i_3 \leq n+1) \\ &\vdots \\ (x_1, \dots, x_{n+1}) &\mapsto \overline{\Phi}_n g(f(x_1), f(x_2), \dots, f(x_{n+1})). \end{aligned}$$

Let  $A_n$  be the additive group generated by  $B_n R_n^n$ . Then

$$\Delta A_n \subset A_{n+1}.$$

**Proof.** We prove:  $h \in B_n R_n^n \Rightarrow \Delta h \in A_{n+1}$ . Write  $h = br$  where  $b \in B_n$ ,  $r \in R_n^n$ . By the Product Rule 2.1 we have for all  $(x_1, x_2, \dots, x_{n+2}) \in \nabla^{n+2} X$

$$\begin{aligned} \Delta h(x_1, x_2, \dots, x_{n+2}) &= b(x_2, x_3, \dots, x_{n+2}) \Delta r(x_1, x_2, \dots, x_{n+2}) + \\ &+ r(x_1, x_3, \dots, x_{n+2}) \Delta b(x_1, x_2, \dots, x_{n+2}). \end{aligned}$$

We have:

- (i)  $b \in B_n$  so  $(x_1, \dots, x_{n+2}) \mapsto b(x_2, x_3, \dots, x_{n+2})$  is in  $B_{n+1}$ .
- (ii)  $r \in R_n^n$  so  $(x_1, \dots, x_{n+2}) \mapsto r(x_1, x_3, \dots, x_{n+2})$  is in  $R_{n+1}^n$  (in the previous proof we had  $r \in S_n^k \Rightarrow$  the map  $(x_1, \dots, x_{n+2}) \mapsto r(x_1, x_3, \dots, x_{n+1})$  is in  $S_{n+1}^k$ , and (ii) follows from this).
- (iii)  $r \in R_n^n$  so  $\Delta r \in R_{n+1}^n$  (Previous Lemma).
- (iv)  $b$  has the form

$$(x_1, x_2, \dots, x_{n+1}) \mapsto \overline{\Phi}_j g(f(x_{i_1}), \dots, f(x_{i_{j+1}}))$$

for some  $j \in \{2, \dots, n+1\}$  and so

$$\Delta b(x_1, x_2, \dots, x_{n+2}) = \frac{b(x_1, x_3, x_4, \dots, x_{n+2}) - b(x_2, x_3, \dots, x_{n+2})}{x_1 - x_2}$$

vanishes if  $i_1 > 1$  (and then  $\Delta b$  is the null function), while if  $i_1 = 1$  it equals

$$\begin{aligned} &\frac{\overline{\Phi}_j g(f(x_1), f(x_{i_2+1}), \dots, f(x_{i_{j+1}+1})) - \overline{\Phi}_j g(f(x_2), f(x_{i_2+1}), \dots, f(x_{i_{j+1}+1}))}{x_1 - x_2} \\ &= \overline{\Phi}_{j+1} g(f(x_1), f(x_2), f(x_{i_2+1}), \dots, f(x_{i_{j+1}+1})) \Phi_1 f(x_1, x_2). \end{aligned}$$

(if  $f(x_1) = f(x_2)$  we have 0 at both sides). So we see that  $\Delta b \in B_{n+1} R_{n+1}^1$ .

Combining (i) - (iv) we get  $\Delta h \in B_{n+1} R_{n+1}^n + R_{n+1}^n B_{n+1} R_{n+1}^1 \subset B_{n+1} R_{n+1}^{n+1} + B_{n+1} R_{n+1}^{n+1} \subset A_{n+1}$ .

**Corollary 2.4.** With the notations as in the previous lemma we have  $\Phi_n(g \circ f) \in A_n$  ( $n \in \mathbb{N}$ ).

**Proof.** We proceed by induction on  $n$ . For the case  $n = 1$  we write, for  $(x_1, x_2) \in \nabla^2 X$ ,

$$\Phi_1(g \circ f)(x_1, x_2) = (x_1 - x_2)^{-1} (g(f(x_1)) - g(f(x_2))) = \overline{\Phi}_1 g(f(x_1), f(x_2)) \Phi_1 f(x_1, x_2).$$

Hence,  $\overline{\Phi}_1(g \circ f) \in B_1 S_1 \subset B_1 R_1^1 \subset A_1$ . To prove the step  $n \rightarrow n+1$  observe that by the induction hypothesis,  $\Phi_n(g \circ f) \in A_n$ . By Lemma 2.3,  $\Phi_{n+1}(g \circ f) = \Delta \Phi_n(g \circ f) \in A_{n+1}$ .

**Remark.** From Corollary 2.4 it follows easily that the composition of two  $C^n$ -functions is again a  $C^n$ -function, a result that already was obtained in [3], 77.5.

**Proposition 2.5.** (Continuity of  $g \mapsto g \circ f$ ) Let  $n \in \mathbb{N}_0$ , let  $f \in C^n(X \rightarrow K)$  and let  $g \in C^n(Y \rightarrow K)$  where  $Y$  has no isolated points,  $Y \supset f(X)$ . Then  $\|g \circ f\|_{n,X} \leq \|g\|_{n,Y} \max_{0 \leq j \leq n} \|f\|_{j,X}^j$ .

**Proof.** We may assume  $\|g\|_{n,Y} < \infty$ . It suffices to prove  $\|\Phi_n(g \circ f)\|_{\nabla^{n+1}X} \leq \|g\|_{n,Y} \|f\|_{n,X}^n$ . Now  $\|\Phi_0(g \circ f)\|_{\nabla^1 X} = \max_{x \in X} |g(f(x))| \leq \|g\|_{0,Y} = \|g\|_{0,Y} \|f\|_{0,X}^0$  which proves the case  $n = 0$ . For  $n \geq 1$  we apply Corollary 2.4 which says that  $\Phi_n(g \circ f) \in A_n$  i.e.  $\Phi_n(g \circ f)$  is a sum of functions in  $B_n S_n^n$ . By the definition of  $B_n$  we have

$$(*) \quad h \in B_n \Rightarrow \|h\|_{\nabla^{n+1}X} \leq \|g\|_{n,Y}$$

Similarly

$$k \in S_n \Rightarrow \|k\|_{\nabla^{n+1}X} \leq \max_{1 \leq i \leq n} \|\Phi_i f\|_{\nabla^{i+1}X} \leq \|f\|_{n,X}$$

so that

$$(**) \quad k \in S_n^n \Rightarrow \|k\|_{\nabla^{n+1}X} \leq \|f\|_{n,X}^n.$$

Combination of (\*) and (\*\*) yields  $\|\Phi_n(g \circ f)\|_{\nabla^{n+1}X} \leq \|g\|_{n,Y} \|f\|_{n,X}^n$ .

Proposition 2.5 enables us to prove

**Proposition 2.6.** Let  $n \in \mathbb{N}_0$  and let  $A$  be a closed subalgebra of  $C^n(X \rightarrow K)$ . Suppose  $A$  separates the points of  $X$  and contains the constant functions. Then  $A$  contains all locally constant functions  $X \rightarrow K$ .

**Proof.** 1. We first prove that  $f \in A$ ,  $U \subset K$ ,  $U$  clopen implies  $\xi_{f^{-1}(U)} \in A$ . In fact,  $f(X)$  is compact so there exist a  $\delta \in (0, 1)$  and finitely many disjoint balls  $B_1, \dots, B_m$  of radius  $\delta$  covering  $f(X)$  where, say,  $B_1, \dots, B_q$  lie in  $U$ , and  $B_{q+1}, \dots, B_m$  are in  $K \setminus U$ . Let  $\varepsilon > 0$ . By the Key Lemma 1.2 there exists, for each  $i \in \{1, \dots, m\}$  a polynomial  $P_i$  such that  $\|\xi_{B_i} - P_i\|_{n,B} < \varepsilon$ , where  $B := \bigcup_{i=1}^m B_i$ . Then  $P := \sum_{i=1}^q P_i$  is a polynomial and

$$\|P - \xi_U\|_{n,B} = \|P - \xi_{B^0}\|_{n,B} = \left\| \sum_{i=1}^q (P_i - \xi_{B_i}) \right\|_{n,B} < \varepsilon, \text{ where } B^0 := \bigcup_{i=1}^q B_i.$$

By Proposition 2.5

$$\|(P - \xi_U) \circ f\|_{n,X} \leq \|P - \xi_U\|_{n,B} \max_{0 \leq j \leq n} \|f\|_{j,X}^j \leq \varepsilon \max_{0 \leq j \leq n} \|f\|_{j,X}^j$$

and we see that there exists a sequence  $P_1, P_2, \dots$  of polynomials such that  $\|P_k \circ f - \xi_U \circ f\|_{n,X} \rightarrow 0$ . Since  $A$  is an algebra with an identity we have  $P_k \circ f \in A$  for all  $k$ . Then  $\xi_{f^{-1}(U)} = \xi_U \circ f = \lim_{k \rightarrow \infty} P_k \circ f \in A$ .

2. Now consider

$$\mathcal{B} := \{V \subset X, \xi_V \in A\}.$$

It is very easy to see that  $\mathcal{B}$  is a ring of clopen subsets of  $X$  and that  $\mathcal{B}$  covers  $X$ . To show that  $\mathcal{B}$  separates the points of  $X$  let  $x \in X, y \in X, x \neq y$ . Then there is an  $f \in A$  for which  $f(x) \neq f(y)$ . Set  $U := \{\lambda \in K : |\lambda - f(x)| < |f(x) - f(y)|\}$ . Then  $U$  is clopen in  $K$ . By the first part of the proof,  $f^{-1}(U) \in \mathcal{B}$ . But  $x \in f^{-1}(U)$  whereas  $y \notin f^{-1}(U)$ . By [1], Exercise 2.H  $\mathcal{B}$  is the ring of all clopens of  $X$ . It follows easily that all locally constant functions are in  $A$ .

To arrive at the Weierstrass-Stone Theorem 2.10 we need a final technical lemma.

**Lemma 2.7.** *Let  $a_1, \dots, a_m \in X$ , let  $\delta_1, \dots, \delta_m$  be in  $(0, 1)$  such that  $B(a_1, \delta_1), \dots, B(a_m, \delta_m)$  form a disjoint covering of  $X$ . Let  $n \in \mathbb{N}_0$ ,  $h \in C^n(X \rightarrow K)$  and suppose  $D_j h(a_i) = 0$  and  $|\overline{\Phi}_{n-j} D_j h(x_1, \dots, x_{n-j+1})| \leq \varepsilon$  for all  $i \in \{1, \dots, m\}$ ,  $x_1, \dots, x_{n+1} \in B(a_i, \delta_i) \cap X$ ,  $j \in \{0, 1, \dots, n\}$ . Then  $\|h\|_{n,X} \leq \varepsilon$ .*

**Proof.** We first prove that  $\|h\|_{n,X} \leq \varepsilon$  (see Proposition 0.4(iii)). Let  $i \in \{1, \dots, m\}$ . Set  $B_i = B(a_i, \delta_i)$ . By Taylor's formula (Proposition 0.3(iv)) we have for  $x \in X \cap B_i$ :

$$|h(x)| = \left| \sum_{s=0}^{n-1} (x - a_i)^s D_s h(a_i) + (x - a_i)^n \rho_1 h(x, a_i) \right| = |x - a_i|^n |\overline{\Phi}_n h(x, a_i, a_i, \dots, a_i)| \leq \delta_i^n \varepsilon.$$

Similarly we have for  $j \in \{0, \dots, n-1\}$  and  $x \in X \cap B_i$ :  $|D_j h(x)| =$

$$\left| \sum_{t=0}^{n-1-j} (x - a_i)^t D_t D_j h(a_i) + (x - a_i)^{n-j} \rho_1 (D_j h)(x, a_i) \right|.$$

Now using Proposition 0.3(iii) we see that  $D_t D_j h(a_i) = 0$  so that

$$(*) \quad |D_j h(x)| = |x - a_i|^{n-j} |\overline{\Phi}_{n-j} D_j h(x, a_i, \dots, a_i)| \leq \delta_i^{n-j} \varepsilon.$$

It follows that  $\|h\|_X, \|D_1 h\|_X, \dots, \|D_{n-1} h\|_X$  are all  $\leq \varepsilon$ . Now let  $x, y \in X$ . If  $x, y$  are in the same  $B_i$  then  $|\rho_1 h(x, y)| = |\overline{\Phi}_n h(x, y, y, \dots, y)| \leq \varepsilon$  by assumption. If  $x \in B_i, y \in B_s$  and  $i \neq s$  then  $|x - y| \geq \delta := \max(\delta_i, \delta_s)$  and by Taylor's formula

$$h(x) = \sum_{t=0}^{n-1} (x - y)^t D_t h(y) + (x - y)^n \rho_1 h(x, y)$$

we obtain, using (\*),

$$\begin{aligned} |\rho_1 h(x, y)| &\leq \frac{|h(x) - h(y)|}{|(x - y)^n|} \vee \frac{|D_1 h(y)|}{|x - y|^{n-1}} \vee \dots \vee \frac{|D_{n-1} h(y)|}{|x - y|} \\ &\leq \frac{\delta^n \varepsilon}{\delta^n} \vee \frac{\delta_s^{n-1} \varepsilon}{\delta^{n-1}} \vee \dots \vee \frac{\delta_s \varepsilon}{\delta} \leq \varepsilon \end{aligned}$$

and we have proved  $\|h\|_{n,X} \leq \varepsilon$ .

Now to prove that even  $\|h\|_{n,X} \leq \varepsilon$  observe that by Proposition 0.4(iii)

$$\|h\|_{n,X} = \|h\|_{n,X} \vee \|D_1 h\|_{n-1,X} \vee \cdots \vee \|D_n h\|_{0,X}.$$

To prove, for example, that  $\|D_1 h\|_{n-1,X} \leq \varepsilon$  we observe that  $D_1 h \in C^{n-1}(X \rightarrow K)$  and that for  $i \in \{1, \dots, m\}$  and  $j \in \{0, 1, \dots, n-2\}$  we have  $D_j D_1 h(a_i) = (j+1)D_{j+1} h(a_i) = 0$  and for all  $x_1, \dots, x_n \in B(a_i, \delta_i)$  and  $j \in \{0, 1, \dots, n-2\}$

$$|\bar{\Phi}_{n-1-j} D_j (D_1 h)(x_1, \dots, x_{n-j})| = |(j+1)| |\bar{\Phi}_{n-1-j} D_{j+1} h(x_1, \dots, x_{n-j})| \leq \varepsilon$$

by assumption. So the conditions of our Lemma (with  $D_1 h$ ,  $n-1$  in place of  $h$ ,  $n$  respectively) are satisfied and by the first part of the proof we may conclude that  $\|D_1 h\|_{n-1,X} \leq \varepsilon$ . In a similar way we prove that  $\|D_2 h\|_{n-2,X} \leq \varepsilon, \dots, \|D_n h\|_{0,X} \leq \varepsilon$  and it follows that  $\|h\|_{n,X} \leq \varepsilon$ .

**Proposition 2.8.** *Let  $n \in \mathbb{N}_0$  and let  $A$  be a closed subalgebra of  $C^n(X \rightarrow K)$  containing the locally constant functions. Let  $g \in C^n(X \rightarrow K)$  and suppose for each  $a \in X$  there exists an  $f_a \in A$  with  $D_i g(a) = D_i f_a(a)$  for  $i \in \{0, 1, \dots, n\}$ . Then  $g \in A$ .*

**Proof.** Let  $\varepsilon > 0$ . For each  $a \in X$  choose an  $f_a \in A$  with  $f_a(a) = g(a)$ ,  $D_1 f_a(a) = D_1 g(a), \dots, D_n f_a(a) = D_n g(a)$ . By continuity there exists a  $\delta_a > 0$  such that, with  $h_a := f_a - g$ ,  $|\bar{\Phi}_{n-j} D_j h_a(x_1, \dots, x_{n-j+1})| \leq \varepsilon$  for all  $j \in \{0, 1, \dots, n\}$  and  $x_1, \dots, x_{n-j+1} \in B(a, \delta_a)$ . The  $B(a, \delta_a)$  cover  $X$  and by compactness there exists a finite disjoint subcovering  $B(a_1, \delta_{a_1}), \dots, B(a_m, \delta_{a_m})$ . Set

$$f := \sum_{i=1}^m f_{a_i} \chi_{B(a_i, \delta_{a_i}) \cap X}$$

Then, by our assumption on  $A$ ,  $f \in A$ . By Lemma 2.7, applied to  $h := f - g$  and where  $\delta_1, \dots, \delta_m$  are replaced by  $\delta_{a_1}, \dots, \delta_{a_m}$  respectively, we then have  $\|f - g\|_{n,X} \leq \varepsilon$ . We see that  $g \in \bar{A} = A$ .

**Remark.** It follows directly that the local polynomial functions  $X \rightarrow K$  form a dense subset of  $C^n(X \rightarrow K)$ .

**Proposition 2.9.** *Let  $n \in \mathbb{N}$  and let  $A$  be a  $K$ -subalgebra of  $C^n(X \rightarrow K)$  containing the constant functions. Suppose  $f'(a) \neq 0$  for some  $f \in A$ ,  $a \in X$ . Then there is a  $g \in A$  with  $g(a) = 0$ ,  $g'(a) = 1$  and  $D_2 g(a) = D_3 g(a) = \cdots = D_n g(a) = 0$ .*

**Proof.** By considering the function  $f'(a)^{-1}(f - f(a))$  it follows that we may assume that  $f(a) = 0$ ,  $f'(a) = 1$ . Then

$$(*) \quad f = (\mathcal{X} - a)h$$

where  $h$  is continuous,  $h(a) = 1$ . To obtain the statement by induction with respect to  $n$  we only have to consider the induction step  $n - 1 \rightarrow n$  and, to prove that, we may assume that  $D_2f(a) = \dots = D_{n-1}f(a) = 0$ . From (\*) we obtain

$$f^n = (\mathcal{X} - a)^n h^n$$

and by uniqueness of the Taylor expansion of the  $C^n$ -function  $f^n$  we obtain  $f^n(a) = D_1f^n(a) = \dots = D_{n-1}f^n(a) = 0$  and  $D_nf^n(a) = h^n(a) = 1$ . We see that  $g := f - D_nf(a)f^n$  is in  $A$  and that  $g(a) = 0$ ,  $g'(a) = 1$ ,  $D_2g(a) = \dots = D_{n-1}g(a) = 0$  and  $D_ng(a) = D_nf(a) - D_nf(a)D_nf^n(a) = 0$ .

**Theorem 2.10. (Weierstrass-Stone Theorem for  $C^n$ -functions).** *Let  $n \in \mathbb{N}_0$  and let  $A$  be a closed subalgebra that separates the points of  $X$  and that contains the constant functions. Suppose also that for each  $a \in X$  there exists an  $f \in A$  with  $f'(a) \neq 0$ . Then  $A = C^n(X \rightarrow K)$ .*

**Proof.** By Proposition 2.9, for each  $a \in X$  there exists an  $f \in A$  with  $f(a) = 0$ ,  $f'(a) = 1$ ,  $D_if(a) = 0$  for  $i \in \{2, \dots, n\}$ . The function  $g := \mathcal{X}$  satisfies  $g(a) = 0$ ,  $g'(a) = 1$ ,  $D_ig(a) = 0$  for  $i \in \{2, \dots, n\}$  so applying Proposition 2.8 (observe that  $A$  contains the locally constant functions by Proposition 2.6) we obtain that  $\mathcal{X} \in A$ . But then all polynomials are in  $A$  and  $A = C^n(X \rightarrow K)$  by the Weierstrass Theorem 1.4.

#### Remarks.

1. The case  $n = 0$  yields, at least for those  $X$  that are embeddable into  $K$ , the well known Kaplansky Theorem proved in [1], 6.15.
2. We leave it to the reader to establish a  $C^\infty$ -version of Theorem 2.10.

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