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**GENERATING FUNCTION AND ORTHOGONALITY PROPERTY
OF A CLASS OF POLYNOMIALS OCCURRING
IN QUANTUM MECHANICS**

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ABSTRACT : In this paper, we present a generating function and an orthogonality property of a class of polynomials occurring in quantum mechanics.

Key words : Generating function, Orthogonality property, Hermite Polynomials, Quantum mechanics.

AMS (MOS) : Subject classification : 33C25 , 81

INTRODUCTION : The object of this paper is to present a generating function and an orthogonality property of the polynomials ${}_1F_1(-n; b + 3/2; x^2)$, which occurs in the radical wave function of isotropic harmonic oscillator [4, p. 36, (6.60)].

The generating function for the polynomials ${}_1F_1(-n; b + 3/2; x^2)$ has been obtained as a particular case of the generating function of B -polynomials, which has recently been defined by the author [2]. We obtain the orthogonality property of the polynomials ${}_1F_1(-n; b + 3/2; x^2)$ as a bonus in our attempts to establish an orthogonality property of B -polynomials. We shall use the symbol $H_n^b(x)$ to denote the polynomials ${}_1F_1(-n; b + 3/2; x^2)$.

It is interesting to note that the polynomials $H_n^b(x)$ appear to lead to the generalization of the Hermite polynomials $H_n(x)$ [5, p. 380, (25)].

We visualize at least three orthogonality properties of the B -polynomials for different weight functions on different intervals. However, we have not been successful to establish any of them. The proofs are difficult in view of the general nature of B -polynomials.

In what follows for sake of brevity, the symbol a_p is used to denote a_1, \dots, a_p , the symbol $1 - a_p - m$ is used to denote $1 - a_1 - m, \dots, 1 - a_p - m$ and the notation $\prod_{j=1}^p (a_j)_m$ stands for the product $(a_1)_m \dots (a_p)_m$. Further, the expression

$${}_pF_q \left[\begin{matrix} a_p; z \\ b_q \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n z^n}{(b_1)_n \dots (b_q)_n n!} \quad (1.0)$$

is known as the generalized hypergeometric series or generalized hypergeometric function. Here p and q are positive integers or zero, and we assume that the variable z , the numerator parameters a_1, \dots, a_p and the denominator parameters b_1, \dots, b_q take on complex values, provided that no $b_j (j = 1, \dots, q)$ is zero or a negative integer.

Recently [2], we have defined the B -polynomials :

$$B_m(x) = \frac{\prod_{j=1}^p (a_j)_m}{\prod_{j=1}^q (b_j)_m} r + q + 1 {}^rF_s + p \left[\begin{matrix} c_r, 1 - b_q - m, -m; \frac{\beta}{\alpha} x (-1)^{p-q-1} \\ d_s, 1 - a_p - m \end{matrix} \right] (\alpha)^m, \quad (1.1)$$

by means of the generating function :

$${}_pF_q \left[\begin{matrix} a_p; \alpha t \\ b_q \end{matrix} \right] {}^rF_s \left[\begin{matrix} c_r; \beta x t \\ d_s \end{matrix} \right] = \sum_{m=0}^{\infty} \frac{(\alpha t)^m}{m!} \frac{\prod_{j=1}^p (a_j)_m}{\prod_{j=1}^q (b_j)_m} \cdot r + q + 1 {}^rF_s + p \left[\begin{matrix} c_r, 1 - b_q - m, -m; \frac{\beta}{\alpha} x (-1)^{p-q-1} \\ d_s, 1 - a_p - m \end{matrix} \right] \quad (1.2)$$

The generating function of the polynomials $H_n^b(x)$:

In (1.2), putting $\alpha = \beta = 1, p = q = r = 0, s = 1, d_1 = b + 3/2$, and setting t^2 for t and $-x^2$ for x , we obtain the generating function for ${}_1F_1(-m; b + 3/2; x^2)$:

$${}_0F_0(-; -; t^2) {}_0F_1(-; b + 3/2; -t^2 x^2) = \sum_{m=0}^{\infty} \frac{t^{2m}}{m!} {}_1F_1(-m; b + 3/2; x^2) \quad (1.3)$$

In (1.3), setting ${}_0F_0(-; -; t^2) = e^{t^2}$, ${}_0F_1(-; b + 3/2; -t^2 x^2) = (tx)^{b/2+1/4} \Gamma(b + 3/2) J_{b+1/2}(2\sqrt{tx})$ and ${}_1F_1(-m, b + 3/2; x^2) = H_m^b(x)$, we have

$$e^{t^2} (tx)^{b/2+1/4} \Gamma(b + 3/2) J_{b+1/2}(2\sqrt{tx}) = \sum_{m=0}^{\infty} \frac{t^{2m}}{m!} H_m^b(x) \quad (1.4)$$

The following formulae are required in the proofs :

The integral :

$$\begin{aligned} & \int_{-\infty}^{\infty} x^{2u} e^{-x^2} p^F q \left[\begin{matrix} a_p; zx^2 \\ b_q \end{matrix} \right] dx \\ &= \Gamma(u + 1/2) p_{+1} F_q \left[\begin{matrix} a_p; zx^2 \\ b_q \end{matrix} \right], \end{aligned} \quad (1.5)$$

where $p < q + 1$ (or $p = q + 1$ and $|z| < 1$), $u = 0, 1, 2, \dots$

The integral (1.5) can easily be established by expressing the hypergeometric function in the integrand as [1, p. 322, (10.1)] and interchanging the order of integration and summation, which is justified due to the absolute convergence of the integral and summation involved in the process, and evaluating the inner-integral with the help of the following integral :

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx = \Gamma(n + 1/2), \quad n = 0, 1, 2, \dots \quad (1.6)$$

The integral :

$$\begin{aligned} & \int_{-\infty}^{\infty} x^{2u} e^{-x^2} p^F q \left[\begin{matrix} a_p; zx^2 \\ b_q \end{matrix} \right] r F_s \left[\begin{matrix} c_r; yx^2 \\ d_s \end{matrix} \right] dx \\ &= \sum_{m=0}^{\infty} \frac{\prod_{j=1}^r (c_j)_m}{\prod_{j=1}^s (d_j)_m} \frac{y^m}{m!} \Gamma(m + u + 1/2) p_{+1} F_q \left[\begin{matrix} a_p; m + u + 1/2; x \\ b_q \end{matrix} \right], \end{aligned} \quad (1.7)$$

where in addition to the conditions of (1.5), $r < s + 1$ (or $r = s + 1$ and $|y| < 1$).

To derive (1.7), we use the series representation of ${}_rF_s$, interchange the order of integration and summation and evaluate the resulting integral with the help of (1.5).

The Vandermonde's theorem [3, p. 110, (4.1.2)] :

$${}_2F_1 \left[\begin{matrix} -n, b; 1 \\ c \end{matrix} \right] = \frac{(c-b)_n}{(c)_n}, n = 0, 1, 2, \dots; \quad (1.8)$$

The modified form of the relation [1, p. 308, (9.37)] :

$$H_{2n}(x) = (-1)^n (2)^{2n} (1/2)_n F_1 \left[\begin{matrix} -n; x^2 \\ 1/2 \end{matrix} \right] \quad (1.9)$$

The modified form of the relation [1, p. 312, (6)] :

$$H_{2n+1}(x) = (-1)^n 2^{2n+1} (3/2)_n F_1 \left[\begin{matrix} -n; x^2 \\ 3/2 \end{matrix} \right] \quad (1.10)$$

The Legendre duplication formula [1, p. 58, (2.24)]

$$2^{2x-1} \Gamma(x) \Gamma(x+1/2) = \sqrt{\pi} \Gamma(2x) \quad (1.11)$$

The following well known relations [1, pp. 275, 323] :

$${}_0F_0(-; -; x) = e^x \quad (1.12)$$

$${}_0F_1 \left[\begin{matrix} -; -\frac{x^2}{4} \\ 1/2 \end{matrix} \right] = \cos x \quad (1.13)$$

$${}_0F_1 \left[\begin{matrix} -; -\frac{x^2}{4} \\ 3/2 \end{matrix} \right] = \sin x \quad (1.14)$$

$$(-k)_n = \begin{cases} 0, n > k \\ (k = 1, 2, 3, \dots) \\ (-1)^n n!, k = n \end{cases} \quad (1.15)$$

2. ORTOGONALITY PROPERTY OF THE POLYNOMIALS $H_n^b(x)$.

The polynomials $H_n^b(x)$ are orthogonal with weight $x^{2(b+1)} e^{-x^2}$ on the interval $(-\infty, \infty)$, i.e.

$$\int_{-\infty}^{\infty} x^{2(b+1)} e^{-x^2} H_n^b(x) H_k^b(x) dx = \begin{cases} 0, k \neq n \\ \frac{\Gamma(b+3/2)n!}{(b+3/2)_n}, k = n \end{cases} \quad (2.1)$$

where $b = -1, 0, 1, 2, \dots$

PROOF . In (1.7), setting $y = z = 1, u = b+1, p = q = r = s = 1, a_1 = -n, b_1 = b+3/2; c_1 = -k, d_1 = b+3/2$, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} x^{2(b+1)} e^{-x^2} {}_1F_1(-n; b+3/2; x^2) {}_1F_1(-k; b+3/2; x^2) dx \\ &= \sum_{m=0}^{\infty} \frac{(-k)_m}{(b+3/2)_m} \frac{1}{m!} \Gamma(m+b+3/2) {}_2F_1 \left[\begin{matrix} -n, m+b+3/2; 1 \\ b+3/2 \end{matrix} \right] \end{aligned} \quad (2.2)$$

Now, using the notation $H_n^b(x)$ for ${}_1F_1(-n; b + 3/2; x^2)$ and Vandermonde's theorem (1.8), (2.2) reduces to the form :

$$\begin{aligned} & \int_{-\infty}^{\infty} x^{2(b+1)} e^{-x^2} H_n^b(x) H_k^b(x) dx \\ &= \sum_{m=0}^{\infty} \frac{\Gamma(b + 3/2)(-k)_m(-m)_n}{m!(b + 3/2)_n} \end{aligned} \quad (2.3)$$

From (1.15), it is evident that all terms of the series (2.3) are zero for $m > k \neq n$ and $m < n \neq k$.

If $k = n = m$, we have

$$\int_{-\infty}^{\infty} x^{2(b+1)} e^{-x^2} \left\{ H_n^b(x) \right\}^2 dx = \frac{\Gamma(b + 3/2)n!}{(b + 3/2)_n} \quad (2.4)$$

This proves (2.1)

3. THE POLYNOMIALS $H_n^b(x)$ AND THE HERMITE POLYNOMIALS $H_n(x)$.

(a) Generating functions

(i) In (1.3), putting $b = -1$, and applying (1.9), (1.11), (1.12) and (1.13), it reduces to the generating function [1, p. 174, 2(a)] for the Hermite polynomials.

(ii) In (1.3), setting $b = 0$, and using (1.10), (1.11), (1.12) and (1.14), it yields the generating function [1, p. 174, 2(b)] for the Hermite polynomials.

(b) Orthogonality properties

(i) In (2.1), putting $b = -1$, and applying (1.9), (1.11), (1.12) and (1.13), we obtain the following orthogonality property of the Hermite polynomials :

$$\int_{-\infty}^{\infty} e^{-x^2} H_{2n}(x) H_{2k}(x) dx = \begin{cases} 0, & k \neq n \\ 2^{2n}(2n)! \sqrt{\pi}, & k = n \end{cases} \quad (3.1)$$

(ii) In (2.1), setting $b = 0$, and using (1.10), (1.11), (1.12) and (1.14), it yields the following orthogonality property of the Hermite polynomials :

$$\int_{-\infty}^{\infty} e^{-x^2} H_{2n+1}(x) H_{2k+1}(x) dx = \begin{cases} 0, & k \neq n \\ 2^{2n+1}(2n+1)! \sqrt{\pi}, & k = n \end{cases} \quad (3.2)$$

From (3.1) and (3.2), the orthogonality property of the Hermite polynomials [1, pp. 170-171, (5.17) - (5.22)] follows.

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